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The fractional order, inertial discrete transfer function using Atangana-Baleanu and FOBD operators

Krzysztof OPRZĘDKIEWICZ

In the paper a new, fractional order, discrete transfer function model of an elementary inertial plant is proposed. The model uses Atangana-Baleanu and discrete Fractional Order Backward Difference operators to describe of the fractional derivative. Such a transfer models have not be presented yet. The analytical formula of the step response for time-continuous transfer function is given. The similarity of the proposed model to "classic" one using Caputo operator is also considered. The stability and the convergence of the discrete transfer function are analyzed. Theoretical results are expanded by simulations. The proposed discrete, approximated model is accurate and its numerical complexity is low. It can be useful in modeling of different physical phenomena, for example thermal processes.

Key words: fractional order transfer function, Grünwald-Letnikov definition, Atangana Baleanu operator, FOBD operator, convergence

1. Introduction

The new fractional operator with nonsingular kernel has been proposed by Atangana and Baleanu in the paper [2]. The approximations od the Atangana-Bealeanu operator (AB operator) are analyzed in papers [11, 13].

Interesting collections of results presenting the use of AB operator in modeling of different physical, biological, and social phenomena can be found e.g. in papers [7] or [8]. There are models of population growth, logistic equation, blood alcohol models.

Some recent results presenting the use of AB operator can be found in the papers: [3] considers the modeling of COVID-19 dynamics in Ethiopia, in the

Author (e-mail: kop@agh.edu.pl) is with Departmeent of Automatic Control and Robotics, Faculty of Electrotechnics, Automatic Control, Informatics and Biomedical Engineering, AGH University of Science and Technology, al. A Mickiewicza 30 30-059 Krakow Poland.

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paper [19] deals with the use of AB operator in nonlinear fractional differential equations, the paper [1] presents the application of AB operator in advection–dispersion equations. The use of AB operator to modeling of heat transfer was considered for example in [18] and [12, 14].

It is characteristic that all known models employing the AB operator have the form of a state equation. A transfer function model using this operator has not been proposed yet.

This paper proposes two versions of a new, fractional order transfer function model using AB operator. Firstly, the time-continuous model and its step response is given. This step response is compared to the step response of the Fractional Order (FO) inertial plant using Caputo definition.

Next the approximated, discrete time version of the transfer function using Fractional Order Backward Difference (FOBD) approximation is proposed and analyzed. The new proposed model can be used to modeling of different real physical phenomena.

The paper is organized as follows. Preliminaries give theoretical background to present of main results. Next the proposed transfer function in both forms is discussed. Furthermore, theoretical results are expanded by simulations.

2. Preliminaries

2.1. Basics of fractional calculus

Elementary ideas from fractional calculus can be found in many books, e.g. [6, 9, 16] or [17]. Here only some definitions necessary to present of main results will be recalled.

First of all the fractional-order, integro-differential operator (see e.g. [6, 10, 17]) needs to be given. Is is as follows:

Definition 1. *(The elementary fractional order operator) The fractional-order integro-differential operator is defined as follows:*

$$
{}_{t_s}D^{\alpha}_{t_f} f(t) = \begin{cases} \dfrac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} & \alpha > 0, \\ f(t) & \alpha = 0, \\ \displaystyle\int_{t_s}^{t_f} f(\tau)(\mathrm{d}\tau)^{\alpha} & \alpha < 0, \end{cases} \tag{1}
$$

where t_s and t_f denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.

Next recall the Gamma Euler function (see e.g. [10]):

Definition 2. *(The Gamma function)*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t. \tag{2}$$

Mittag-Leffler function is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in solution of FO state equation. The one parameter Mittag-Leffler function is defined as follows:

Definition 3. *(The one parameter Mittag-Leffler function)*

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)} . \tag{3}$$

The two parameter Mittag-Leffler function is defined as follows:

Definition 4. *(The two parameters Mittag-Leffler function)*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}. \tag{4}$$

For $\beta = 1$ the two parameter function (4) turns to one parameter function (3).

The fractional-order, integro-differential operator can be described by different definitions, given by Grünvald and Letnikov (GL defintion), Riemann and Liouville (RL definition) and Caputo (C definition). In the further consideration C and GL definitions will be used. They are given below ([5, 15]).

Definition 5. *(The Caputo definition of the FO operator)*

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^\infty \frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha+1-N}} \mathrm{d}\tau \tag{5}$$

where $N - 1 < \alpha < N$ denotes the non-integer order of operation and $\Gamma(.)$ is the complete Gamma function expressed by (2).

