BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
DOI: 10.24425/bpasts.2024.150813

THIS IS AN EARLY ACCESS ARTICLE.
This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication.

$M^X/G/1/\infty$ single-server queueing system with random volume customers and multiple vacations

O. TIKHONENKO [1] * and **M. ZIÓŁKOWSKI** [2] **

[1] Faculty of Designing. University of Social Science and Humanities – USWPS, Warsaw, Poland
[2] Institute of Information Technology. Warsaw University of Life Sciences – SGGW, Poland

Abstract. In the present paper, we investigate the model of a single-server queueing system with unlimited queue (of $M^X/G/1/\infty$-type), random volume customers, unlimited memory buffer and multiple vacations. In analyzed system, arriving customers (that form Poisson entrance flow of groups of customers) transport some information measured in bytes so they are assumed to be additionally characterized by some non-negative random volume. Customer's service time generally depends on his volume. Information delivered by a customer is written out into dedicated unlimited memory buffer until customer ends his service. In addition, in considered system the mechanism of multiple vacations is implemented which means that server can have some breaks (rests) for a random period of time but breaks begin only in the moments when there is no customer present in the system. The above-mentioned mechanism has obvious influence on customer's waiting time and, in consequence, on customers' total volume. For the introduced model, we obtain general formula for the steady-state customers' total volume distribution in the terms of Laplace–Stieltjes transforms as well as formulae defining its first two moments. Analysis of some interesting, practical special cases of the model and numerical computations are attached as well together with examples of possible applications of the model regarding real telecommunication or computer systems.

Key words: single-server queueing system, queueing systems with random volume customers, queueing systems with vacations, total volume, Laplace–Stieltjes transform.

1. INTRODUCTION

Queueing theory is a scientific area that was started by A. K. Erlang in the 20's of the previous century. In the beginning, research was concentrated mainly on mathematical models describing telephone exchanges working process.

In first simple models analyzes, some additional assumptions were introduced that made mathematical computations much easier. These models did not take into consideration all technical aspects of analyzed real-life telecommunication systems but approximated satisfactorily their working process. First publications were those connected with queueing systems of the $M/M/n/m$-type according to Kendall's modified notation. In such models, we assume that customers form Poisson arrival flow, customer's service time is exponentially distributed, system has n servers working independently without breaks and m additional places in waiting room (queue) i.e. when all servers are busy, the arriving customers wait patiently (if there is at least one free place in the waiting room) until there is a free server that can start their service (usually according to FIFO discipline). Results obtained in this case are: number of customers distribution (at least in the steady state), waiting time distribution function and loss probability (in this case steady-state loss probability is obviously equal to value p_{n+m}, where p_k, $k = 0,\dots,n+m$ is the steady-state number of customers present in the system). Mathematical analysis of the above-mentioned models is not complicated as it demands primarily the use of Markov chains with continuous time [1].

*e-mail: otikhonenko@swps.edu.pl
**e-mail: marcin_ziolkowski@sggw.edu.pl

Later on, main results for the system of the $M/G/1/\infty$-type were obtained by Felix Pollaczek and Aleksandr Khinchin [2, 3]. Here we have only one server working, customer's service time is distributed in any way and waiting room is unlimited. In this model, mathematical background is a little bit more complicated as analysis needs the use of more complicated random processes and integral transforms (especially Laplace or Laplace–Stieltjes transform). But interestingly, for that simple model we cannot obtain even exact formula for the steady-state number of customers distribution, the main result is presented in the term of a generating function: $P(z) = \sum_{k=0}^{\infty} p_k z^k$. During whole previous century, many interesting papers and monographs were published that faced the problem of real-life telecommunication systems analysis. Investigated systems were modeled by more and more complex stochastic processes - see e.g. [4, 5]. Besides many authors modified well-known models, introducing additional practical assumptions, and obtained new interesting results.

The example of such modifications are queueing systems with vacations. Here we assume that server can have some breaks. Taking breaks from work makes sense because sometimes real devices should be repaired or simply turned off to save energy (especially when using batteries). First papers dealing with this problem were [6–9]. In those articles (investigating queueing models of the $M/G/1/n$ and $M/G/1/\infty$-types) we can find some classical results for models of queueing systems with vacations connected mainly with number of customers distribution and waiting time distribution function. Later on, there appeared many published papers that were devoted to above-mentioned models and their modifications e.g

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication.

[10–12]. Over the last 20 years the problem of such models analyzing is more and more often present in works of many scientists from different countries [13–20] because analyzed models well describe problems occurring in real computer devices (especially those used in computer networks e.g. routers) related to the need to have a certain energy saving mode - see additionally [21–24]. In cited papers some more interesting characteristics were obtained such like number of served packets in some time instant t (in non-steady state) or distribution of time to the first overload of the buffer (in the case when it is limited).

In the end of 20-th century, together with the headway in computer science, the role of queueing theory increased because it was clear that well-known classical models (after some changes or introduced generalizations) may be adapted to new real-life computer systems. By this way, it started the new direction in queueing theory called theory of queueing systems with random volume customers. In analyzed models, it was additionally assumed that each arriving customer has some random volume (as transporting some information measured in bytes) and his service time generally depends on this volume. Moreover, system owns additional memory buffer (limited or unlimited) in which above-mentioned information is written out until customer ends his service. The purpose of analysis for such systems is much wider and mathematical background is more complicated. The main aim of research is to find characteristics of the customers' total volume (the sum of the volumes of all customers present in the system) and formulae defining loss characteristics (in the case when memory buffer is limited). Papers devoted to this issue often used classical models, introducing to them only small corrections [25,26] and authors did not take into consideration substantial influence of the character of dependency between customer's service time and his volume on total customers' volume characteristics (or took it only on the level of marginal distributions). But it turned out quickly that analysis would need new mathematical approach. As examples of initial works analyzing analogous models and introducing new methodological approach, we can indicate articles [27,28] and monograph [29].

In recent years, many new papers appeared also dealing with this problem e.g. [30–35]. Because of constant headway in the area of computer science, the number of publications and their citations have been still increasing and the scientific soundness of these works has been becoming meaningful because obtained results can be successfully used in the real-life computer or telecommunication systems analysis or designing process. Customer's volume is a practical concept present e.g. in the process of data packets servicing (sending) where packets are described by their sizes having influence on time length of their delivering to the ending user. It is worth noting that there have been very few publications so far on queueing systems with random volume customers, implementing an additional vacation mechanism, that could study the characteristics of the customers' total volume in such systems, as well as the influence of the main system parameters (arrival rate, service time distribution, vacation period distribution) on these features. Some initial research can be found in works [36, 37] that investigate systems with unreliable servers of the $M/G/\infty$ and $M/G/1/\infty$-types but analyzed models cannot be exactly understood as models with vacations. Therefore, our main motivation was to build a new modification of the classical single-server queueing system with a vacation policy assuming the additional random character of customer's size (volume) and obtain important, from the practical point of view, performance characteristics of such systems (especially those related to customers' total volume distribution). Indeed, based on these characteristics (especially first steady-state two moments of customers' total volume), it is possible to calculate approximations of needed sizes of memory buffers of real-life computer or telecommunication systems (with limited memory buffers) in which we have additional vacation mechanism. It may be useful in the process of such systems initial designing.

In this paper, we investigate the classical $M^X/G/1/\infty$ queueing system, for which we additionally assume that each customer belonging to an arriving group is characterized by some random volume (size) and system has unlimited memory buffer in which information delivered by customers is written out. Customer's service time is generally dependent on his volume and, in addition, the server rests for a random time in the case when it is empty (so we have some mechanism of multiple vacations implemented). The goal of our research is obtaining main characteristics of the customers' total volume, analyzing chosen practical special cases of the model as well as showing possible applications of calculated results in the process of real-life computer or telecommunication systems designing and performance evaluation.

