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Static anti-windup compensator based on BMI optimization for discrete-time systems with directional change in controls avoidance

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Abstract. In the paper, a design method of a static anti-windup compensator for systems with input saturations is proposed. First, an anti-windup controller is presented for system with cut-off saturations, and, secondly, the design problem of the compensator is presented to be a non-convex optimization problem easily solved using bilinear matrix inequalities formulation. This approach guarantees stability of the closed-loop system against saturation nonlinearities and optimizes the robust control performance while the saturation is active.

Keywords: anti-windup compensation; bilinear matrix inequalities; cut-off constraint; no directional change in controls.

1. INTRODUCTION

In this paper, we explore directional change phenomenon from the viewpoint of compensation of negative effects of control vector saturation. This approach is crucial for maintaining the stability and performance of discrete-time control systems under saturation conditions with directional change in controls avoidance. The paper extends the bilinear matrix inequality (BMI) formulation presented in paper [1] which has been the first stage in this research, to a more demanding control scenario. Static compensation is a research topic still currently developed, such as in [2] for nonlinear systems, satisfying Lipschitz conditions, [3] for nonlinear time-delayed systems, or [4] for time-delayed systems.

One can list a number of strategies present to avoid changes in direction of a control vector, such as vector control techniques to drive inverters and motor drives, to fit the requirement of precise control of voltage, current and flux space vectors by avoiding direction changes in a control vector [5].

The other might be a switched control strategy to incorporate control constraints to preserve the initial control direction and, at the same time, to prevent the state vector from leaving the domain of control authority, to keep the system both stable and work under constraints [6].

In electrical machines, especially in induction motors, there is a space vector modulation algorithm to keep the direction of a control vector unaltered by flux or torque ripple reduction, due to varying switching frequency citeREFERENCE3.

It is not only the case of stability as the problem, but also it is related to dynamic decoupling problem, tackled out in, e.g. [8,9], where the preservation of direction of a control vector is bound with coupling its components to suit certain aims.

The problem of decoupling becomes even more impeding if nonlinear multiple-input multiple-output plants are considered, see [10] with uncertainty taken into account. The approach presented in the current paper can also be extended to uncertainty-related case by adding a polytopic uncertainty information, to present the conditions by a set of BMIs for the polytopic information [11–13].

Recent research of the author of the paper, see the D.Sc. monograph [14] or [15], or the research of the other authors, [16–18], provide some detailed information on control quality improvement with simultaneous anti-windup compensation scheme, also for the case when directional change is to be avoided. However, there was a gap in the current literature concerning extension of this research to discrete-time models, leading to an optimization-based way to obtain optimal static anti-windup compensator. The optimization problems presented in the paper concern all minimisation tasks of a linear function subject to BMI constraints [17, 19]. These problems will not be solvable using linear matrix inequality approach, and require other software, such as PENBMI (PENOPT suite) [19], TOMLAB-PENBMI or other. The basic formulation of the problem is coded down in Yalmip [20–22].

The results reported at the previous stage of the research [1], present design of static compensators for discrete-time models with cut-off constraints, whereas the results for no directional change in controls are presented in this paper. It is a first try to formalize the conditions, by introducing a scaling factor to the conditions, to lead to an off-line calculation of the mapping between the saturation level (expressed by the scaling factor) versus compensator feedback matrix. The major contribution of the paper, is not only the introduction of a scaling factor, but also deriving the BMI conditions.

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2. DIRECTIONAL CHANGE IN CONTROLS

2.1. Introduction

The directional change in controls issue can be easily depicted for a system with two control inputs, each with a prescribed cut-off level. As it can be observed in Fig. 1 the calculated control vector $\underline{v}_t$ has a different direction than the constrained control vector $\underline{u}_t$ [15].

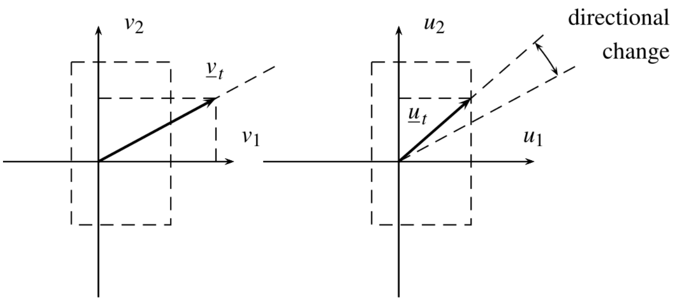


Fig. 1. Directional change issue

A change in control vector direction should not be connected to coupling avoidance task only, but rather in maintaining the information stored in the original, i.e., calculated control vector, to preserve balance between separate control inputs, what could be met in various robotics-correlated problems. The directional change problem can be avoided/miminized by the use of a posteriori anti-windup compensators, or a priori ones, see the description in [15].