For the Caputo operator the Laplace transform can be defined (see for example [9]):

Definition 6. *(The Laplace transform of the Caputo operator)*

$$\mathcal{L}({}_0^C D_t^\alpha f(t)) = s^\alpha F(s), \qquad \alpha < 0,$$
$$\mathcal{L}({}_0^C D_t^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}{}_0 D_t^k f(0), \tag{6}$$
$$\alpha > 0, \ \ n-1 < \alpha \leqslant n \in N.$$

Consequently, the inverse Laplace transform for non integer order function is expressed as follows ([10]):

$$\mathcal{L}^{-1}[s^\alpha F(s)] ={}_0 D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{t^{k-1}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+) \tag{7}$$
$$n-1 < \alpha < n, \ \ n \in \mathbb{Z}.$$

The GL derivative along the time from function $g(t)$ is defined as follows [5, 15]:

Definition 7. *(The Grünwald-Letnikov definition)*

$${}_0^{GL} D_t^\alpha g(t) = \lim_{h\to 0} h^{-\alpha} \sum_{l=0}^{\left[\frac{t}{h}\right]} (-1)^l \binom{\alpha}{l} g(t-lh). \tag{8}$$

In (8) $0.0 < \alpha \leqslant 1.0$ is the fractional order along the time, h is the sample time, $[.]$ is the nearest integer value, $\binom{\alpha}{l}$ is the binomial coefficient:

$$\binom{\alpha}{l} = \left\{ \begin{array}{cc} 1, & l = 0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-l+1)}{l!}, & l > 0 \end{array} \right\}. \tag{9}$$

2.2. Elementary FO transfer function

The elementary, scalar input-output differential equation using elementary fractional operator (1) takes the following form:

$$T_{\alpha 0} D_t^\alpha y(t) = -y(t) + u(t). \tag{10}$$

where T_α is the time constant, $u(t)$ is the control signal and $y(t)$ is the output.

Assume homogenous initial condition. Applying (6) in (10) gives the elementary, fractional order transfer function:

$$G_C(s) = \frac{1}{T_\alpha s^\alpha + 1}. \tag{11}$$

For this transfer function its impulse and step responses are as beneath (see e.g. [5, p. 11]):

$$g_C(t) = \frac{t^{\alpha-1}}{T_\alpha} E_\alpha\left(-\frac{t^\alpha}{T_\alpha}\right), \tag{12}$$

$$y_C(t) = 1(t) - E_\alpha\left(-\frac{t^\alpha}{T_\alpha}\right). \tag{13}$$

In (12) and (13) $E_\alpha(.)$ is the one parameter Mittag-Leffler function (3).

2.3. The FOBD approximation

The GL definition is the limit case for $h \to 0$, $\Delta x \to 0$ of the FOBD, commonly employed in discrete FO calculations (see e.g. [16, p. 68]).

Definition 8. *(The Fractional Order Backward Difference along the time – FOBDT)*

$$\Delta^\alpha g(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} g(t - lh). \tag{14}$$

In (14) L denotes a memory length necessary to correct approximation of a non integer order operator. Unfortunately, good accuracy of approximation requires to use a long memory L which can make implementation difficult.

Denote coefficients $(-1)^l\binom{\alpha}{l}$ by d_l:

$$d_l = (-1)^l \binom{\alpha}{l}. \tag{15}$$

The coefficients (15) can be also computed using the following, equivalent, recursive formula (e.g. [5, p. 12]), useful in numerical calculations:

$$\begin{aligned} d_0 &= 1, \\ d_l &= \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1, \ldots, L. \end{aligned} \tag{16}$$

In [4] it is given that:

$$\sum_{l=1}^{\infty} d_l = 1 - \alpha, \tag{17}$$

$$\sum_{l=0}^{\infty} d_l = 0. \tag{18}$$

Using (15) the operator (14) can be expressed in shorter form:

$$\Delta^{\alpha} g(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} d_l g(t - lh), \tag{19}$$

and consequently its discrete transfer function $G_{FOBD}(z^{-1})$ takes the following form:

$$G_{FOBD}(z^{-1}) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} d_l z^{-l}. \tag{20}$$

2.4. Discrete systems: selected results

Let recall two theorems from theory of discrete time dynamic systems, necessary to present of main results: there are Final Value Theorem (FVT) and necessary condition of the asymptotic stability of a system described by a discrete transfer function $G^{+}(z)$.

Theorem 1. *(Final Value Theorem for discrete time)*

Let $g(k)$ is a discrete function of time, defined in k time instants and $G(z)$ is its z-transform. Assume that $G^{+}(z)$:

1) has no poles outside the unit circle,

2) has maximally one pole on the unit circle: $z = 1$,

then:

$$\lim_{k\to\infty} g(k) = \lim_{z\to 1} (z - 1)G(z). \tag{21}$$

Theorem 2. *(Necessary condition of the asymptotic stability of the discrete polynomial)*

Consider the characteristic polynomial of a discrete system:
$w(z) = a_N z^N + \ldots + a_1 z + a_0.$
The necessary condition of its asymptotic stability is as follows:

$$w(1) > 0 \ \wedge \ (-1)^N w(-1) > 0 \ \wedge \ |a_0| < a_N. \tag{22}$$

2.5. The Atangana-Baleanu fractional operator

The fractional order derivative Atangana-Baleanu operator is obtained via replacing the exponential kernel in the Caputo-Fabrizio (CF) operator by the Mittag-Leffler kernel. It is defined using the C or RL definition of fractional order derivative. Using these definitions we obtain the Atangana-Baleanu-Caputo (ABC) or Atangana-Baleanu-Riemann (ABR)operator respectively [2]:

Definition 9. *(The Atangana-Baleanu-Caputo (ABC) operator)*

$$ {}^{ABC}{}_{a}D_t^{\alpha}(f(t)) = M(\alpha)\int\limits_a^t f'(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)\mathrm{d}x, \tag{23}$$

where $E_{\alpha}(.)$ is the one parameter Mittag-Leffler function, M_{α} is the normalization function equal:

$$M(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}\,. \tag{24}$$

In (24) $\Gamma(.)$ *is the Gamma function.*

Definition 10. *(The Atangana-Baleanu-Riemann (ABR) operator)*

$$ {}^{ABR}{}_{a}D_t^{\alpha}(f(t)) = M(\alpha)\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_a^t f(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)\mathrm{d}x. \tag{25}$$

where $E_{\alpha}(.)$ is the one parameter Mittag-Leffler function, $M(\alpha)$ is the normalization function expressed by (24), $\Gamma(.)$ *is the Gamma function.*

The Laplace transforms for the ABC and ABR derivatives are as follows:

Definition 11. *(The Laplace transform of the ABC operator)*

$$\mathcal{L}\{{}^{ABC}{}_{a}D_t^{\alpha}(f(t))\}(s) = \frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\{f(t)\}(s)-s^{\alpha-1}f(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\,. \tag{26}$$

Definition 12. *(The Laplace transform of the ABR operator)*

$$\mathcal{L}\{{}^{ABR}{}_{a}D_t^{\alpha}(f(t))\}(s) = \frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\{f(t)\}(s)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\,. \tag{27}$$

For the homogenous initial condition: $f(0) = 0$ both Laplace transforms are equal:

$$\mathcal{L}\{{}^{ABR}{}_{a}D_t^{\alpha}(f(t))\}(s) = \mathcal{L}\{{}^{ABC}{}_{a}D_t^{\alpha}(f(t))\}(s). \tag{28}$$

In further considerations it will be used the common notation ^{AB} to denote this operator in both versions, becuase the initial conditions are equal zero during analysis of a transfer function. To simplify, introduce the following short notation:

$$\mathcal{L}\{{}^{AB}{}_{a}D_t^{\alpha}(f(t))\}(s) = \frac{b_{\alpha}s^{\alpha}}{s^{\alpha}+a_{\alpha}} \tag{29}$$

where:

$$a_{\alpha} = \frac{\alpha}{1-\alpha}, \tag{30}$$

$$b_\alpha = \frac{M(\alpha)}{1-\alpha}. \tag{31}$$

The form a_α and b_α require to assume that $0.0 \leqslant \alpha < 1.0$.

3. Main results

3.1. The time-continuous transfer function

The application of the AB operator (23) or (25) in the elementary FO differential equation (10) yields:

$$T_\alpha \left({}_a^{AB}D_t^\alpha y(t) \right) + y(t) = u(t). \tag{32}$$

Assume homogenous initial condition. Using of (29) in (32) we obtain:

$$\frac{T_\alpha b_\alpha s^\alpha}{s^\alpha + a_\alpha} Y(s) + Y(s) = U(s). \tag{33}$$

Consequently the transfer function $G(s) = \frac{Y(s)}{U(s)}$ takes the following form:

$$G_{AB}(s) = \frac{s^\alpha + a_\alpha}{T_{AB} s^\alpha + a_\alpha}\,. \tag{34}$$

where a_α and b_α are expressed by (30) and (31), respectively, and:

$$T_{AB} = 1 + T_\alpha b_\alpha\,. \tag{35}$$

The step response of the transfer function is described by the following proposition:

Proposition 1. *(The step response of the transfer function using AB operator) Consider the FO transfer function $G_{AB}(s)$ described by (34).*

Its step response takes the following form:

$$y_{AB}(t) = 1(t) + \left(\frac{1}{T_{AB}} - 1 \right) E_\alpha \left(-\frac{a_\alpha t^\alpha}{T_{AB}} \right), \tag{36}$$

where T_{AB} is expressed by (35).

Proof. The transfer function (34) can be expressed as the sum of two following transfer functions:

$$G_{AB1}(s) = \frac{s^\alpha}{T_{AB} s^\alpha + 1}\,, \tag{37}$$

$$G_{AB2}(s) = \frac{a_\alpha}{T_{AB} s^\alpha + 1}\,. \tag{38}$$

The step response we are looking for is the sum of step responses of both components (37), (38). Denote these responses as $y_{AB1}(t)$ and $y_{AB2}(t)$, respectively. They are equal:

$$y_{AB1,2}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}G_{AB1,2}(s)\right\}. \tag{39}$$

The step response $y_{AB1}(t)$ is obtained using Equation (1.34), page 11 in book [5]. It takes the following form:

$$y_{AB1}(t) = \frac{1}{T_{AB}}E_{\alpha}\left(\frac{-a_{\alpha}t^{\alpha}}{T_{AB}}\right). \tag{40}$$

Next, the step response $y_{AB2}(t)$ we obtain using (13):

$$y_{AB2}(t) = 1(t) - E_{\alpha}\left(\frac{-a_{\alpha}t^{\alpha}}{T_{AB}}\right). \tag{41}$$

After adding (40) to (41) we obtain (36) and the proof is completed. □

Next the steady-state response of the considered transfer function (34) is described by the following remark.