The rest of the paper is organized as follows. In Sec. 2, we obtain main results concerning the most important customer's total volume characteristics. In this section, we prove formula for the steady-state customer's total volume in analyzed queueing system in the terms of Laplace–Stieltjes transforms and formulae defining first two moments of this random variable. The next Sec. 3 contains exact analysis of interesting special cases of the model together with numerical examples and possible applications for the real-life computer or telecommunication systems (e.g. calculating approximations of loss characteristics for analogous systems but with limited memory buffer size). In the last section, we present important conclusions and final remarks.

2. EXACT ANALYSIS OF A SINGLE-SERVER QUEUEING SYSTEM WITH RANDOM VOLUME CUSTOMERS AND MULTIPLE VACATIONS

Let us consider the queueing system $M^X/G/1/\infty$ i.e. single-server queueing system with infinite queue to which the groups of customers are arriving in random moments of time. Let a be a parameter of a Poisson entrance flow of such groups (which means that time between neighboring moments of groups of customers arrival is exponentially distributed with parameter a). Let θ be a random number of customers in a group and $g_k = \mathbf{P}\{\theta = k\}$, $k = 1,2,\ldots$ be distribution of this random variable, whereas $G(z) = \sum_{k=1}^{\infty} g_k z^k$ be generating function of the number of customers in a group. Denote by $B(t)$ the distribution function (DF) of customer's service time ξ. Note that in classical queueing models, customer's service time distribution function $B(t)$ can be defined in any way if only it is left-side continuous, non-decreasing and satisfy conditions: $B(t)=0$ for $t\leq 0$ and $\lim_{t\to\infty}B(t)=1$. In the case where we additionally assume that customers are characterized by some random non-negative volume (size) ζ, having distribution function $L(x)$ (this assumption will appear in the next section), we usually treat customer's service time as dependent on this volume. In some, very special models, these random variables may be treated as independent but in most computer or telecommunication systems they are dependent and the character of the above-mentioned dependency may be different e.g. customer's service time can be proportional to his volume - then $B(t)=L(t/c)$, where $c(c>0)$ is some constant coefficient (e.g. in computer networks, data packet's service time is proportional to its size), or character of dependency can be more complex, then we have to define two-dimensional common distribution function of customer's volume and his service time $F(x,t)$. Evidently, in such cases we have: $F(x,t)=L(x)B(t)$ if RV ζ and ξ are independent, and in other cases we can calculate $B(t)$ using relation: $B(t)=\lim_{x\to\infty}F(x,t)$. It is clear that definition (formula) of function $F(x,t)$ contains the character of this dependency. Moreover, we are able to obtain relation for $B(t)$ basing on it. Now we additionally introduce the multiple vacation mechanism to this queueing system in the following way: if at the moment of service completion the system becomes empty, the server is turned off (rests) for a random period of time κ whose DF is equal to $H(y)$. After this time elapses, the server starts handling requests again if the system is not empty at that moment. If, on the other hand, at the end of the time κ of the server being in the switched-off state, there is no single request in the system, then the server is switched off again for time κ, and so on.

A. Number of customers distribution

First we determine the generating function (GF) $P(z)=\sum_{k=0}^{\infty}\mathbf{P}\{\eta=k\}z^k$ of the number of customers η present in analyzed system in steady state. The final result (that can be understood as the generalization of Pollaczek–Khinchine formula [3] for systems with multiple vacations) is rather well known and was published before (see e.g. [13]) but in this paper we show the different (and probably less complicated) way of its obtaining. Moreover, partial results obtained in this subsection will be used in the next one to calculate customers' total volume characteristics for the analyzed model (main new results presented in this paper).

Theorem 1. *For the analyzed single-server queueing system with multiple vacations, the following relation holds:*

$$P(z)=\frac{1-a\beta_1G'(1)}{a\kappa_1}\times$$
$$\times\frac{(1-z)\beta(a-aG(z))\left[1-\kappa(a-aG(z))\right]}{(1-G(z))\left[\beta(a-aG(z))-z\right]},\tag{1}$$

where $\beta_1=\mathbf{E}\xi$, $\kappa_1=\mathbf{E}\kappa$ and $\beta(q)=\int_0^\infty e^{-qt}dB(t)$, $\kappa(q)=$ $=\int_0^\infty e^{-qy}dH(y)$ are Laplace–Stieltjes transforms (LSTs) of distribution functions $B(t)$ and $H(y)$, respectively.

Proof. *Let us introduce the following notations: $\eta(t)$ is the number of customers in the system at time instant t, $\nu(t)$ is the indicator function, i.e. $\nu(t)=1$ if the server is occupied at time t, and $\nu(t)=0$ if the server "has a rest" at this time.*

Denote by $\xi^*(t)$ the length of time interval from customers (being served at time t) service beginning to the instant t, if $\nu(t)=1$, or the length of time interval from the last shutdown of the server to the instant t, if $\nu(t)=0$.

In the following, we will analyze the Markov stochastic process $(\eta(t),\nu(t),\xi^*(t))$, which will be characterized by functions having the following probabilistic sense:

$$P_k(1,y,t)\,dy=\mathbf{P}\{\eta(t)=k,\nu(t)=1,\xi^*(t)\in[y,y+dy)\},$$
$$k=1,2,\ldots;\tag{2}$$

$$P_k(0,y,t)\,dy=\mathbf{P}\{\eta(t)=k,\nu(t)=0,\xi^*(t)\in[y,y+dy)\},$$
$$k=0,1,\ldots;\tag{3}$$

$$P_k(m,t)=\mathbf{P}\{\eta(t)=k,\nu(t)=m\}=\int_0^t P_k(m,y,t)\,dy,$$
$$m=0,1;\ k=1,2,\ldots;\tag{4}$$

$$P_0(t)=P_0(0,t)=\mathbf{P}\{\eta(t)=0\};\tag{5}$$

$$P_k(t)=\mathbf{P}\{\eta(t)=k\}=P_k(1,t)+P_k(0,t),$$
$$k=1,2,\ldots;\tag{6}$$

For simplicity, we assume that the densities $b(y)$ and $h(y)$ of customer's service time ξ and vacation period κ exist, respectively. Then we introduce notations $\mu(y)=\frac{b(y)}{1-B(y)}$, $\gamma(y)=\frac{h(y)}{1-H(y)}$ (having the sense of service and "rest" intensities, respectively – see e.g. [29]). Note that all our results can be obtained without this assumption (it really does not have influence on final results because we can change the way of writing equations describing the system's behavior or consider densities in generalized sense with the use of Dirac-delta distribution).