2.2. Optimization problem formulation of directional change

A task of taking constraints into account, which are imposed on the calculated control vector, is connected with solving the optimization problem by the direction-preserving (DP) algorithm. Let the constraints be given by affine functions in the form

$$\boldsymbol{H}\underline{u}_t \leq \underline{\alpha}. \tag{1}$$

One needs to seek the vector $\underline{u}_t$ which is as close as possible (in the sense of some norm) to $\underline{v}_t$, what in turn having introduced a weighting matrix $\boldsymbol{Q}_{\mathrm{DP}} > 0$ can be stated as

$$\begin{aligned} \min_{\underline{u}_t} \quad & (\underline{u}_t - \underline{v}_t)^T \boldsymbol{Q}_{\mathrm{DP}}^{-1} (\underline{u}_t - \underline{v}_t) \\ \text{s.t.} \quad & \boldsymbol{H}\underline{u}_t \leq \underline{\alpha}, \\ & \underline{u}_t \text{ of the same direction as } \underline{v}_t . \end{aligned} \tag{2}$$

As an example, let us consider the case with $m = 2$ control inputs and constraints in the form $-\alpha \leq u_{1,t} \leq \alpha$, $-\alpha \leq u_{2,t} \leq \alpha$. On the basis of (1) one gets:

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \underline{\alpha} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{bmatrix}.$$

The problem (2) can be transformed to finding a scalar multiplier γ which enters as a product into $\underline{v}_t$ to satisfy the requirements (1), thus:

$$\begin{aligned} \min_{\gamma} \quad & (\underline{u}_t - \underline{v}_t)^T \boldsymbol{Q}_{\mathrm{DP}}^{-1} (\underline{u}_t - \underline{v}_t) \\ \text{s.t.} \quad & \boldsymbol{H}\underline{u}_t \leq \underline{\alpha}, \\ & \underline{u}_t = \gamma \underline{v}_t , \end{aligned} \tag{3}$$

which simplifies to the form

$$\begin{aligned} \min_{\gamma} \quad & (\gamma - 1)^2 \underline{v}_t^T \boldsymbol{Q}_{\mathrm{DP}}^{-1} \underline{v}_t \\ \text{s.t.} \quad & \gamma \boldsymbol{H}\underline{v}_t \leq \underline{\alpha}. \end{aligned} \tag{4}$$

The problem (4) is convex (minimisation of a convex function over a convex set). As can be seen, when no constraints are active, one gets $\gamma^* = 1$ as the optimal solution. Otherwise $0 \leq \gamma^* < 1$ (the calculated control vector gets contracted).

Usually the matrix $\boldsymbol{H}$ is not of full rank, and the problem cannot be solved analytically via a Lagrange dual problem, as per its connection with the inversion of a matrix $\boldsymbol{X}^T \boldsymbol{H}^T \boldsymbol{Y} \boldsymbol{H} \boldsymbol{X}$ (where the introduced matrices are of appropriate dimensions), and the other solution methods must apply, such as KKT condition-based.

First, it can be assumed that the constraints from (4) can be presented in a compact form as a set of n_{const} inequalities

$$\gamma \boldsymbol{H}_{\mathrm{const}} \underline{v}_t \leq \underline{b},$$

where n_{const} is the smallest number of constraints describing the feasible set and:

$$\boldsymbol{H}_{\mathrm{const}} \underline{v}_t = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n_{\mathrm{const}}} \end{bmatrix}, \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_{\mathrm{const}}} \end{bmatrix}.$$

It is important to verify what value of γ makes all constraints inactive only in the case whenever $b_i \leq h_i$ ($1 \leq i \leq n_{\mathrm{const}}$), where

$$\gamma = \min \left(\min_{\substack{b_i < h_i \\ 1 \leq i \leq n_{\mathrm{const}}}} \frac{b_i}{h_i}, \ 1 \right)$$

corresponds to the multiplier γ, which should be used to multiply the calculated control vector $\underline{v}_t$, to avoid directional change in controls (satisfying $b_i < h_i$ guaranteed the inactive and equality conditions are eliminated). In summary one can formulate the complete DP algorithms in which the inner min function defines the minimal ratio (provided it exists), and the outer function – the lesser of two: $\frac{b_i}{h_i}$ and 1.

2.3. DP algorithm

1) for given $\boldsymbol{H}$, $\underline{v}_t$ and defined constraints transform (4) into

$$\begin{aligned} \min_{\gamma} \quad & (\gamma - 1)^2 \underline{v}_t^T \boldsymbol{Q}_{\mathrm{DP}}^{-1} \underline{v}_t \\ \text{s.t.} \quad & \gamma \boldsymbol{H}_{\mathrm{const}} \underline{v}_t \leq \underline{b}; \end{aligned} \tag{5}$$

2) evaluate the optimal multiplier of $\underline{v}_t$, as

$$\gamma^* = \min\left(\min_{\substack{b_i<h_i\\1\le i\le n_{\text{const}}}} \frac{b_i}{h_i},\ 1\right); \tag{6}$$

3) calculate the constrained control vector

$$\underline{u}_t = \gamma^* \underline{v}_t \tag{7}$$

and use it to impose constraints on the control vector.

The proposed DP algorithm can be used to any general set of constraints defined by affine functions. In the other cases one has to approximate the feasible set by a set defined by inequalities with affine functions.