Remark 1. *(The steady-state response of the time-continuous transfer function)*

Consider the transfer function using AB operator (34). *Its steady-state response is equal:*

$$y_{ss} = 1. \tag{42}$$

Proof. The Laplace transform of the step response of the transfer function (34) is as follows:

$$Y(s) = \frac{1}{s}G_{AB}(s). \tag{43}$$

The steady-state value of (43) is obtained using Final Value Theorem (FVT):

$$y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} G_{AB}(s) = \lim_{s\to 0}\frac{s^{\alpha}+a_{\alpha}}{T_{\alpha}b_{\alpha}s^{\alpha}+a_{\alpha}} = 1. \tag{44}$$

An interesting issue is to compare the proposed transfer function (34) with the well known transfer function using Caputo operator (11). To do this the following “quasi-norms” H describing the distance between step responses of both transfer functions are proposed:

$$H_{\max} = \max_{0\geqslant t\geqslant T_f}\left|y_C(t) - y_{AB}(t)\right|, \tag{45}$$

$$H_2 = \int_0^{T_f}\left(y_C(t) - y_{AB}(t)\right)^2 \mathrm{d}t, \tag{46}$$

where T_f is the final time of the response caclulation.

The fundametal parameter describing dynamics of each system is damping rate ξthat can be easily defined for scalar systems considered here. It is equal for each considered transfer function:

$$\xi_C = \frac{1}{T_\alpha}, \tag{47}$$

$$\xi_{AB} = \frac{a_\alpha}{T_{AB}}. \tag{48}$$

Association of both damping rates is described by the following remark.

Remark 2. *(The damping rate of AB transfer function)*

Consider the inertial transfer function using C operator (11) *with damping rate* (47) *and inertial transfer function using AB operator* (34) *with damping rate* (48).

Both damping rates are associated as beneath:

$$\lim_{\alpha \to 1} \xi_{AB} = \xi_C. \tag{49}$$

Proof.

$$\lim_{\alpha \to 1} \xi_{AB} = \lim_{\alpha \to 1} \frac{a_\alpha}{T_{AB}}. \tag{50}$$

Recalling (30) and (35) yields:

$$\lim_{\alpha \to 1} \xi_{AB} = \lim_{\alpha \to 1} \frac{\frac{\alpha}{1-\alpha}}{1 + T_\alpha \frac{M(\alpha)}{1-\alpha}} = \lim_{\alpha \to 1} \frac{\alpha}{1 - \alpha + T_\alpha M(\alpha)}. \tag{51}$$

Next, using (24) we obtain:

$$\lim_{\alpha \to 1} M(\alpha) = 1. \tag{52}$$

Taking into consideration (52) in (51) gives:

$$\lim_{\alpha \to 1} \xi_{AB} = \frac{1}{T_\alpha} \tag{53}$$

which completes the proof. □

The remarks (42) and (49) allow to conclude that:

$$\begin{aligned} \lim_{\alpha \to 1} H_{\max} &= 0, \\ \lim_{\alpha \to 1} H_2 &= 0. \end{aligned} \tag{54}$$

3.2. The approximated, discrete transfer function

The discrete-time transfer function using FOBD operator with fixed memory length L is obtained by employing of (20) in (34):

$$G_{ABL}(z^{-1}) = \frac{N_L(z^{-1})}{D_L(z^{-1})}, \tag{55}$$

where:

$$N_L(z^{-1}) = h^{-\alpha} \sum_{l=0}^{L} d_l z^{-l} + a_\alpha, \tag{56}$$

$$D_L(z^{-1}) = h^{-\alpha} T_{AB} \sum_{l=0}^{L} d_l z^{-l} + a_\alpha. \tag{57}$$

For each memory length it is expressed as:

$$G_{AB\infty}(z^{-1}) = \frac{N_\infty(z^{-1})}{D_\infty(z^{-1})}, \tag{58}$$

where:

$$N_\infty(z^{-1}) = h^{-\alpha} \sum_{l=0}^{\infty} d_l z^{-l} + a_\alpha, \tag{59}$$

$$D_\infty(z^{-1}) = h^{-\alpha} T_{AB} \sum_{l=0}^{\infty} d_l z^{-l} + a_\alpha. \tag{60}$$

The Z transform of the step response of both considered transfer functions takes the following form:

$$Y_{L,\infty}(z^{-1}) = \frac{G^{+}_{FOBD}(z^{-1})}{1 - z^{-1}} \tag{61}$$

and consequently the step response of the discrete, approximated transfer function is as follows:

$$y_{L,\infty}(k) = \mathcal{Z}^{-1}\{Y^{+}_{L,\infty}(z^{-1})\}, \tag{62}$$

where $k = 1, 2, \ldots$ denotes discrete time instants. The formula (62) can be solved numerically using e.g. MATLAB, which was used for numerical validation of results discussed in the next section.

The steady-state responses to the Heaviside function of transfer functions (55) and (58) are described by the following remarks.