It can be easily shown, analyzing the system's behavior, that the introduced functions satisfy the following equations ($\delta_{k,0}$ and $\delta_{k,1}$ are defined as follows: $\delta_{k,0}=1$ if $k=0$ and $\delta_{k,0}=$ $=0$, otherwise and, analogously, $\delta_{k,1}=1$ if $k=1$ and $\delta_{k,1}=0$, otherwise) :

$$\frac{\partial P_k(0,y,t)}{\partial t}+\frac{\partial P_k(0,y,t)}{\partial y}=-(a+\gamma(y))P_k(0,y,t)+$$
$$+\left(1-\delta_{k,0}\right)a\sum_{i=0}^{k-1}P_i(0,y,t)g_{k-i},\ k=0,1,\ldots;\tag{7}$$

$$\frac{\partial P_k(1,y,t)}{\partial t}+\frac{\partial P_k(1,y,t)}{\partial y}=-(a+\mu(y))P_k(1,y,t)+$$
$$+\left(1-\delta_{k,1}\right)a\sum_{i=1}^{k-1}P_i(1,y,t)g_{k-i},\ k=1,2,\ldots;\tag{8}$$

$$P_k(1,0^+,t)=$$
$$=\int_0^t P_{k+1}(1,y,t)\mu(y)dy+\int_0^t P_k(0,y,t)\gamma(y)dy,$$
$$k=1,2,\ldots; \tag{9}$$

$$P_0(0,0^+,t)=$$
$$=\int_0^t P_1(1,y,t)\mu(y)dy+\int_0^t P_0(0,y,t)\gamma(y)dy,\ t>0; \tag{10}$$

$$P_k(0,0^+,t)=0,\ k=1,2,\ldots. \tag{11}$$

For the analyzed model, the steady state exists only if the following conditions hold: $\kappa_1=\mathbf{E}\kappa<\infty$ *and* $\rho=a\beta_1 G'(1)=$ $=a\,\mathbf{E}\xi G'(1)<1$. *In such situation,* $\eta(t)\Rightarrow\eta$ *if* $t\to\infty$ *in the sense of a weak convergence, where* η *is the number of customers present in the system in steady state. Moreover, then the following limits exist:*

$$p_k(m,y)=\lim_{t\to\infty}P_k(m,y,t),\ m=0,1;\ k=1,2,\ldots; \tag{12}$$

$$p_0(0,y)=\lim_{t\to\infty}P_0(0,y,t); \tag{13}$$

$$p_0=\lim_{t\to\infty}P_0(t); \tag{14}$$

$$p_k(1)=\lim_{t\to\infty}P_k(1,t),\ k=1,2,\ldots;$$
$$p_k(0)=\lim_{t\to\infty}P_k(0,t),\ k=1,2,\ldots; \tag{15}$$

$$p_k=\lim_{t\to\infty}P_k(t)=p_k(1)+p_k(0),\ k=1,2,\ldots; \tag{16}$$

Thus, in steady state, from equations (7)–(11), we obtain the following equations:

$$\frac{\partial p_k(0,y)}{\partial y}=-(a+\gamma(y))\,p_k(0,y)+$$
$$+(1-\delta_{k,0})\,a\sum_{i=0}^{k-1}p_i(0,y)g_{k-i},\ k=0,1,\ldots; \tag{17}$$

$$\frac{\partial p_k(1,y)}{\partial y}=-(a+\mu(y))\,p_k(1,y)+$$
$$+(1-\delta_{k,1})\,a\sum_{i=1}^{k-1}p_i(1,y)g_{k-i},\ k=1,2,\ldots; \tag{18}$$

$$p_k(1,0^+)=$$
$$=\int_0^\infty p_{k+1}(1,y)\mu(y)dy+\int_0^\infty p_k(0,y)\gamma(y)dy,$$
$$k=1,2,\ldots; \tag{19}$$

$$p_0(0,0^+)=$$
$$=\int_0^\infty p_1(1,y)\mu(y)dy+\int_0^\infty p_0(0,y)\gamma(y)dy; \tag{20}$$

$$p_k(0,0^+)=0,\ k=1,2,\ldots. \tag{21}$$

Introduce now the following GFs:

$$p_{(0)}(z,y)=\sum_{k=0}^\infty p_k(0,y)z^k,\ p_{(1)}(z,y)=\sum_{k=1}^\infty p_k(1,y)z^k,$$
$$p_{(1)}(z,0)=\sum_{k=1}^\infty p_k(1,0^+)z^k. \tag{22}$$

Multiplying the equation with number k *in (17) by* z^k, $k=1,2,\ldots$, *and adding the resulting equations, we get the following equation for GF* $p_{(0)}(z,y)$:

$$\frac{\partial p_{(0)}(z,y)}{\partial y}=-(a-aG(z)+\gamma(y))\,p_{(0)}(z,y). \tag{23}$$

Its solution has the form:

$$p_{(0)}(z,y)=[1-H(y)]\,e^{-(a-aG(z))y}p_{(0)}(z,0),$$

where evidently $p_{(0)}(z,0)=p_0(0,0^+)$, *whereas*

$$p_{(0)}(z,y)=[1-H(y)]\,e^{-(a-aG(z))y}p_0(0,0^+). \tag{24}$$

Analogously, from (18) we obtain the following equation for GF $p_{(1)}(z,y)$:

$$\frac{\partial p_{(1)}(z,y)}{\partial y}=-(a-aG(z)+\mu(y))\,p_{(1)}(z,y). \tag{25}$$

Its solution has the form:

$$p_{(1)}(z,y)=[1-B(y)]\,e^{-(a-aG(z))y}p_{(1)}(z,0). \tag{26}$$

In the same way, from (19) we obtain equation for GF $p_{(1)}(z,0)$:

$$p_{(1)}(z,0)=\frac{1}{z}\int_0^\infty p_{(1)}(z,y)\mu(y)dy-\int_0^\infty p_1(1,y)\mu(y)dy+$$
$$+\int_0^\infty p_{(0)}(z,y)\gamma(y)dy-\int_0^\infty p_0(0,y)\gamma(y)dy, \tag{27}$$

whereas, taking into consideration (20), we have:

$$p_{(1)}(z,0)=$$
$$=\frac{1}{z}\int_0^\infty p_{(1)}(z,y)\mu(y)dy+\int_0^\infty p_{(0)}(z,y)\gamma(y)dy-p_0(0,0^+). \tag{28}$$

If we substitute in (28) the functions $p_{(0)}(z,y)$ *and* $p_{(1)}(z,y)$ *from (24) and (26), respectively, we obtain:*

$$p_{(1)}(z,0)=$$
$$=\frac{p_{(1)}(z,0)}{z}\beta(a-aG(z))-p_0(0,0^+)\,[1-\kappa(a-aG(z))].$$

Hence

$$p_{(1)}(z,0)=\frac{p_0(0,0^+)z[1-\kappa(a-aG(z))]}{\beta(a-aG(z))-z}. \tag{29}$$

Substituting the value of $p_{(1)}(z,0)$ *in this form to (26), we obtain:*

$$p_{(1)}(z,y)=\frac{p_0(0,0^+)z[1-\kappa(a-aG(z))]}{\beta(a-aG(z))-z}\times$$

$$\times\left[1-B(y)\right]e^{-(a-aG(z))y}. \tag{30}$$

For GF $p(z,y)=\sum_{k=0}^{\infty}p_k(y)z^k$, where $p_k(y)=p_k(0,y)+p_k(1,y)$, $k=0,1,\ldots$ (note that $p_0(1,y)=0$), we obviously have: $p(z,y)=p_{(0)}(z,y)+p_{(1)}(z,y)$. Introduce now the following functions:

$$P_{(0)}(z)=\int_0^{\infty}p_{(0)}(z,y)dy=$$
$$=p_0(0,0^+)\int_0^{\infty}\left[1-H(y)\right]e^{-(a-aG(z))y}dy=$$
$$=p_0(0,0^+)\frac{1-\kappa(a-aG(z))}{a(1-G(z))} \tag{31}$$

and

$$P_{(1)}(z)=\int_0^{\infty}p_{(1)}(z,y)dy=$$
$$=p_0(0,0^+)\frac{z\left[1-\kappa(a-aG(z))\right]}{\beta(a-aG(z))-z}\times$$
$$\times\int_0^{\infty}\left[1-B(y)\right]e^{-(a-aG(z))y}dy=$$
$$=p_0(0,0^+)\frac{z\left[1-\kappa(a-aG(z))\right]\left[1-\beta(a-aG(z))\right]}{a(1-G(z))\left[\beta(a-aG(z))-z\right]}. \tag{32}$$