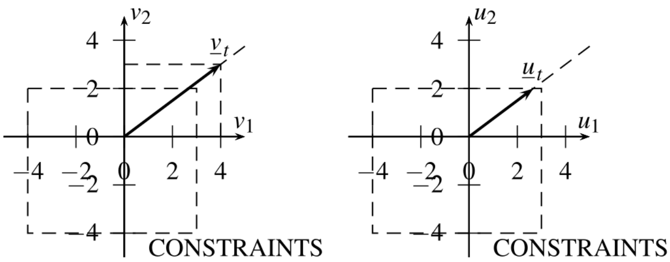
As an example, let $\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\underline{\alpha} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 4 \end{bmatrix}$, $\underline{v}_t = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ be given. As can be seen, the first two constraints are active, and:

$$\boldsymbol{H}_{\text{const}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{H}_{\text{const}}\underline{v}_t = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

with

$$\min_{\substack{b_i<h_i\\1\le i\le n_{\text{const}}}} \frac{b_i}{h_i} = \min\left(\frac{3}{4}, \frac{2}{3}\right) = \frac{2}{3},$$

resulting in $\gamma^* = \min\left(\frac{2}{3}, 1\right) = \frac{2}{3}$ and $\underline{u}_t = [\frac{8}{3}, 2]^T$ satisfies the constraints and is of the same direction as $\underline{v}_t$ (see Fig. 2).



(a) (b)

Fig. 2. Control vector: (a) calculated, (b) DP-constrained

3. DISCRETE-TIME MODEL OF THE CONTROL SYSTEM

The plant is modeled by a set of difference equations

$$\underline{x}_{\mathscr{P},t+1} = \boldsymbol{A}_{\mathscr{P}}\underline{x}_{\mathscr{P},t} + \boldsymbol{B}_{\mathscr{P}}\underline{u}_t, \tag{8}$$

$$\underline{y}_t = \boldsymbol{C}_{\mathscr{P}}\underline{x}_{\mathscr{P},t} + \boldsymbol{D}_{\mathscr{P}}\underline{u}_t, \tag{9}$$

with the matrices $\boldsymbol{A}_{\mathscr{P}} \in \mathscr{R}^{n\times n}$, $\boldsymbol{B}_{\mathscr{P}} \in \mathscr{R}^{n\times m}$, $\boldsymbol{C}_{\mathscr{P}} \in \mathscr{R}^{p\times n}$, $\boldsymbol{D}_{\mathscr{P}} \in \mathscr{R}^{p\times m}$, whereas the controller is described as

$$\underline{x}_{c,t+1} = \boldsymbol{A}_c\underline{x}_{c,t} + \boldsymbol{B}_c\underline{e}_t + \underline{\xi}_t, \tag{10}$$

$$\underline{v}_t = \boldsymbol{C}_c\underline{x}_{c,t} + \boldsymbol{D}_c\underline{e}_t, \tag{11}$$

with $\boldsymbol{A}_c \in \mathscr{R}^{n_c\times n_c}$, $\boldsymbol{B}_c \in \mathscr{R}^{n_c\times p}$, $\boldsymbol{C}_c \in \mathscr{R}^{m\times n_c}$, $\boldsymbol{D}_c \in \mathscr{R}^{m\times p}$. The static anti-windup compensation action is carried out by calculation of the $\underline{\xi}$ to alter interior states of the controller (n_c states). The general block diagram is the same as in [1] and has been omitted here for the sake of brevity.

The state-space vector of the model of the plant has n components, $\underline{x}_{\mathscr{P}} \in \mathscr{R}^n$, whereas the output vector (to track the reference vector $\underline{r} \in \mathscr{R}^p$) has p elements as well, $\underline{y} \in \mathscr{R}^p$, and finally control vectors have m components, namely $\underline{u}, \underline{v} \in \mathscr{R}^m$. As has already been stated the full-state anti-windup compensator modifies n_c states of the controller, i.e. $\underline{\xi} \in \mathscr{R}^{n_c}$.

special version of a cut-off saturation is considered, related to the DP algorithm, which for the amplitude-constrained can be presented as a result of the operation $\gamma\underline{v}_t$, where $\gamma \le 1$ corresponds to the saturation level of all the components of the calculated control vector.

As in [1], the compensator modifies controller states via $\underline{\xi}$,

$$\underline{\xi} = \boldsymbol{\Lambda}\underline{\eta} = \boldsymbol{\Lambda}(\underline{u} - \underline{v}). \tag{12}$$

Taking (8)–(11) into account, and introducing q operator, the state-space description in a linear-fractional form is given by [16]

$$\begin{aligned} q\underline{x} &= \begin{bmatrix} q\underline{x}_{\mathscr{P}} \\ q\underline{x}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\mathscr{P}}\underline{x}_{\mathscr{P}} + \boldsymbol{B}_{\mathscr{P}}\underline{u} \\ \boldsymbol{A}_c\underline{x}_c + \boldsymbol{B}_c(\underline{w} - \underline{y}) + \underline{\xi} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}_{\mathscr{P}}\underline{x}_{\mathscr{P}} + \boldsymbol{B}_{\mathscr{P}}\underline{u} \\ \boldsymbol{A}_c\underline{x}_c + \boldsymbol{B}_c\underline{w} - \boldsymbol{B}_c\left(\boldsymbol{C}_{\mathscr{P}}\underline{x}_{\mathscr{P}} + \boldsymbol{D}_{\mathscr{P}}\underline{u}\right) + \underline{\xi} \end{bmatrix} \\ &= \mathscr{A}\,\underline{x} + \mathscr{B}_u\underline{u} + \mathscr{B}_w\underline{w} + \mathscr{B}_\xi\underline{\xi}, \end{aligned} \tag{13}$$

where $\underline{w} = \underline{r}$, $\underline{z} = \underline{e}$ and:

$$\mathscr{A} = \begin{bmatrix} \boldsymbol{A}_{\mathscr{P}} & \boldsymbol{0}^{n\times n_c} \\ -\boldsymbol{B}_c\boldsymbol{C}_{\mathscr{P}} & \boldsymbol{A}_c \end{bmatrix}, \tag{14}$$