Remark 3. *(The steady-state response of the discrete, fixed memory length transfer function)*

Consider the discrete transfer function with fixed memory length L (55)*. Its steady-state response is equal:*

$$y_{ssL} = \frac{S_L + h^\alpha a_\alpha}{T_{AB} S_L + h^\alpha a_\alpha}\,, \tag{63}$$

where:

$$S_L = \sum_{l=0}^{L} d_l\,. \tag{64}$$

Proof. The Z transform of the step response of transfer function (55) is given by (61). The use of Final Value Theorem for discrete system yields:

$$Y_{ssL} = \lim_{k\to\infty} y(k) = \lim_{z^{-1}\to 1} (1 - z^{-1})Y(z^{-1}) = \lim_{z^{-1}\to 1} G_{ABL}(z^{-1}). \tag{65}$$

Calculating of the limit (65) with the use of (64) gives directly (63) and the proof is completed. □

Remark 4. *(The steady-state response of the discrete, each memory length transfer function)*

Consider the discrete transfer function with each memory length (58)*. Its steady-state response is equal:*

$$y_{ss\infty} = 1. \tag{66}$$

Proof. To prove (66) remember that:

$$\sum_{l=0}^{\infty} d_l = 0. \tag{67}$$

Taking (67) to (65) gives (66). □

3.3. The stability of the approximated discrete transfer function

The proposed approximated transfer function (55) or (58) is the integer-order, discrete transfer function. This allows to analyze its stability with the use of well known tools. The necessary condition of its asymptotic stability is described by the following proposition.

Proposition 2. *(The necessary condition of the asymptotic stability of the transfer function* (55)*,* (58)*)*

Consider the discrete transfer function (55)*,* (58)*, being the discrete FOBD approximation of the FO transfer function* (34) *with fractional order* $0.0 < \alpha < 1.0$.

The denominator of the FOBD approximation (57) *or* (60) *meets the necessary condition of the asymptotic stability* (22) *for each value of sample time h and memory length L.*

Proof. The proof consists in examination all the conditions given in (22).

The characteristic polynomial of the transfer function (34) as a function of z with respect to (16) takes the following form:

$$w(z) = \left(\frac{T_{AB}}{h^\alpha} + a_\alpha\right) z^L - \alpha z^{L-1} + \sum_{l=0}^{L} d_l z^{L-l}. \tag{68}$$

The first condition is written as:

$$w(1) = \frac{T_{AB}}{h^\alpha} + a_\alpha - \alpha + \sum_{l=0}^{L} d_l\,. \tag{69}$$

The sum in (69) for $L \to \infty$ goes to zero (see (18)). Remember that $a_\alpha = \frac{\alpha}{1-\alpha}$. After elementary transformation we obtain:

$$w(1) = \frac{T_{AB}}{h^\alpha} + \frac{\alpha^2}{1-\alpha}\,. \tag{70}$$

Expression (69) is positive for each $T_{AB}, h > 0$ and $0.0 < \alpha < 1.0$, thus the 1'st condition from (22) is met.

The second condition from (22) takes the following form:

$$(-1)^L w(-1) = (-1)^L \left(\left(\frac{T_{AB}}{h^\alpha} + a_\alpha\right)(-1)^L - \alpha(-1)^{L-1} + \sum_{l=2}^{L}(-1)^{L-l} d_l\right). \tag{71}$$

The sum in the bracket can be also ingnored. This yields:

$$(-1)^L w(-1) = (-1)^L \left(\left(\frac{T_{AB}}{h^\alpha} + a_\alpha\right)(-1)^L - \alpha(-1)^{L-1}\right). \tag{72}$$

For even L the condition (72) turns to:

$$(-1)^L w(-1) = \left(\left(\frac{T_{AB}}{h^\alpha} + a_\alpha\right) - \alpha(-1)\right) = \frac{T_{AB}}{h^\alpha} + a_\alpha + \alpha. \tag{73}$$

For odd L we obtain:

$$(-1)^L w(-1) = -\left(-\left(\frac{T_{AB}}{h^\alpha} + a_\alpha\right) - \alpha\right) = \frac{T_{AB}}{h^\alpha} + a_\alpha + \alpha. \tag{74}$$

Both expressions (73) and (74) are positive for each value of L, $T_{AB}, h > 0$ and $0.0 < \alpha < 1.0$.

Finally the last condition from (22) needs to be analyzed. It takes the following form:

$$|d_L| < \frac{T_{AB}}{h^\alpha} + a_\alpha \,. \tag{75}$$

The coefficients d_l, expressed by (16) are descreasing function of L. In the range $l = 1, \ldots, L$ the maximum value has the coefficient $|d_1| = \alpha$ and it can be employed to testing of the condition (75) due to it gives the strongest limitation. With respect to (16) and (30) it is as follows:

$$\alpha < \frac{T_{AB}}{h^\alpha} + \frac{\alpha}{1-\alpha} \iff \frac{T_{AB}}{h^\alpha} + \frac{\alpha^2}{1-\alpha} > 0. \tag{76}$$

The condition (76) is met any for value of $L, T_{AB}, h > 0$ and $0.0 < \alpha < 1.0$. This finishes the proof. □

3.4. The convergence of the discrete approximation

The Rate of Convergence (ROC) of the proposed, discrete, approximated model can be defined as follows:

Definition 13. *(The Rate of Convergence)*

ROC of the discrete transfer function (55) *constructed with the fixed memory length L is equal to its steady state value* (63)*:*

$$ROC_L = y_{ssL} \tag{77}$$

where y_{ssL} is described by (63).

