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Existence results and stabilization of homogeneous conformable fractional order systems

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In this paper, we give results on the stability of homogeneous conformable fractional order systems using assumptions on a family of compact sets and provide the stabilization of an affine control system and given an explicit homogeneous feedback control with the requirement that a control Lyapunov function exists and satisfying a homogeneous condition and we present existence and uniqueness theorems for sequential linear conformable fractional differential equations.

Key words: conformable fractional order systems, homogeneous fractional systems, stabilization, fractional problems solutions, Lyapunov function, sequential linear fractional differential equations

1. Introduction

The fractional calculus seems to be originally introduced in 1695 in a letter written by Leibniz to L'Hospital where he suggested to generalize his celebrated formula of the kth derivative of a product (where $k \in \mathbb{N}^*$ is a positive integer) to any positive real $k \succ 0$. In another letter to Bernoulli, Leibniz mentioned derivatives of general order. Since then, numerous renowned mathematicians introduced several notions of fractional operators. We can cite the works of Euler (1730's), Fourier (1820's), Liouville (1830's), Riemann (1840's), Sonin (1860's), Grunwald (1860's), Letnikov (1860's), Caputo (1960's), etc. [4]. All these notions are not disconnected. In most cases it can be proved that two different notions actually coincide or are correlated by an explicit formula.

If you write Fractional Derivative (FD) on google search only gives you around 24900 results (0.29 seconds). The majority of these articles concerning

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nonlinear systems with non-integer derivatives remain open problems in many fields such as physics, engineering, medicine, robotics, biology, mechanics, electronics, etc. [4, 12] due to the fractional nature and non-linearity of these systems. Thus, stabilization is one of the major concern of both researchers and engineers.

The methods of stabilizing linear systems are numerous as well as the existence and uniqueness of the solution.

However, in practice, a significant number of industrial or agricultural processes are inherently distributed in space and in the earth so that their behaviors depend on spatial position as well as time or the time water distrubition.These systems are usually described by a set of nonlinear fractional differential equations (FDEs) with homogeneous or mixed boundary conditions. Over the past decades, the fractional differential theory of FDE systems has been developed by Khalil et al. [18] and introduced a new fractional derivative called the conformable fractional derivative. This new concept is very interesting. Later, this theory was developed by T.Abdeljawad in [2], who gave definitions of left and right conformable derivatives of higher order, exponential functions, Gronwall's inequality, Laplace transform for conformable fractional calculus, etc.

Existing works on the nonlinear design of FDE systems can be classified into two types:

– The first: the stability analysis of nonlinear systems and many problems have been studied in this regard, where some fundamental results have been obtained. The stability of nonlinear systems has received increased attention due to its important role in the fields of science and engineering. Many of monographs and articles are devoted to fractional nonlinear systems. Lyapunov's method of functions (or direct Lyapunov's method) is known to be extended to many classes of equations of perturbed motion, including distributed parameter systems and sets of equations in metric spaces [21, 23, 26]. In the classical derivatives, homogeneous systems have attracted attention in recent years as a means of studying the stability or stabilizability of general nonlinear systems, which has been introduced by Rothschild and Stein [25]. Homogeneous systems offer many desirable properties. Due to homogeneity, asymptotic stability of the origin implies global asymptotic stability as well as the existence of a C^1 Lyapunov function which is also homogeneous [24]. Many approaches in homogeneous system design rely on this theory [11, 16, 24].

In [19] the authors proposed a homogenous feedback control design by Lyapunov functions In [9] the authors prove the existence of Lyapunov homogeneous function for homogeneous fractional systems. In addition, they prove that local and global behaviors are the same. and they study the uniform Mittag-Leffler stability of homogeneous fractional time-varying systems. It is therefore interesting to study the homogeneity in the conformable fractional derivative system. In this

paper, we study the stability and the stabilization of homogeneous conformable fractional order systems and provide the stabilization of an multi-linear control system via an explicit homogeneous feedback control with the requirement that a Lyapunov function exists and satisfying a homogeneous condition.
– The second: the existence and uniqueness of the solution to the FDE systems in the interval $[t_0-T, t_0+T]$ with values in $\overline{B(y_0, r_0)}$ [22]. There are fewer researches on the existence and uniqueness of the solution used fixed points theorem's of Banach or of Schauder. Accordingly, studying the existence and uniqueness of the solution to the FDE system becomes important. In this paper, we will study the existence and uniqueness of the solution to nonlinear fractional order system with approximate solution via Cauchy-Arzela-Peano and Ascoli theorem's.

Briefly speaking, the main contribution of this study includes three aspects.