$$\mathscr{B}_u = \begin{bmatrix} \boldsymbol{B}_{\mathscr{P}} \\ -\boldsymbol{B}_c\boldsymbol{D}_{\mathscr{P}} \end{bmatrix}, \tag{15}$$

$$\mathscr{B}_w = \begin{bmatrix} \boldsymbol{0}^{n\times p} \\ \boldsymbol{B}_c \end{bmatrix}, \tag{16}$$

$$\mathscr{B}_\xi = \begin{bmatrix} \boldsymbol{0}^{n\times n_c} \\ \boldsymbol{I}^{n_c\times n_c} \end{bmatrix}. \tag{17}$$

Following [12, 17], like in [1] one can write

$$\begin{aligned} \underline{v} &= \boldsymbol{C}_c\underline{x}_c + \boldsymbol{D}_c\underline{e} = \boldsymbol{C}_c\underline{x}_c + \boldsymbol{D}_c(\underline{w} - \underline{y}) \\ &= \mathscr{C}_v\underline{x} + \mathscr{D}_{vu}\underline{u} + \mathscr{D}_{vw}\underline{w} + \mathscr{D}_{v\xi}\underline{\xi}, \end{aligned} \tag{18}$$

where:

$$\mathscr{C}_v = [-\boldsymbol{D}_c\boldsymbol{C}_{\mathscr{P}}, \boldsymbol{C}_c], \qquad \mathscr{D}_{vu} = -\boldsymbol{D}_c\boldsymbol{D}_{\mathscr{P}}, \tag{19}$$

$$\mathscr{D}_{vw} = \boldsymbol{D}_c, \qquad \mathscr{D}_{v\xi} = \boldsymbol{0}^{m\times n_c}. \tag{20}$$

The output vector of a linear-fractional transformation form, from Fig. 3, becomes

$$\begin{aligned} \underline{z} &= \underline{e} = \underline{w} - \underline{y} = \underline{w} - \boldsymbol{C}_{\mathscr{P}} \underline{x}_{\mathscr{P}} - \boldsymbol{D}_{\mathscr{P}} \underline{u} \\ &= \mathscr{C}_z \underline{x} + \mathscr{D}_{zu} \underline{u} + \mathscr{D}_{zw} \underline{w} + \mathscr{D}_{z\xi} \underline{\xi}, \end{aligned} \tag{21}$$

with:

$$\mathscr{C}_z = \left[-\boldsymbol{C}_{\mathscr{P}}, \boldsymbol{0}^{p \times n_c} \right], \qquad \mathscr{D}_{zu} = -\boldsymbol{D}_{\mathscr{P}}, \tag{22}$$

$$\mathscr{D}_{zw} = \boldsymbol{I}^{p \times p}, \qquad \mathscr{D}_{z\xi} = \boldsymbol{0}^{p \times n_c}. \tag{23}$$

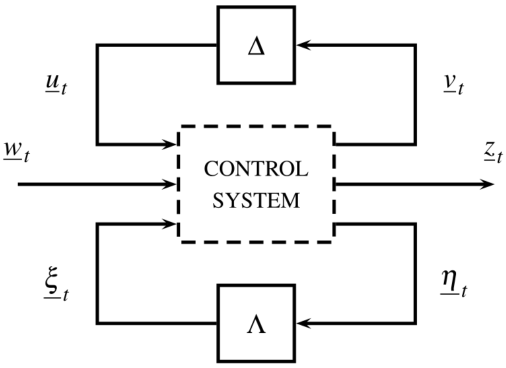


Fig. 3. Linear-fractional transformation-related description of the discrete-time model of the control system

As per $\eta = \underline{u} - \underline{v}$, and according to [1], the $\underline{\xi}$ can be removed from the linear-fractional form by introduction of

$$\underline{\xi} = -\boldsymbol{X} \mathscr{C}_v \underline{x} + \boldsymbol{X} \left(\boldsymbol{I} - \mathscr{D}_{vu} \right) \underline{u} - \boldsymbol{X} \mathscr{D}_{vw} \underline{w}, \tag{24}$$

$$\boldsymbol{X} = \left(\boldsymbol{I}^{n_c \times n_c} + \boldsymbol{\Lambda} \mathscr{D}_{v\xi} \right)^{-1} \boldsymbol{\Lambda}. \tag{25}$$

Shortly, by substitution of (24) to (13), (18) and (21), one obtains:

$$D\underline{x} = \boldsymbol{A} \underline{x} + \boldsymbol{B}_u \underline{u} + \boldsymbol{B}_w \underline{w}, \tag{26}$$

$$\underline{v} = \boldsymbol{C}_v \underline{x} + \boldsymbol{D}_{vu} \underline{u} + \boldsymbol{D}_{vw} \underline{w}, \tag{27}$$

$$\underline{z} = \boldsymbol{C}_z \underline{x} + \boldsymbol{D}_{zu} \underline{u} + \boldsymbol{D}_{zw} \underline{w},, \tag{28}$$

where:

$$\boldsymbol{A} = \mathscr{A} - \mathscr{B}_\xi \boldsymbol{X} \mathscr{C}_v, \tag{29}$$