It is obvious that $\lim_{L\to\infty} ROC_L = 1$. The ROC is a function of parameters of FOBD: sample time h and memory length L. It is also a function of parameters of the model: fractional order α and time constant T_α.

The value of the sample time h assuring the minimum value Δ_L of ROC is described by the following proposition:

Proposition 3. *(The value of sample time h assuring the minimum value of ROC)*

Consider the discrete FO transfer function (55). *The minimum value of the sample time h assuring the minimum, predefined value of Δ_L is described as follows:*

$$h \geqslant \left(\frac{S_L(1-\Delta_L T_{AB})}{a_\alpha(\Delta_L - 1)}\right)^{\frac{1}{\alpha}} . \tag{78}$$

Proof. The minimum predefined value of ROC using (63) is expressed as follows:

$$\begin{aligned} \Delta_L \geqslant \frac{S_L + h^\alpha a_\alpha}{T_{AB} S_L + h^\alpha a_\alpha} \iff \\ \iff h^\alpha a_\alpha (\Delta_L - 1) \geqslant S_L (1 - \Delta_L T_{AB}) \iff \\ \iff h^\alpha \geqslant \left(\frac{S_L (1 - \Delta_L T_{AB})}{a_\alpha (\Delta_L - 1)} \right) \iff \\ \iff h \geqslant \left(\frac{S_L (1 - \Delta_L T_{AB})}{a_\alpha (\Delta_L - 1)} \right)^{\frac{1}{\alpha}}. \end{aligned}$$

4. Simulations

4.1. The time continuous transfer function

Firstly, the step responses using analytical formula (36) were examined. Time trends obtained using MATLAB for different values of fractional order α and time constants T_α are shown in Figures 1 and 2. These analytical responses will be used as a reference to estimate the quality of the discrete approximation.

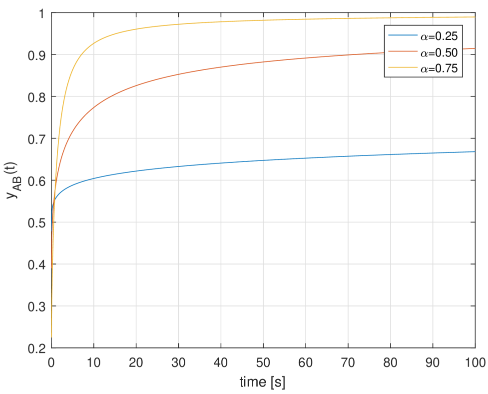

Figure 1: The analytical step responses of the proposed transfer function for $T_\alpha = 1s$ and different α computed with the use of (36)

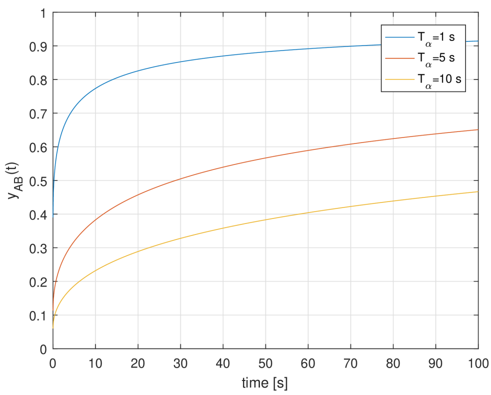

Figure 2: The step responses of the proposed transfer function for $\alpha = 0.5$ and different T_α computed with the use of (36)

Secondly, the step responses were compared with step responses of the transfer function using C operator, computed using (13). The comparison was run for time constant $T_\alpha = 1s$ and final time $T_f = 100s$. The quasi norms (45) and (46) for varying α are given in Table 1. The comparison of the step responses for the same parameters is presented in Fig. 3.

Table 1: Quasi norms (45) and (46) for $T_\alpha = 1s$, $T_f = 100s$ and various fractional orders α

α	$H_{\max}$	H_2
0.25	0.4780	1.1417
0.50	0.3900	0.1888
0.75	0.2248	0.0167
0.95	0.0490	7.9049e-04