Hence, we obtain that

$$P(z)=P_{(0)}(z)+P_{(1)}(z)=$$
$$=p_0(0,0^+)\frac{(1-z)\beta(a-aG(z))\left[1-\kappa(a-aG(z))\right]}{a(1-G(z))\left[\beta(a-aG(z))-z\right]}.$$

From the fact that $P(1)=1$, we obtain $\frac{p_0(0,0^+)}{a}=\frac{1-a\beta_1G'(1)}{a\kappa_1}$, where $\beta_1=\mathbf{E}\xi$ and $\kappa_1=\mathbf{E}\kappa$, what ends the proof. □

Now, we can also easily calculate the value of probability that the system is empty:

$$p_0=\mathbf{P}\{\eta=0\}=P(0)=\frac{(1-a\beta_1G'(1))\left[1-\kappa(a)\right]}{a\kappa_1}.$$

Corollary 1. *In the case when the entrance flow is single Poissonian, we have $G(z)=z$ and obtain:*

$$P(z)=\frac{(1-a\beta_1)\beta(a-az)\left[1-\kappa(a-az)\right]}{a\kappa_1\left[\beta(a-az)-z\right]}.$$

For example, if the vacation period is exponentially distributed (say, with parameter r), we obtain:

$$P(z)=\frac{(1-a\beta_1)(1-z)\beta(a-az)}{\left[1+a\kappa_1(1-z)\right]\left[\beta(a-az)-z\right]},$$

where $\kappa_1=1/r$. If, in addition, customer's service time is also exponentially distributed (say, with parameter μ), we have:

$$P(z)=\frac{1-a\beta_1}{(1-a\beta_1z)\left[1+a\kappa_1(1-z)\right]},$$

where again $\kappa_1=1/r$ and additionally $\beta_1=1/u$.

B. Total customers' volume distribution

Assume additionally that each customer is characterized by some random volume ζ (where ζ is non-negative random variable (RV)) which does not depend on other customers' volumes, and customer's service time ξ generally depends on his volume. Let $F(x,t)=\mathbf{P}\{\zeta<x,\xi<t\}$ be the joint DF of random variables ζ and ξ. Denote by

$$\alpha(s,q)=\int_0^{\infty}\int_0^{\infty}e^{-sx-qt}\mathrm{d}F(x,t)$$

the double LST of DF $F(x,t)$. Then, we have obviously that $L(x)=\mathbf{P}\{\zeta<x\}=F(x,\infty)$ and $B(t)=\mathbf{P}\{\xi<t\}=F(\infty,t)$. The respective LSTs of RVs ζ and ξ have the following forms:

$$\varphi(s)=\int_0^{\infty}e^{-sx}\mathrm{d}L(x)=\alpha(s,0),$$
$$\beta(q)=\int_0^{\infty}e^{-qt}\mathrm{d}B(t)=\alpha(0,q).$$

Denote by $\sigma(t)$ the total volume of all customers present in the system at time instant t (i.e. the sum of the volumes of such customers). In steady state, we have evidently that $\sigma(t)\Rightarrow\sigma$, where σ is the steady-state total volume. Let $D(x)=\mathbf{P}\{\sigma<x\}$ be DF of RV σ and $\delta(s)=\int_0^{\infty}e^{-sx}\mathrm{d}D(x)$ be its LST. Now we will prove general formula for the function $\delta(s)$ (what is the most important new result presented in this paper).

Theorem 2. *For the analyzed single-server queueing system with multiple vacations and random volume customers, we have the following relation:*

$$\delta(s)=\frac{1-\rho}{\chi}\cdot\frac{1-\kappa(a-aG(\varphi(s)))}{1-G(\varphi(s))}\times$$
$$\times\left\{1+\frac{\varphi(s)-\alpha(s,a-aG(\varphi(s)))}{\beta(a-aG(\varphi(s)))-\varphi(s)}\right\}, \tag{33}$$

where $\rho=a\beta_1G'(1)$, $\chi=a\kappa_1$.

Proof. *Denote by $L_*^k(x)$ the k-fold Stieltjes convolution of DF $L(x)$ at the point x and by $A*B(x)$ the Stieltjes convolution of DF $A(x)$ and $B(x)$ at the point x.*

It is clear that (based on total probability theorem in continuous form) that

$$D(x)=\sum_{k=0}^{\infty}\int_0^{\infty}p_k(0,y)L_*^k(x)dy+$$
$$+\sum_{k=1}^{\infty}p_k(1,y)\left[L_*^{k-1}*E_y(x)\right]dy, \tag{34}$$

where the meaning of the conditional distribution $E_y(x)$ can be found e.g. in [29]. If we use LST to both sides of above-mentioned equation, we obtain:

$$\delta(s)=\sum_{k=0}^{\infty}(\varphi(s))^k\int_0^{\infty}p_k(0,y)dy+$$
$$+\sum_{k=1}^{\infty}(\varphi(s))^{k-1}\int_0^{\infty}p_k(1,y)e_y(s)dy=$$
$$=\int_0^{\infty}p_{(0)}(\varphi(s),y)dy+\frac{1}{\varphi(s)}\int_0^{\infty}p_{(1)}(\varphi(s),y)e_y(s)dy, \tag{35}$$

where

$$e_y(s) = [1-B(y)]^{-1}\int_{x=0}^{\infty} e^{-sx}\int_{u=y}^{\infty} dF(x,u), \tag{36}$$

(see again [29]). Functions $p_{(0)}(z,y)$ and $p_{(1)}(z,y)$ were defined in (24) and (30), respectively. Then we obtain

$$\int_0^\infty p_{(0)}(\varphi(s),y)\,dy = P_{(0)}(G(\varphi(s))) =$$
$$= p_0(0,0^+)\frac{1-\kappa(a-aG(\varphi(s)))}{a(1-G(\varphi(s)))}. \tag{37}$$

From (30) and (36) we obtain:

$$\int_0^\infty p_{(1)}(\varphi(s),y)\,e_y(s)dy =$$
$$= \frac{p_0(0,0^+)\varphi(s)\,[1-\kappa(a-aG(\varphi(s)))]}{\beta(a-aG(\varphi(s)))-\varphi(s)}\times$$
$$\times\int_0^\infty [1-B(y)]e^{-(a-aG(\varphi(s)))y}e_y(s)dy.$$

The integral in the above-mentioned formula can be calculated as follows:

$$\int_0^\infty [1-B(y)]e^{-(a-aG(\varphi(s)))y}e_y(s)dy =$$
$$= \int_{y=0}^\infty e^{-(a-aG(\varphi(s)))y}\left(\int_{x=0}^\infty e^{-sx}\int_{u=y}^\infty dF(x,u)\right)dy =$$
$$= \int_{x=0}^\infty e^{-sx}\int_{u=0}^\infty\left(\int_{y=0}^u e^{-(a-aG(\varphi(s)))y}dy\right)dF(x,u) =$$
$$= \frac{1}{a(1-G(\varphi(s)))}\times$$
$$\times\int_0^\infty\int_0^\infty e^{-sx}\left[1-e^{-(a-aG(\varphi(s)))u}\right]dF(x,u) =$$
$$= \frac{\varphi(s)-\alpha(s,a-aG(\varphi(s)))}{a(1-G(\varphi(s)))}.$$

Then we obtain:

$$\int_0^\infty p_{(1)}(\varphi(s),y)\,e_y(s)dy =$$
$$= p_0(0,0^+)\varphi(s)\,[1-\kappa(a-aG(\varphi(s)))]\times$$
$$\times\frac{[\varphi(s)-\alpha(s,a-aG(\varphi(s)))]}{a(1-G(\varphi(s)))\,[\beta(a-aG(\varphi(s)))-\varphi(s)]}. \tag{38}$$

Hence, from (35), (37) and (38) we finally obtain, after some calculations, relation (33). □