1. Conformable fractional derivative and its properties.
2. Existence and uniqueness of solution Fractional Differential System conformable.
3. Stabilization Fractional Differential System conformable.

The rest of this paper is arranged as follows. Section 2 gives some necessary definitions and properties of the conformable fractional calculus which are used in this paper and can be found in [2, 17]. Existence and uniqueness of solution are presented in Section 3. Stabilization of Fractional Differential System conformable is described in Section 4. Section 5 presents numerical example for affine function, and Section VI gives a brief conclusion.

2. Preliminary

Prior to presenting the main results, we recall definitions and theorems which will be used intensively in our study.

2.1. Conformable fractional derivative and its properties

Definition 1. *Let f a function defined on $[t_0, \infty)$, the conformable fractional derivative of f starting from t_0 of order α is defined by*

$$T_\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t - t_0)^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{1}$$

for all $t > t_0$, $\alpha \in (0, 1]$. If $T_\alpha f(t)$ exists $\forall t \in (t_0, b)$ for some $b > t_0$ and $\lim_{t \to t_0^+} T_\alpha f(t)$ exists, then by definition

$$T_\alpha f(t_0) = \lim_{t \to t_0^+} T_\alpha f(t).$$

If the above fractional derivative exists, we say that f is α-differentiable.

Theorem 1. *[3] If f is differentiable, the conformable derivative of order $\alpha \in (0,1]$, denoted by T_α of f α-differentiable at a point $t > t_0$ is defined by:*

$$T_\alpha f(t) = (t-t_0)^{1-\alpha}\frac{d}{dt}f(t). \tag{2}$$

Proof. By the fact that f differentiable we assume $h = \epsilon(t-t_0)^{1-\alpha}$

$$\begin{aligned} T_\alpha f(t) &:= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-t_0)^{1-\alpha}) - f(t)}{\varepsilon} \\ &= (t-t_0)^{1-\alpha}\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = (t-t_0)^{1-\alpha}\frac{d}{dt}f(t). \end{aligned}$$

Corollary 1. *[6] Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t \geqslant t_0$. Then*

1) $T_\alpha(af+bg)(t) = aT_\alpha(f)(t) + bT_\alpha(g)(t)$;

2) $T_\alpha(fg)(t) = T_\alpha(f)(t)g(t) + f(t)T_\alpha(g)(t)$;

3) $T_\alpha\left(\frac{f}{g}\right)(t) = \frac{T_\alpha(f)(t)g(t) - f(t)T_\alpha(g)(t)}{g^2}$ *if $g(t) \neq 0$ for all $t \geqslant t_0$.*

Proof. The proof can be easily done using the same principle as the ordinary derivative. □

Example Let $\alpha \in (0,1]$ and $t \geqslant t_0$

1) $T_\alpha(f)(t) = 0$ for all constant functions $f(t) = c (c \in R)$;

2) $T_\alpha((t-t_0)^p)(t) = p(t-t_0)^{p-\alpha}$ for all $p \in R$;

3) $T_\alpha(e^{\frac{1}{\alpha}(t-t_0)^\alpha})(t) = e^{\frac{1}{\alpha}(t-t_0)^\alpha}$;

4) $T_\alpha(\sin(\frac{1}{\alpha}(t-t_0)^\alpha)(t) = \cos(\frac{1}{\alpha}(t-t_0)^\alpha)$;

5) $T_\alpha(\cos(\frac{1}{\alpha}(t-t_0)^\alpha)(t) = -\sin(\frac{1}{\alpha}(t-t_0)^\alpha)$.

Proof.

1) and 2) follows from Theorem 1,

3) $T_\alpha(e^{\frac{1}{\alpha}(t-t_0)^\alpha}) = (t-t_0)^{1-\alpha}\frac{d}{dt}e^{\frac{1}{\alpha}(t-t_0)^\alpha} = (t-t_0)^{1-\alpha}(t-t_0)^{\alpha-1}e^{\frac{1}{\alpha}(t-t_0)^\alpha}$
$= e^{\frac{1}{\alpha}(t-t_0)^\alpha}$,

4) and 5) are the same that as 3).

Definition 2. *The conformable fractional integral starting from t_0 of a function f of order $0 < \alpha \leqslant 1$ is defined by*

$$I_\alpha(f)(t) = \int_{t_0}^{t}(s-t_0)^{\alpha-1}f(s)\,\mathrm{d}s. \tag{3}$$

If the above integral exists we say that f is α-integrable.

The relationship that exists between functions that are α-differentiable and α-integrable is summarized in the following result.