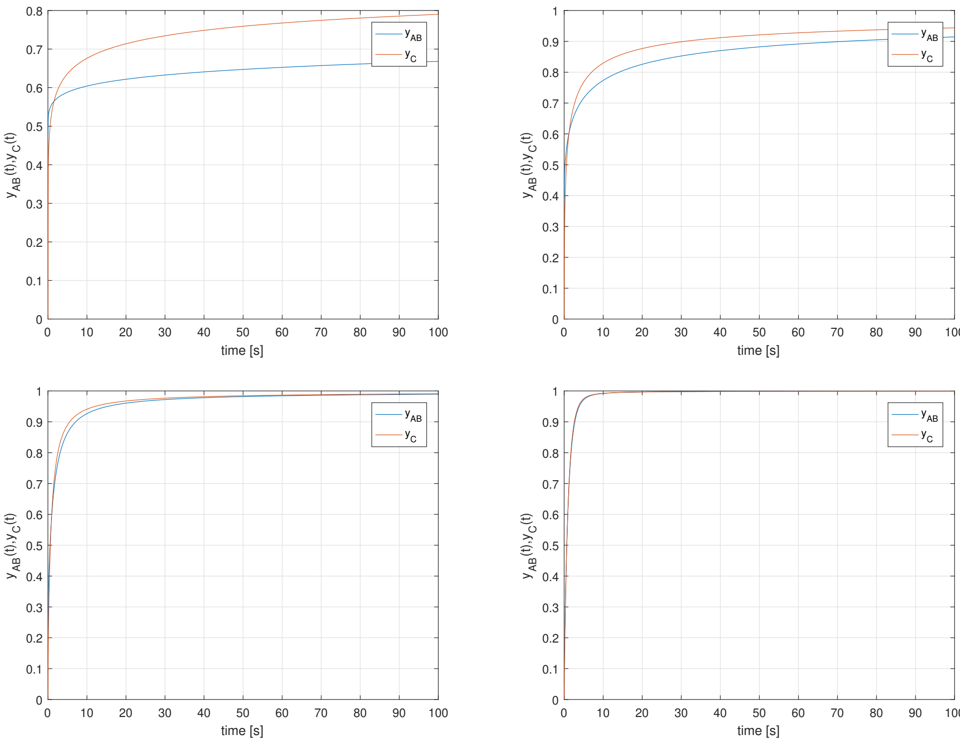

Figure 3: The comparing of step responses $y_C(t)$ vs $y_{AB}(t)$ for $T_\alpha = 1s$, $T_f = 100s$ and $\alpha = 0.25$, 0.50, 0.75, 0.95 (top-bottom)

4.2. The approximated transfer function using FOBD

In this section the discrete transfer function using FOBD approximation was examined. Its accuracy was estimated using known Integral Absolute Error (IAE) cost function:

$$IAE = h \sum_{k=1}^{K} |y_{AB}(kh) - y_L(kh)|, \tag{79}$$

where $y_{AB}(kh)$ is the analytical response (36) and $y_L(kh)$ is the step response of approximation (62). For fixed α and T this cost function is a function of memory length L and sample time h. Its 3D plot for $L = 100–500$ and $h = 0.1–10$ s is shown in Fig. 4.

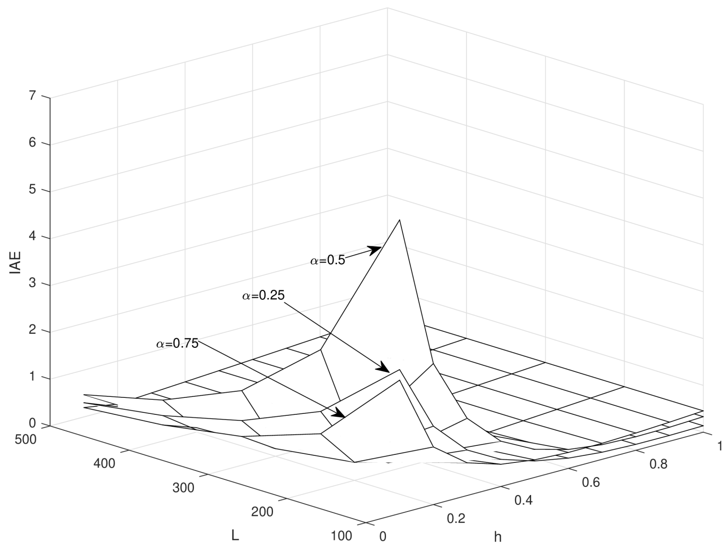

Figure 4: The IAE cost function as a function of memory length L and sample time h for $\alpha = 0.25$, 0.50, 0.75

Comparison of $y_{AB}(t)$ and $y_L(kh)$ for different values of fractional order α is shown in Fig. 5 and respective ISE values are given in Table 2.

Table 2: The cost function (79) for $T_\alpha = 1$s, $h = 1$s, $L = 100$ and various fractional orders α

α	0.25	0.50	0.75
ISE	0.1372	0.3312	0.4554

In the next step the ROC coefficient (77) as a function of memory length L and sample time h was numerically estimated. Its 3D plots for various values of fractional order α are given in Fig. 6.

Next, the value of the sample time h assuring the predefined value Δ_L of ROC is to be calculated with the use of (78). Assume that the required value of $\Delta_L = 0.90$, the memory length $L = 200$. The parameters of the plant are: $\alpha = 0.5$ and $T_\alpha = 5$ s. The use of (78) yields: $h \geqslant 7.6524$ s.

To verify this result assume h= 8s for the same values of other parameters. The use of (63) gives the value of $ROC = 0.9019$.