Basing on proven theorem, we can obtain formulae for first two moments of steady-state customers' total volume. We use here well-known properties of the Laplace–Stieltjes transform, namely $\delta_1 = -\delta'(0)$, $\delta_2 = \delta''(0)$:

$$\delta_1 = a\alpha_{11}G'(1) + \frac{a\beta_1\varphi_1 G''(1)}{2(1-\rho)} +$$
$$+\frac{\beta_2\varphi_1\rho^2}{2\beta_1^2(1-\rho)} + \frac{\varphi_1\kappa_2\rho}{2\beta_1\kappa_1}, \tag{39}$$

$$\delta_2 = a\alpha_{21}G'(1) +$$
$$+\varphi_1\rho^3\left(\frac{3\alpha_{11}\beta_2+\beta_3\varphi_1}{3\beta_1^3(1-\rho)} + \frac{\beta_2\varphi_1\kappa_2}{2\beta_1^3\kappa_1(1-\rho)}\right)+$$
$$+\frac{\beta_2^2\varphi_1^2\rho^4}{2\beta_1^4(1-\rho)^2} + \frac{\rho^2(\beta_2\varphi_2+2\alpha_{12}\varphi_1(1-\rho))}{2\beta_1^2(1-\rho)}+$$
$$+\frac{\alpha_{11}\varphi_1\kappa_2\rho^2}{\beta_1^2\kappa_1} + \frac{\varphi_1^2\kappa_3\rho^2}{3\beta_1^2\kappa_1} + \frac{a\beta_1\varphi_2 G''(1)}{2(1-\rho)}+$$
$$+\frac{a\alpha_{11}\varphi_1 G''(1)}{1-\rho} + \frac{a\varphi_1^2\kappa_2 G''(1)}{2\kappa_1(1-\rho)} + \frac{(a\beta_1\varphi_1 G''(1))^2}{2(1-\rho)^2}+$$
$$+\frac{\varphi_2\kappa_2\rho}{2\beta_1\kappa_1} + \frac{a\beta_2\varphi_1^2 G''(1)\rho}{\beta_1(1-\rho)^2} + \frac{a\beta_1\varphi_1^2 G'''(1)}{3(1-\rho)}. \tag{40}$$

In formulae (39) and (40) β_i denotes i-th moment of the random variable ξ, φ_i – i-th moment of the random variable ζ, κ_i – i-th moment of the random variable κ and α_{ij} – mixed $(i+j)$-th moment of the random vector (ζ,ξ).

Corollary 2. *In the case when the entrance flow is single Poissonian (each group contains only one customer, then $G(z)=z$ and $\rho = a\beta_1$), we obtain:*

$$\delta(s) = \frac{1-\rho}{\chi}\cdot\frac{1-\kappa(a-a\varphi(s))}{1-\varphi(s)}\times$$
$$\times\left\{1+\frac{\varphi(s)-\alpha(s,a-a\varphi(s))}{\beta(a-a\varphi(s))-\varphi(s)}\right\}. \tag{41}$$

Then, formulae defining first two moments are the following:

$$\delta_1 = a\alpha_{11} + \frac{1}{2}a\varphi_1\left(\frac{a\beta_2}{1-\rho}+\frac{\kappa_2}{\kappa_1}\right), \tag{42}$$

$$\delta_2 = a\alpha_{21} + a^2\alpha_{12}\varphi_1 + \frac{a(3\varphi_2\kappa_2+2a\varphi_1^2\kappa_3)}{6\kappa_1}+$$
$$+\frac{a^4\beta_2^2\varphi_1^2}{2(1-\rho)^2} + \frac{a^2\beta_2\varphi_2}{2(1-\rho)} + \frac{a^2\alpha_{11}\varphi_1\kappa_2}{\kappa_1}+$$
$$+\frac{a^3\alpha_{11}\varphi_1\beta_2}{1-\rho} + \frac{a^3\beta_3\varphi_1^2}{3(1-\rho)} + \frac{a^3\beta_2\varphi_1^2\kappa_2}{2\kappa_1(1-\rho)}. \tag{43}$$

3. SPECIAL CASES ANALYSIS AND NUMERICAL RESULTS

In this section, we investigate some simple special cases of the analyzed model, illustrating obtained results by numerical examples. For simplicity, we assume customers' arrival flow is single Poissonian, i.e. we will use in our computations only formula (41).

A. Customer's service time and his volume are independent

Assume additionally that customer's service time and his volume are independent. Then, basing on (41), we obtain the following formula:

$$\delta(s) = \frac{1-\rho}{\chi}\cdot\frac{1-\kappa(a-a\varphi(s))}{1-\varphi(s)}\times$$
$$\times\left\{1+\frac{\varphi(s)(1-\beta(a-a\varphi(s)))}{\beta(a-a\varphi(s))-\varphi(s)}\right\}. \tag{44}$$

Table 1. Exemplary δ_1 calculations for $M/M/1/\infty(V)$ system with exponentially distributed vacation period, $a = 1$, $f = 1$ (customer's service time and his volume are independent)

δ_1	$r = 1.1$	$r = 1.6$	$r = 2.1$	$r = 2.6$	$r = 3.1$
$\mu = 1.1$	10.9091	10.6250	10.4762	10.3846	10.3226
$\mu = 1.6$	2.5758	2.2917	2.1429	2.0513	1.9893
$\mu = 2.1$	1.8181	1.5341	1.3853	1.2937	1.2317
$\mu = 2.6$	1.5341	1.2500	1.1012	1.0096	0.9476
$\mu = 3.1$	1.3853	1.1012	0.9524	0.8608	0.7988
$\mu = 3.6$	1.2937	1.0096	0.8608	0.7692	0.7072

Now we demonstrate interesting numerical computations in case where customer's volume is exponentially distributed with parameter f, his service time is exponentially distributed with parameter μ and vacation period is exponentially distributed with parameter r. After rather simple calculations, we finally obtain:

$$\delta(s) = \frac{(1-\rho)(s+f)^2}{[s(1+\chi)+f](s+f-\rho f)}, \tag{45}$$

where $\rho = \frac{a}{\mu}$. In this case we may also obtain the exact form of steady-state customers' total volume distribution (basing on Laplace transform inversion):

$$D(x) = \mathscr{L}^{-1}(\delta(s)/s) =$$

$$= 1 - \frac{(1-\rho)\chi^2 e^{-(1+\chi)^{-1}fx} - \rho^2 e^{-(1-\rho)fx}}{\chi(1-\rho)-\rho}. \tag{46}$$

Formulae defining first two moments of customers' total volume are the following:

$$\delta_1 = \frac{1}{f}\left(\chi + \frac{\rho}{1-\rho}\right), \tag{47}$$

$$\delta_2 = \frac{2\left[\chi(1+\chi(1-\rho))(1-\rho)+\rho\right]}{f^2(1-\rho)^2}. \tag{48}$$

In Tab. 1. we present exemplary results for δ_1, for fixed values $a = 1$, $f = 1$ (changing r and μ), whereas in Fig. 1. we present 3-dimensional graph of function $\delta_1(r,\mu)$. Of course, the values of $1/f$ and δ_1 have to be measured in the same information units (e.g. B, kB, MB and so on).