$$\boldsymbol{B}_u = \mathscr{B}_u + \mathscr{B}_\xi \boldsymbol{X} \left(\boldsymbol{I} - \mathscr{D}_{vu} \right), \tag{30}$$

$$\boldsymbol{B}_w = \mathscr{B}_w - \mathscr{B}_\xi \boldsymbol{X} \mathscr{D}_{vw}, \tag{31}$$

$$\boldsymbol{C}_v = \mathscr{C}_v - \mathscr{D}_{v\xi} \boldsymbol{X} \mathscr{C}_v, \tag{32}$$

$$\boldsymbol{D}_{vu} = \mathscr{D}_{vu} + \mathscr{D}_{v\xi} \boldsymbol{X} \left(\boldsymbol{I} - \mathscr{D}_{vu} \right), \tag{33}$$

$$\boldsymbol{D}_{vw} = \mathscr{D}_{vw} - \mathscr{D}_{v\xi} \boldsymbol{X} \mathscr{D}_{vw}, \tag{34}$$

$$\boldsymbol{C}_z = \mathscr{C}_z - \mathscr{D}_{z\xi} \boldsymbol{X} \mathscr{C}_v, \tag{35}$$

$$\boldsymbol{D}_{zu} = \mathscr{D}_{zu} + \mathscr{D}_{z\xi} \boldsymbol{X} \left(\boldsymbol{I} - \mathscr{D}_{vu} \right), \tag{36}$$

$$\boldsymbol{D}_{zw} = \mathscr{D}_{zw} - \mathscr{D}_{z\xi} \boldsymbol{X} \mathscr{D}_{vw}. \tag{37}$$

When the constraints are imposed with respect to the DP algorithm, thus $\underline{u} = \gamma \underline{v}$ holds, then on the basis of (26), (27) and (28) a new description coherent with the linear-fractional transformation form – LFT – for is obtained (LTF does not change for a system with constraints):

$$\begin{aligned} D\underline{x} &= \boldsymbol{A} \underline{x} + \gamma \boldsymbol{B}_u \underline{v} + \boldsymbol{B}_w \underline{w} \\ &= \left(\boldsymbol{A} + \gamma \boldsymbol{B}_u \boldsymbol{C}_v \right) \underline{x} + \gamma \boldsymbol{B}_u \boldsymbol{D}_{vu} \underline{u} \\ &\quad + \left(\gamma \boldsymbol{B}_u \boldsymbol{D}_{vw} + \boldsymbol{B}_w \right) \underline{w} \\ &= \boldsymbol{A}^\gamma \underline{x} + \boldsymbol{B}_u^\gamma \underline{u} + \boldsymbol{B}_w^\gamma \underline{w}, \end{aligned} \tag{38}$$

$$\begin{aligned} \underline{z} &= \boldsymbol{C}_z \underline{x} + \gamma \boldsymbol{D}_{zu} \underline{v} + \boldsymbol{D}_{zw} \underline{w} \\ &= \left(\boldsymbol{C}_z + \gamma \boldsymbol{D}_{zu} \boldsymbol{C}_v \right) \underline{x} + \gamma \boldsymbol{D}_{zu} \boldsymbol{D}_{vu} \underline{u} \\ &\quad + \left(\gamma \boldsymbol{D}_{zu} \boldsymbol{D}_{vw} + \boldsymbol{D}_{zw} \right) \underline{w} \\ &= \boldsymbol{C}_z^\gamma \underline{x} + \boldsymbol{D}_{zu}^\gamma \underline{u} + \boldsymbol{D}_{zw}^\gamma \underline{w}, \end{aligned} \tag{39}$$

where:

$$\begin{aligned} \boldsymbol{A}^\gamma &= \boldsymbol{A} + \gamma \boldsymbol{B}_u \boldsymbol{C}_v, & \boldsymbol{B}_u^\gamma &= \gamma \boldsymbol{B}_u \boldsymbol{D}_{vu}, \\ \boldsymbol{B}_w^\gamma &= \gamma \boldsymbol{B}_u \boldsymbol{D}_{vw} + \boldsymbol{B}_w, & \boldsymbol{C}_z^\gamma &= \boldsymbol{C}_z + \gamma \boldsymbol{D}_{zu} \boldsymbol{C}_v, \\ \boldsymbol{D}_{zu}^\gamma &= \gamma \boldsymbol{D}_{zu} \boldsymbol{D}_{vu}, & \boldsymbol{D}_{zw}^\gamma &= \gamma \boldsymbol{D}_{zu} \boldsymbol{D}_{vw} + \boldsymbol{D}_{zw}. \end{aligned} \tag{40}$$

4. ON IMPOSING PERFORMANCE REQUIREMENTS WITH RESPECT TO THE CLOSED-LOOP SYSTEM

The basic performance requirement imposed on the system is the stability property, which is below, and following the derivations in [1], is cast to a matrix inequality-related form. Secondly, and in order to avoid repetitions, an induced norm helps to formulate the other constraint, eventually leading to a BMI-constrained optimization task. On the basis of the representation of a system with a DP algorithm, see (38), (39), the basic performance requirement is the mean-square stability condition, related to the existence of a Lyapunov function

$$V(\underline{x}_{t+1}) - V(\underline{x}_t) = \underline{x}_{t+1}^T \boldsymbol{P} \underline{x}_{t+1} - \underline{x}_t^T \boldsymbol{P} \underline{x}_t < 0, \tag{41}$$