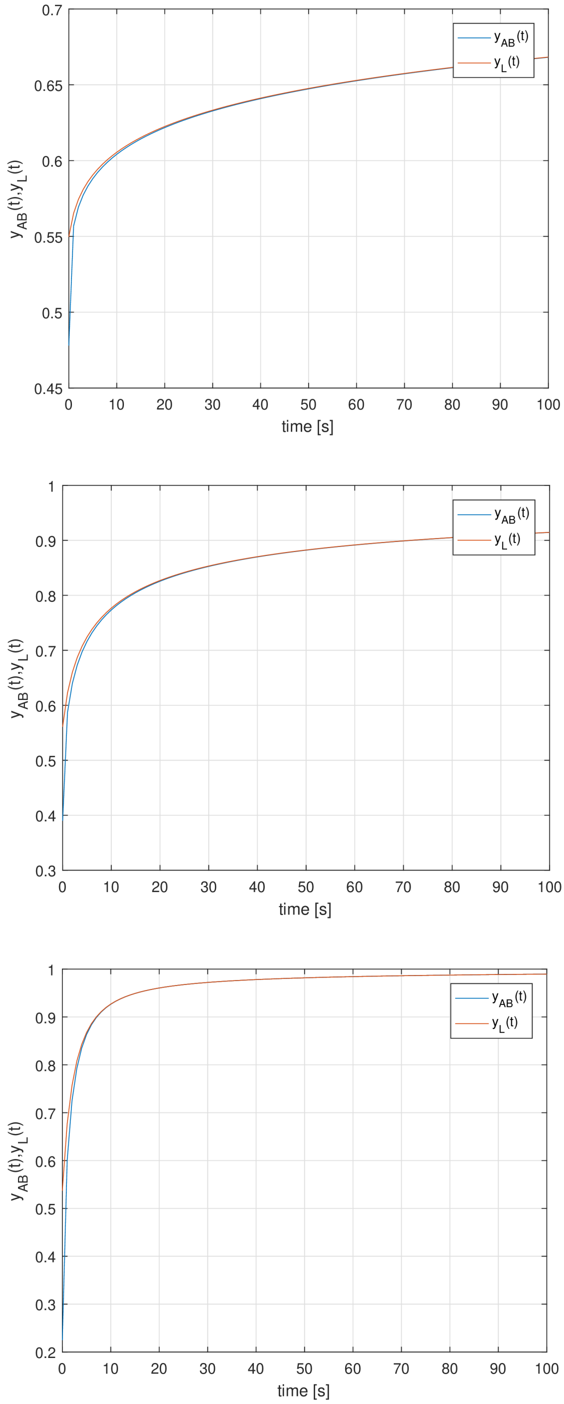

Figure 5: The comparing of step responses $y_{AB}(t)$ vs $y_L(kh)$ for $T_\alpha = 1$ s, $T_f = 100$ s, $L = 100$, $h = 1$ s and $\alpha = 0.25, 0.50, 0.75, 0.95$ (top-bottom)

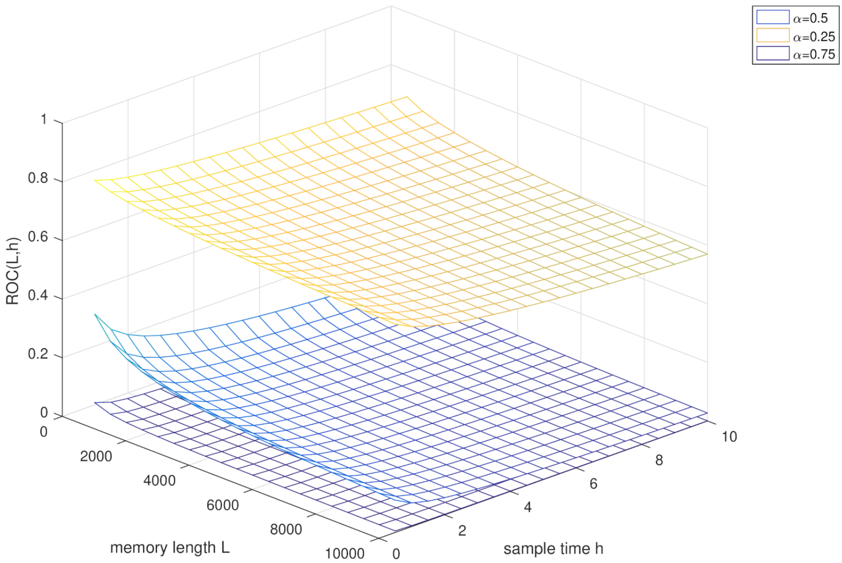

Figure 6: The ROC (77) as a function of memory length L and sample time h for different orders α

5. Discussion of results and final conclusions

For the time continuous transfer function (34) the most important remark is that for fractional order α close to 1.0 its step response is tending to the step response of the "classic" FO transfer function using C operator (11). This is illustrated by Fig. 3 and Table 1. From this observation it can be concluded that the use of the proposed transfer function in modeling has a sense only for fractional orders α significantly smaller than 1.0. For fractional order α close to 1.0 the "classic" transfer function (11) assures very similar behavior in the sense of the step response and it is simplier to implement.

The analysis of the discrete transfer function using FOBD approximation (55), (58) shows valuable advantage of the proposed model. Namely, its good accuracy achieved for long sample time h is associated with low memory length L. It can be observed in the 3D plots 4 and 6. This property can be very useful during its digital implementation at bounded platform (e.g. PLC or microcontroller).

Simultaneously, the shortening of the sample time h requires to increasing of the memory length L to obtain the same accuracy and convergence.

The spectrum of further investigations of the proposed transfer function covers first of all its use to describe of real physical phenomena, e.g. thermal processes previously considered by author.

Next, theoretical considerations presented in this paper should be supplemented with deeper analysis of a stability. Here necessary and sufficient stability condition is expected to prove. Interesting is also the expanding of the fractional order to range $1.0 < \alpha < 2.0$.

An another issue is the construction of the approximated transfer function with the use of the CFE approximation instead of the FOBD. This should further reduce the numerical complexity of the model.

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