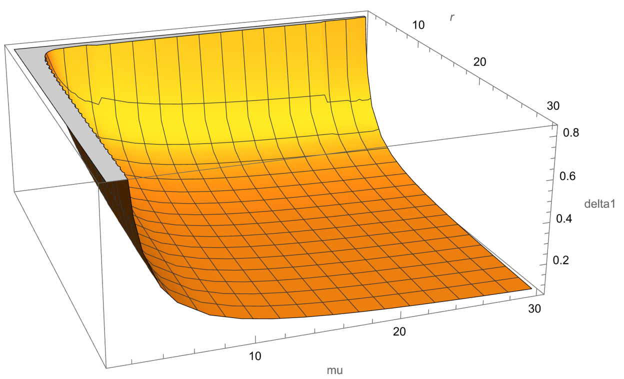

Fig. 1. Graph presenting function $\delta_1(r,\mu)$ in the $M/M/1/\infty(V)$ queueing system with exponentially distributed vacation period (customer's service time and his volume are independent)

Table 2. Loss probability approximations for $M/M/1/\infty(V)$ system with exponentially distributed vacation period, $a = 1$, $\mu = 3$, $r = 1$, $f = 1$ (MB^{-1}) (customer's service time and his volume are independent)

$V[MB]$	p_{LOSS}	$V[MB]$	p_{LOSS}
0.0	1.0000	5.0	0.1285
0.5	0.8411	5.5	0.1023
1.0	0.6996	6.0	0.0813
1.5	0.5769	6.5	0.0644
2.0	0.4722	7.0	0.0510
2.5	0.3841	7.5	0.0403
3.0	0.3109	8.0	0.0318
3.5	0.2506	8.5	0.0251
4.0	0.2012	9.0	0.0197
4.5	0.1610	9.5	0.0155

Obtained results have also very important meaning for real-life computer systems. As it was shown e.g. in [29], based on exact formula for $D(x)$, we are able to approximate loss characteristics for analogous systems but having limited (by constant value V) total volume (in this case V does not mean the number of positions in the waiting room (that is infinite) but limitation of total volume of all customers present in the system). Arriving customers must have enough small volume that guarantees they will be accepted to the system. Note that the value of V is usually measured in information units like bytes (B), kilobytes (kB), megabytes (MB) etc. Here we use some approximate formula:

$$p_{LOSS} \approx 1 - \int_0^V D(V-x)l(x)\mathrm{d}x, \tag{49}$$

where $l(x)$ is the density function of the customer's volume. Substitute now, for example, $a = 1$, $\mu = 3$, $r = 1$, $f = 1$ (which means that mean value of customer's volume equals $\frac{1}{f} = 1$) and use formulae (49) and (46). Then we obtain results presented in Tab. 2. and graphically illustrated in Fig. 2 (here V and $\mathbf{E}\zeta = \frac{1}{f}$ are measured in the same information units e.g. MB). Analyzed system is usually denoted in literature as $M/M/1/(\infty,V)$ or $M/M/1/\infty(V)$ [29].

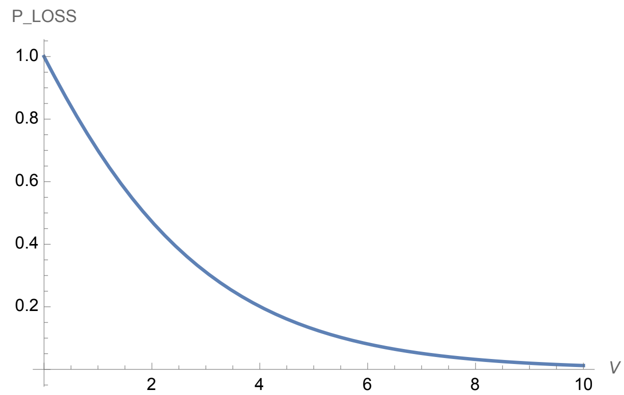

Fig. 2. Graph presenting the approximations of loss probability in the $M/M/1/\infty(V)$ queueing system with exponentially distributed vacation period (customer's service time and his volume are independent)

B. Customer's service time proportional to his volume

Assume now that customer's service time is proportional to his volume: $\xi = c\zeta$, $c > 0$. Then (from (41)) we obtain:

$$\delta(s) = \frac{1-\rho}{\chi} \cdot \frac{1-\kappa(a-a\varphi(s))}{1-\varphi(s)} \times \\ \times \left\{ 1 + \frac{\varphi(s) - \varphi(s + c(a - a\varphi(s)))}{\varphi(c(a - a\varphi(s))) - \varphi(s)} \right\}. \quad (50)$$

In case when customer's volume is exponentially distributed with parameter f and vacation period is exponentially distributed with parameter r, we obtain the following formula:

$$\delta(s) = \frac{(1-\rho)(f+s)^4}{(f+s+\chi s)(f+s-\rho f)\left[(f+s)^2+\rho f s\right]}, \quad (51)$$

where $\rho = \frac{ac}{f}$. Analogously like in previous subsection, we obtain the exact form of steady-state customers' total volume distribution. If $f \neq 2ac$ ($\rho \neq \frac{1}{2}$), we obtain:

$$D(x) = 1 - \frac{1}{\chi - \rho(1+\chi)} \times \\ \times \left[\frac{(1-\rho)\chi^4 e^{-(1+\chi)^{-1} f x}}{(1+2\chi)^2 + \rho\chi(1+\chi)} + \frac{\rho^3 e^{-(1-\rho) f x}}{1-2\rho} \right] - \\ - \frac{\rho(1-\rho)}{\sqrt{\rho(4+\rho)}} \times \\ \times \left\{ \frac{(1-b_1)^2 e^{-b_1 f x}}{[1-b_1(1+\chi)](1-b_1-\rho)} - \frac{(1-b_2)^2 e^{-b_2 f x}}{[1-b_2(1+\chi)](1-b_2-\rho)} \right\}, \quad (52)$$

where $b_1 = \frac{2+\rho-\sqrt{\rho(4+\rho)}}{2}$, $b_2 = \frac{2+\rho+\sqrt{\rho(4+\rho)}}{2}$. Whereas, if $f = 2ac$ ($\rho = \frac{1}{2}$), we obtain:

$$D(x) = 1 - \frac{2\chi^4 e^{-(1+\chi)^{-1} f x}}{(1+2\chi)(1-\chi)^2} + \frac{e^{-2fx}}{9(1+2\chi)} - \\ - \frac{1}{6(1-\chi)} \left[\frac{11-17\chi}{3(1-\chi)} + \frac{fx}{2} \right] e^{-\frac{fx}{2}}. \quad (53)$$

Formula defining first moment δ_1 of customers' total volume is the following:

$$\delta_1 = \frac{1}{f} \left(\chi + \frac{\rho(2-\rho)}{1-\rho} \right). \quad (54)$$

To obtain δ_2 value, we can use general formula (43) with the following substitutions: $\varphi_i = \frac{i!}{f^i}$, $\beta_i = \frac{c^i i!}{f^i}$, $\kappa_i = \frac{i!}{r^i}$, $\alpha_{ij} = \frac{c^j (i+j)!}{f^{i+j}}$. In Tab. 3. we present exemplary results for δ_1 for fixed values $a = 1$, $c = 1$ (changing f and r) whereas in Fig. 3. we present 3–dimensional graph of function $\delta_1(r, f)$.

Substitute now, for example, $a = 1$, $f = 3$, $r = 1$, $c = 1$ and use formulae (49) and (52). Then we obtain results presented in Tab. 4. and graphically illustrated in Fig. 4. These results are connected again with calculating of loss probability approximations. Note that, for such fixed parameters, queueing systems analyzed in subsections 3.1 and 3.2 are identical from the classical point of view (they have the same classical characteristics such like arrival flow parameter and service time distribution function) but total volume characteristics are different (compare tables 1 with 3, 2 with 4 and also graphs 1 with 3, and 2 with 4). It confirms that the character of dependency between customer's service time and his volume has a substantial meaning in the case of total customers' volume characteristics.