from where

$$\begin{bmatrix} \boldsymbol{A}^{\gamma T} \boldsymbol{P} \boldsymbol{A}^\gamma - \boldsymbol{P} & \star & \star \\ \boldsymbol{B}_u^{\gamma T} \boldsymbol{P} \boldsymbol{A}^\gamma & \boldsymbol{B}_u^{\gamma T} \boldsymbol{P} \boldsymbol{B}_u^\gamma & \star \\ \boldsymbol{B}_w^{\gamma T} \boldsymbol{P} \boldsymbol{A}^\gamma & \boldsymbol{B}_w^{\gamma T} \boldsymbol{P} \boldsymbol{B}_u^\gamma & \boldsymbol{B}_w^{\gamma T} \boldsymbol{P} \boldsymbol{B}_w^\gamma \end{bmatrix} < 0, \tag{42}$$

or

$$\begin{bmatrix} -\boldsymbol{P} & \star & \star & \star \\ \boldsymbol{0} & \boldsymbol{0} & \star & \star \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \star \\ \boldsymbol{P} \boldsymbol{A}^\gamma & \boldsymbol{P} \boldsymbol{B}_u^\gamma & \boldsymbol{P} \boldsymbol{B}_w^\gamma & -\boldsymbol{P} \end{bmatrix} < 0. \tag{43}$$

The nomenclature used in this Section is briefly characterized in Table 1.

Table 1
Nomenclature

Symbol	Role
$\boldsymbol{P}$	positive-definite matrix, forming Lyapunov function
$\boldsymbol{\Gamma}$	$\boldsymbol{\Gamma} = [\boldsymbol{\Gamma}_1, \ldots, \boldsymbol{\Gamma}_m]$ used to incorporate $u_i^2 \le v_i^2$ $(i = 1, \ldots, m)$ conditions using the S-procedure
δ	supremum estimate of the norm-induced gain mentioned below
γ	contraction ratio of the calculated control vector
$*$	symmetrical entry of a matrix

Using the S-procedure, and on the basis of (42) one obtains

$$\begin{bmatrix} \boldsymbol{A}^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} - \boldsymbol{P} + \boldsymbol{C}_v{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \star & \star \\ \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} + \boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{B}_u^{\gamma} + \boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} - \boldsymbol{\Gamma} & \star \\ \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{A}^{\gamma} + \boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_u^{\gamma} + \boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_w^{\gamma} + \boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vw} \end{bmatrix} \le 0 \quad (44)$$

that can be transformed into

$$\begin{bmatrix} -\boldsymbol{P} & \star & \star & \star & \star \\ \boldsymbol{0} & -\boldsymbol{\Gamma} & \star & \star & \star \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \star & \star \\ \boldsymbol{P}\boldsymbol{A}^{\gamma} & \boldsymbol{P}\boldsymbol{B}_u^{\gamma} & \boldsymbol{P}\boldsymbol{B}_w^{\gamma} & -\boldsymbol{P} & \star \\ \boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{\Gamma}\boldsymbol{D}_{vw} & \boldsymbol{0} & -\boldsymbol{\Gamma} \end{bmatrix} \le 0\,. \qquad (45)$$

The next step, following a similar derivation for a cut-off constrained system, with the L_2-induced norm between $\underline{z}$ and $\underline{w}$ used, one gets

$$V(\underline{x}_{t+1}) - V(\underline{x}_t) + \underline{z}^T\underline{z} - \delta\underline{w}^T\underline{w} \le 0, \qquad (46)$$

where in the case of a discrete-time system one can write:

$$\begin{bmatrix} \boldsymbol{A}^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} - \boldsymbol{P} + \boldsymbol{C}_z^{\gamma T}\boldsymbol{C}_z^{\gamma} & \star & \star \\ \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} + \boldsymbol{D}_{zu}^{\gamma}{}^T\boldsymbol{C}_z^{\gamma} & \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{B}_u^{\gamma} + \boldsymbol{D}_{zu}^{\gamma}{}^T\boldsymbol{D}_{zu}^{\gamma} & \star \\ \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{A}^{\gamma} + \boldsymbol{D}_{zw}^{\gamma}{}^T\boldsymbol{C}_z^{\gamma} & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_u^{\gamma} + \boldsymbol{D}_{zw}^{\gamma}{}^T\boldsymbol{D}_{zu}^{\gamma} & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_w^{\gamma} + \boldsymbol{D}_{zw}^{\gamma}{}^T\boldsymbol{D}_{zw}^{\gamma} - \delta \end{bmatrix} \le 0. \quad (47)$$

By using the S-procedure again:

$$\begin{bmatrix} \boldsymbol{A}^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} - \boldsymbol{P} + \boldsymbol{C}_z^{\gamma T}\boldsymbol{C}_z^{\gamma} + \boldsymbol{C}_v{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \star & \star \\ \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} + \boldsymbol{D}_{zu}^{\gamma}{}^T\boldsymbol{C}_z^{\gamma} + \boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{B}_u^{\gamma} + \boldsymbol{D}_{zu}^{\gamma}{}^T\boldsymbol{D}_{zu}^{\gamma} + \boldsymbol{D}_{vu}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} - \boldsymbol{\Gamma} & \star \\ \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{A}^{\gamma} + \boldsymbol{D}_{zw}^{\gamma}{}^T\boldsymbol{C}_z^{\gamma} + \boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_u^{\gamma} + \boldsymbol{D}_{zw}^{\gamma}{}^T\boldsymbol{D}_{zu}^{\gamma} + \boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_w^{\gamma} + \boldsymbol{D}_{zw}^{\gamma}{}^T\boldsymbol{D}_{zw}^{\gamma} - \delta + \boldsymbol{D}_{vw}{}^T\boldsymbol{\Gamma}\boldsymbol{D}_{vw} \end{bmatrix} \le 0, \qquad (48)$$

which is synonymous with

$$\begin{bmatrix} \boldsymbol{A}^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} - \boldsymbol{P} & \star & \star & \star & \star \\ \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{A}^{\gamma} & \boldsymbol{B}_u^{\gamma T}\boldsymbol{P}\boldsymbol{B}_u^{\gamma} - \boldsymbol{\Gamma} & \star & \star & \star \\ \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{A}^{\gamma} & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_u^{\gamma} & \boldsymbol{B}_w^{\gamma}{}^T\boldsymbol{P}\boldsymbol{B}_w^{\gamma} - \delta & \star & \star \\ \boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{\Gamma}\boldsymbol{D}_{vw} & -\boldsymbol{\Gamma} & \star \\ \boldsymbol{C}_z^{\gamma} & \boldsymbol{D}_{zu}^{\gamma} & \boldsymbol{D}_{zw}^{\gamma} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} \le 0 \quad (49)$$

or

$$\begin{bmatrix} -\boldsymbol{P} & \star & \star & \star & \star & \\ \boldsymbol{0} & -\boldsymbol{\Gamma} & \star & \star & \star & \star \\ \boldsymbol{0} & \boldsymbol{0} & -\delta & \star & \star & \star \\ \boldsymbol{P}\boldsymbol{A}^{\gamma} & \boldsymbol{P}\boldsymbol{B}_u^{\gamma} & \boldsymbol{P}\boldsymbol{B}_w^{\gamma} & -\boldsymbol{P} & \star & \star \\ \boldsymbol{\Gamma}\boldsymbol{C}_v & \boldsymbol{\Gamma}\boldsymbol{D}_{vu} & \boldsymbol{\Gamma}\boldsymbol{D}_{vw} & \boldsymbol{0} & -\boldsymbol{\Gamma} & \star \\ \boldsymbol{C}_z^{\gamma} & \boldsymbol{D}_{zu}^{\gamma} & \boldsymbol{D}_{zw}^{\gamma} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} \le 0. \qquad (50)$$

In order to derive the final form of the inequality conditions one needs to define:

$$\begin{aligned} \boldsymbol{P}\boldsymbol{A}^{\gamma} &= \boldsymbol{P}(\boldsymbol{A} + \gamma\boldsymbol{B}_u\boldsymbol{C}_v) \\ &= \boldsymbol{P}\big(\mathscr{A} - \mathscr{B}_\xi\boldsymbol{X}\mathscr{C}_v + \gamma\mathscr{B}_u\mathscr{C}_v - \gamma\mathscr{B}_u\mathscr{D}_{v\xi}\boldsymbol{X}\mathscr{C}_v \\ &\quad + \gamma\mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\mathscr{C}_v \\ &\quad - \gamma\mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\mathscr{D}_{v\xi}\boldsymbol{X}\mathscr{C}_v\big), \qquad (51) \\ \boldsymbol{P}\boldsymbol{B}_u^{\gamma} &= \boldsymbol{P}(\gamma\boldsymbol{B}_u\boldsymbol{D}_{vu}) \\ &= \boldsymbol{P}\big(\gamma\mathscr{B}_u\mathscr{D}_{vu} + \gamma\mathscr{B}_u\mathscr{D}_{v\xi}\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu}) \\ &\quad + \gamma\mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\mathscr{D}_{vu} \\ &\quad + \gamma\mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\mathscr{D}_{v\xi}\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\big), \qquad (52) \\ \boldsymbol{P}\boldsymbol{B}_w^{\gamma} &= \boldsymbol{P}(\gamma\boldsymbol{B}_u\boldsymbol{D}_{vw} + \boldsymbol{B}_w) \\ &= \boldsymbol{P}\big(\mathscr{B}_w - \mathscr{B}_\xi\boldsymbol{X}\mathscr{D}_{vw} \\ &\quad + \gamma\mathscr{B}_u\mathscr{D}_{vw} - \gamma\mathscr{B}_u\mathscr{D}_{v\xi}\boldsymbol{X}\mathscr{D}_{vw} \\ &\quad + \gamma\mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\mathscr{D}_{vw} \\ &\quad - \gamma\mathscr{B}_\xi\boldsymbol{X}(\boldsymbol{I} - \mathscr{D}_{vu})\mathscr{D}_{v\xi}\boldsymbol{X}\mathscr{D}_{vw}\big), \qquad (53) \end{aligned}$$

and $\boldsymbol{\Gamma}\boldsymbol{C}_v$, $\boldsymbol{C}_z^{\gamma}$, $\boldsymbol{\Gamma}\boldsymbol{D}_{vu}$, $\boldsymbol{D}_{zu}^{\gamma}$, $\boldsymbol{\Gamma}\boldsymbol{D}_{vw}$, $\boldsymbol{D}_{zw}^{\gamma}$ defined as in [1].