Table 3. Exemplary δ_1 calculations for $M/M/1/\infty(V)$ system with exponentially distributed vacation period, $a = 1$, $c = 1$ (customer's service time proportional to his volume)

δ_1	$r = 1.1$	$r = 1.6$	$r = 2.1$	$r = 2.6$	$r = 3.1$
$f = 1.1$	10.7438	10.4855	10.3503	10.2670	10.2106
$f = 1.6$	2.0005	1.8229	1.7299	1.6727	1.6339
$f = 2.1$	1.0926	0.9573	0.8864	0.8428	0.8133
$f = 2.6$	0.7380	0.6287	0.5715	0.5362	0.5124
$f = 3.1$	0.5509	0.4593	0.4113	0.3817	0.3617
$f = 3.6$	0.4365	0.3576	0.3163	0.2908	0.2736

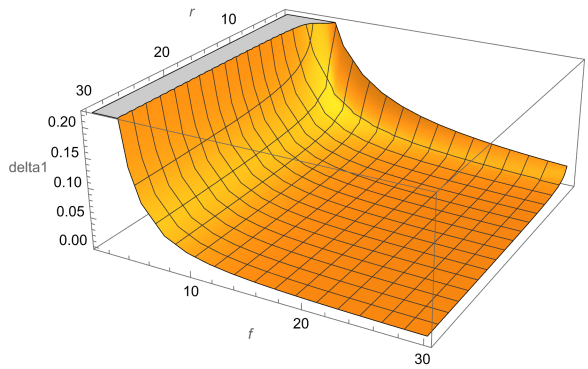

Fig. 3. Graph presenting function $\delta_1(r, f)$ in the $M/M/1/\infty(V)$ queueing system with exponentially distributed vacation period (customer's service time proportional to his volume)

Table 4. Loss probability approximations for $M/M/1/\infty(V)$ system with exponentially distributed vacation period, $a = 1$, $f =$, $r = 1$, $c = 1$ (customer's service time proportional to his volume)

$V[MB]$	P_{LOSS}	$V[MB]$	P_{LOSS}
0.0	1.0000	5.0	0.0021
0.5	0.6159	5.5	0.0010
1.0	0.3659	6.0	0.0005
1.5	0.2078	6.5	0.0003
2.0	0.1140	7.0	0.0001
2.5	0.0609	7.5	$6.04 \cdot 10^{-5}$
3.0	0.0319	8.0	$2.93 \cdot 10^{-5}$
3.5	0.0165	8.5	$1.42 \cdot 10^{-5}$
4.0	0.0084	9.0	$6.84 \cdot 10^{-6}$
4.5	0.0042	9.5	$3.29 \cdot 10^{-6}$

It is worth adding that in the case when we do not have the exact form of $D(x)$ distribution, we may approximate it with the use of gamma distribution based on values δ_1 and δ_2 (see again [29]) and in this way we obtain needed loss probability approximations.

C. Generalization on systems with sectorized memory buffer

Obtained results can be generalized on systems with sectorized memory i.e. we can consider systems in which total vol-

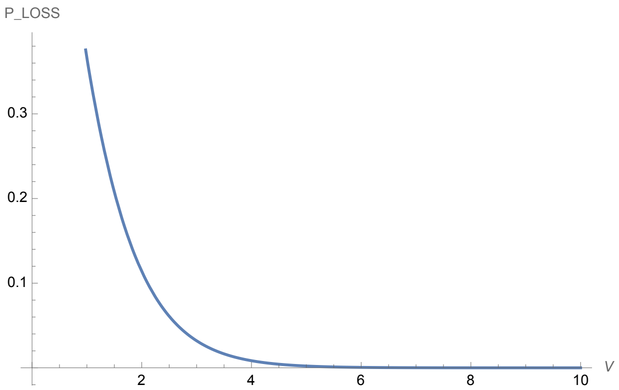

Fig. 4. Graph presenting the approximations of loss probability in the $M/M/1/\infty(V)$ queueing system with exponentially distributed vacation period (customer's service time proportional to his volume)

ume contains some sectors in which specific types of data is located. In such systems, arriving customers are additionally characterized by random volume vectors i.e. their volumes are multidimensional (see e.g. [33]). Assume e.g. that total volume contains n sectors. In the same way as we did it in Sec. 2. we obtain the following formula describing double LST of n-dimensional total customers' volume for the system $M/G/1/\infty$:

$$\delta(s_1,\ldots,s_n) = \frac{1-\rho}{\chi}\cdot\frac{1-\kappa(a-a\varphi(s_1,\ldots,s_n))}{1-\varphi(s_1,\ldots,s_n)}\times$$
$$\times\left\{1+\frac{\varphi(s_1,\ldots,s_n)-\alpha(s_1,\ldots,s_n,a-a\varphi(s_1,\ldots,s_n))}{\beta(a-a\varphi(s_1,\ldots,s_n))-\varphi(s_1,\ldots,s_n)}\right\}, \quad (55)$$

where $\varphi(s_1,\ldots,s_n)$ is n-dimensional LST of the customer's random volume vector, and $\alpha(s_1,\ldots,s_n,q)$ is $(n+1)$-dimensional LST of the customer's random volume vector and his service time.

Using (55), we can obtain formulae defining basic characteristics of customers' total volume e.g. mixed moments (of the $(i_1+\ldots+i_n)$-th order) $\delta_{i_1\ldots i_n}$ as we have obvious relation:

$$\delta_{i_1\ldots i_n} = (-1)^{i_1+\ldots+i_n}\frac{\partial\delta^{i_1+\ldots+i_n}(s_1,\ldots,s_n)}{\partial s_1^{i_1}\ldots\partial s_n^{i_n}}\Big|_{s_1=0,\ldots,s_n=0}.$$

To obtain it, we have to use generalized l'Hospital's rule for functions of many variables explained in [38] and [39] and the help of *Mathematica* environment (see e.g. [40]). Final results are very complicated and contain mixed moments $\alpha_{i_1,\ldots,i_n,j}$ of the random vectors $(\zeta_1,\ldots,\zeta_n,\xi)$, where ζ_i, $i=1,\ldots,n$, are the indications of the customer's random volume vector. But *Mathematica* environment lets calculate these characteristics for fixed character of dependency between customer's service time and his volume vector indications and fixed parameters.

4. CONCLUSIONS AND FINAL REMARKS

d paper, we have investigated the model of a single-server queueing system with random volume customers and multiple vacations. We obtained general formula for steady-state customers' total volume in the terms of Laplace–Stieltjes transforms and formulae defining first two moments of this random variable. Then we showed exact calculations for some chosen special cases of analyzed model together with some numerical results and graphs. We also explained the way of using obtained results to approximate loss probabilities of analogous systems but with limited total volume and paid attention on the influence of the character of dependency between customer's volume and his service time on total volume characteristics. In the end, we showed possible generalization of obtained results on systems with sectorized memory e.g. possibility to calculate mixed moments of total volume vector with the use of generalized l'Hosptal's rule, by the help of *Mathematica* environment.

REFERENCES

[1] A. K. Erlang, "Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges," *Post Office Electrical Engineer's Journal*, vol. 10, pp. 189–197, 1917.

[2] F. Pollaczek, "Über eine aufgabe der wahrscheinlichkeitstheorie," *Mathematische Zeitschrift*, vol. 32, pp. 64–100, 1930 (In German).

[3] A. Y. Khintchine, "Mathematical theory of stationary queues," *Matem. Sbornik*, vol. 39, no. 4, pp. 73–84, 1932 (In Russian).

[4] S. Asmussen, *Applied Probability and Queues, 2d ed.* New York: Springer-Verlag, 2003.

[5] P. P. Bocharov, C. D'Apice, A. V. Pechinkin, and S. Salerno, *Queueing Theory*. Utrecht-Boston: VSP, 2004.