The optimization task enabling one to find the optimal compensator feedback gains is

$$\begin{array}{lll} \min\limits_{\gamma,\boldsymbol{P},\boldsymbol{X},\boldsymbol{\Gamma},\delta} & \delta & \\ \text{s.t.} & (50) & \qquad (54) \\ & \gamma > 0, \boldsymbol{P} > 0, \boldsymbol{\Gamma} > 0, \delta > 0 & \end{array}$$

and it is solved as a sequence of optimization tasks for a fixed value of γ related to how severe the DP constraints are and $\boldsymbol{P}$, $\boldsymbol{X}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$ as decision variables in

$$\begin{array}{lll} \min\limits_{\boldsymbol{P},\boldsymbol{X},\boldsymbol{\Gamma},\delta} & \delta & \\ \text{s.t.} & (50) & \qquad (55) \\ & \boldsymbol{P} > 0, \boldsymbol{\Gamma} > 0, \delta > 0 & \end{array}$$

which has BMI constraints only. The optimal solution corresponds to the minimal value of δ achieved. It can be easily presented as a series of solutions, δ vs. γ to understand the interplay. Strict inequalities in the optimization problems result mainly from the fact the interior-point methods are used to obtain solutions, and with reference to properties of selected matrices.

5. SIMULATION RESULTS

The simulation for the two-input two-output plant has been conducted, with:

$$\boldsymbol{A}_{\mathscr{P}} = \begin{bmatrix} 0.9048 & 0 \\ 0 & 0.9048 \end{bmatrix}, \qquad (56)$$

$$\boldsymbol{B}_{\mathscr{P}} = \begin{bmatrix} 9.516 & 0 \\ 0 & 9.516 \end{bmatrix}, \qquad (57)$$

$$\boldsymbol{C}_{\mathscr{P}} = \begin{bmatrix} 0.4 & -0.5 \\ -0.3 & 0.4 \end{bmatrix}, \qquad (58)$$

$$\boldsymbol{D}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (59)$$

and the controller with $n_c = 1$ given by:

$$\boldsymbol{A}_c = 1\,, \qquad (60)$$

$$\boldsymbol{B}_c = [7.0710, 7.0710]\,, \qquad (61)$$

$$\boldsymbol{C}_c = [0.0318, 0.0247]^T\,, \qquad (62)$$

$$\boldsymbol{D}_c = \begin{bmatrix} 2.0 & 2.5 \\ 1.5 & 2.0 \end{bmatrix}. \qquad (63)$$

It is assumed that for $t = 0$ in accordance with [1] a step change in reference vector takes place between $\underline{w} = \underline{0}$ and $\underline{w} = [0.63, 0.79]^T$, and the control vector is saturated at the level ± 1, see Fig. 4 to observe tracking properties.

6. SUMMARY

The paper presented an optimization-based compensator design method to ensure superior performance of a control system. The proposed method can form a lookup table of various compensators for different values of γ to swiftly change the controller feedback gains for various constraint violation conditions in dynamic states. That would open the door to another interesting research on the topic, based on switched-system formulation, or linear-piecewise description, and will be topic of further research.

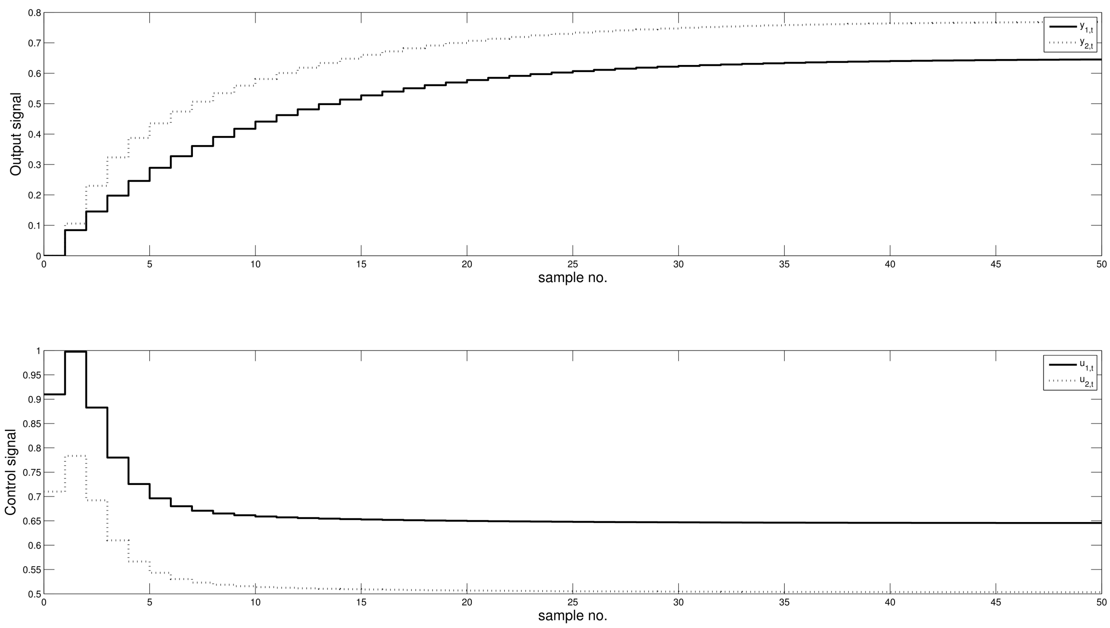


Fig. 4. Tracking performance and constrained control vector

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