[6] M. Scholl and L. Kleinrock, "On the $M/G/1$ queue with rest periods and certain service-independent queueing disciplines," *Operations Research*, vol. 31, no. 4, pp. 705–719, 1983.

[7] T. Lee, "$M/G/1/N$ queue with vacation time and exhaustive service discipline," *Operations Research*, vol. 32, no. 4, pp. 774–784, 1984.

[8] B. T. Doshi, "Queueing systems with vacations-a survey," *Queueing systems*, vol. 1, pp. 29–66, 1986.

[9] T. Lee, "$M/G/1/N$ queue with vacation time and limited service discipline," *Performance Evaluation*, vol. 9, no. 3, pp. 181–190, 1989.

[10] J. Machuveettil and T. Madhusoodanan, "Transient solution for a finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server," *Queueing systems*, vol. 2, pp. 381–386, 1987.

[11] H. Takagi, "Analysis of an $M/G/1/N$ queue with multiple server vacations, and its application to a polling model," *Journal of the Operations Research Society of Japan*, vol. 35, no. 3, pp. 300–315, 1992.

[12] A. Frey and Y. Takahashi, "Explicit solution for $M/GI/1/N$ vacation queue," *Journal of the Operations Research*, vol. 41, no. 3, pp. 430–441, 1998.

[13] U. Yechiali, "On the $M^X/G/1$ queue with a waiting server and vacations," *Sankhyā: The Indian Journal of Statistics*, pp. 159–174, 2004.

[14] D.-A. Wu and H. Takagi, "$M/G/1$ queue with multiple working vacations," *Performance Evaluation*, vol. 63, pp. 654–681, 2006.

[15] R. Kalyanaraman and S. Pazhani Bala Murugan, "A single server queue with additional optional service in batches and server vacation," *Applied Mathematical Sciences*, vol. 2, no. 56, pp. 2765–2776, 2008.

[16] A. Banik, "The infinite-buffer single server queue with a variant of multiple vacation policy and batch markovian arrival process," *Applied Mathematical Modelling*, vol. 33, pp. 3025–3039, 2009.

[17] C. Xiu, N. Tian, and Y. Liu, "The $M/M/1$ queue with single working vacation serving at a slower rate during the start–up period," *Journal of Mathematics Research*, vol. 2, no. 1, pp. 98–102, 2010.

[18] U. Ayesta, O. Boxma, and I. Verloop, "Sojourn times in a processor sharing queue with multiple vacations," *Queueing Systems*, vol. 33, pp. 53–78, 2012.

[19] W. M. Kempa, K. Książek, and R. Marjasz, "On time-dependent queue-size distribution in a model with finite buffer capacity and deterministic multiple vacations with applications to LTE DRX mechanism modeling," *IEEE Access*, vol. 9, pp. 148 374–148 383, 2021.

[20] T.-H. Liu, H.-Y. Hsu, J.-C. Ke, and F.-M. Chang, "Preemptive priority markovian queue subject to server breakdown with imperfect coverage and working vacation interruption," *Computation*, vol. 11, no. 89, 2023.

[21] F. Jiang, D. Huang, and K. Wang, "Design approaches for optimizing power consumption of sensor node with N-policy $M/G/1$ queuing model," in *Proceedings of the 4th International Conference on Queueing Theory and Network Applications, QTNA 09, New York, USA*. Assocation for Computing Machinery, 2009.

[22] J. Li, H. Zhou, D. Zuo, K. Hou, H. Xie, and P. Zhou, "Energy consumption evaluation for wireless sensor network nodes based on queuing Petri net," *International Journal of Distributed Sensor Networks*, vol. 10, no. 4, p. 262848, 2014.

[23] W. Kempa, "Analytical model of a wireless sensor network (WSN) node operation with a modified threshold-type energy saving mechanism," *Sensors*, vol. 19, no. 14, 2019.

[24] M. Gorawski, K. Grochla, R. Marjasz, and A. Frankiewicz, "Energy minimization algorithm for estimation of clock skew and reception window selection in wireless networks," *Sensors*, vol. 21, no. 5, p. 1768, 2021.

[25] M. Schwartz, *Computer-communication Network Design and Analysis*. New York: Prentice-Hall, Englewood Cliffs, 1977.

[26] M. Schwartz, *Telecommunication Networks: Protocols, Modeling and Analysis*. New York: Addison-Wesley Publishing Company, 1987.

[27] A. M. Alexandrov and B. A. Kaz, "Non-homogeneous demands flows service," *Izvestiya AN SSSR. Tekhnicheskaya Kibernetika*, no. 4, pp. 47–53, 1973 (In Russian).

[28] B. Sengupta, "The spatial requirements of an $M/G/1$ queue, or: How to design for buffer space," in *Modelling and Performance Evaluation Methodology. Lecture Notes in Control and Information Science*, F. Baccelli and G. Fayolle, Eds. Heidelberg: Springer, 1984, pp. 547–562.

[29] O. Tikhonenko, *Computer Systems Probability Analysis*. Warsaw, Poland: Akademicka Oficyna Wydawnicza EXIT, 2006 (In Polish).

[30] A. Cascone, R. Manzo, A. Pechinkin, and S. Shorgin, "A $Geom/G/1/n$ queueing system with lifo discipline, service interruptions and resumption, and restrictions on the total volume of demands," in *Proceedings of the world congress on engineering*, vol. 3, 2010.

[31] S. Dudin, C. Kim, O. Dudina, and J. Baek, "Queueing system with heterogeneous customers as a model of a call center with a call-back for lost customers," *Mathematical Problems in Engineering*, vol. 2013, pp. 1–13, 2013.

[32] E. Lisovskaya, S. Moiseeva, and M. Pagano, "Multiclass $GI/GI/\infty$ queueing systems with random resource requirements," in *International Conference on Information Technologies and Mathematical Modelling*. Springer, 2018, pp. 129–142.

[33] O. Tikhonenko, M. Ziółkowski, and W. M. Kempa, "Queueing systems with random volume customers and a sectorized unlimited memory buffer," *International Journal of Applied Mathematics and Computer Science*, vol. 31, no. 3, pp. 471–486, 2021.

[34] M. Ziółkowski and O. Tikhonenko, "Single-server queueing system with limited queue, random volume customers and unlimited sectorized memory buffer," *Bulletin of the Polish Academy of Sciences. Technical Sciences*, vol. 70, no. 6, art. no. e143647, 2022.

[35] M. Ziókowski, "Multiserver loss queueing system with random volume customers, nonidentical servers and a limited sectorized memory buffer," *Bulletin of the Polish Academy of Sciences. Technical Sciences*, vol. 71, no. 5, art. no. e146764, 2023.

[36] O. Tikhonenko and M. Ziółkowski, "Performance evaluation of unreliable system with infinite number of servers," *Bulletin of the Polish Academy of Sciences. Technical Sciences*, vol. 68, no. 2, pp. 289–297, 2020.

[37] O. Tikhonenko and M. Ziółkowski, "Unreliable single-server queueing system with customers of random capacity," in *Computer Networks*, P. Gaj, W. Gumiński, and A. Kwiecień, Eds. Cham: Springer International Publishing, 2020, pp. 153–170.

[38] V. V. Ivlev, "Indeterminate forms of many variables functions. part I," *Education of Mathematics*, vol. 23, no. 4, pp. 90–100, 2002 (In Russian).

[39] V. V. Ivlev, "Indeterminate forms of many variables functions. part II," *Education of Mathematics*, vol. 26, no. 3, pp. 77–85, 2003 (In Russian).

[40] M. Ziółkowski, "On applications of computer algebra systems in queueing theory calculations," *Bulletin of the Polish Academy of Sciences. Technical Sciences*, vol. 72, no. 3, art. no. e150199, 2024.