Lemma 1. *([18]). If f is a continuous function on (t_0, ∞), then $T_\alpha (I_\alpha(f))(t) = f(t)$ for all $t \in (t_0, \infty)$.*

The fundamental theorem of conformable fractional calculus is stated in the following way.

Lemma 2. *([2]) If f is a continuous function on (t_0, ∞), then $I_\alpha (T_\alpha(f))(t) = f(t) - f(t_0)$.*

A geometric property of functions that are α-differentiable discussed in [2,5,18] is stated in the following result.

Lemma 3. *If T_α exists over (t_0, ∞) and $T_\alpha(f)(t) \geqslant 0$ (respectively, $T_\alpha(f)(t) \leqslant 0$) for all $t \in (t_0, \infty)$, then the graph of the function f is increasing (respectively, decreasing) on (t_0, ∞).*

The following result establishes the validity of the chain rule for functions that are α-differentiable.

Lemma 4. *[2]. If $f, g : (t_0, \infty) \to \mathbb{R}$ are α-differentiable functions, where $\alpha \in (0, 1]$, then the function $r(t) = f(g(t))$ is α-differentiable and for all $t \neq t_0$ and if $g(t) \neq 0$ we have:*

$$T_\alpha r(t) = T_\alpha f(g(t)).T_\alpha g(t).g(t)^{\alpha-1}. \tag{4}$$

Remark 1. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ are α-differentiable and $g^{1-\alpha} \in \mathbb{R}^{n\times n}$ is a diagonal matrix as $g^{1-\alpha} = diag(g_1^{1-\alpha}, ..., g_n^{1-\alpha})$. Let $r(t) = f(g(t))$. Then $r(t)$ is α-differentiable for all $t \neq t_0$ and if $\det[g^{1-\alpha}] \neq 0$ we have:*

$$T_\alpha r(t) = \nabla_\alpha f(g).g^{1-\alpha}.T_\alpha g(t), \tag{5}$$

and, if $t = t_0$ we have

$$T_\alpha r(t) = \lim_{t\to t_0^+} \nabla_\alpha f(g).g^{1-\alpha}.T_\alpha g(t), \tag{6}$$

where $\nabla_\alpha f(g)$ denoted the α-Graduim of f with respect to g

$$\nabla_\alpha f(g) = \frac{\partial^\alpha f(g)}{\partial g^\alpha} = \frac{\partial f(g)}{\partial g}.g^{\alpha-1}. \tag{7}$$

Consider now the following conformable fractional derivative nonlinear system:

$$T_\alpha x(t) = f(x(t)),\ x(t_0) = x_0 \tag{8}$$

where $\alpha \in (0,1]$, $x(t) \in \mathbb{R}^n$ is the state vector; $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and $f(0) = 0$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t) = x(t, t_0, x_0)$ of system (8) for each initial condition (t_0, x_0).

2.2. Homogeneity

Definition 3. *For any $r = (r_1, ..., r_n) \in \mathbb{R}^n$ with $r_i > 0,\ i \in \{1, ..., n\}$, and $\lambda > 0$, the dilation vector of $x = (x_1, ..., x_n) \in \mathbb{R}^n$ associated with weight r is defined as*

$$\Delta_\lambda(x) = (\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n).$$

The homogeneous norm of $x \in \mathbb{R}^n$ associated with weight r is defined as

$$\|x\|_r = (\sum_{i=1}^{n} |x|^{\frac{\varrho}{r_i}})^{\frac{1}{\varrho}},\quad \varrho = \prod_{i=1}^{n} r_i.$$

An important property is that

$$\|\Delta_\lambda(x)\|_r = \lambda \|x\|_r.$$

The homogeneous norm is not a standard norm, because the triangle inequality is not satisfied. However, there exists $\overline{\sigma} > 0$ and $\underline{\sigma} > 0$ such that

$$\underline{\sigma}\|x\|_r \leqslant \|x\| \leqslant \overline{\sigma}\|x\|_r.$$

Definition 4.

- **i** – *A continuous function $h : R^n \longrightarrow \mathbb{R}$ is r-homogeneous of degree k if $h(\Delta_\lambda(x)) = \lambda^k h(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.*
- **ii** – *We say that a continuous function $f : \mathbb{R}^n \longrightarrow R^n$ is r-homogeneous of degree k if each f_i, $i \in \{1, ..., n\}$, is r-homogeneous of degree $k + r_i$. i.e. $f(\Delta_\lambda(x)) = \lambda^k \Delta_\lambda(f(x))$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.*
- **iii** – *The system (8) is r-homogeneous of degree k if the vector field f is r-homogeneous of degree k.*

Lemma 5. *[10] If $x(t)$ is a solution of the r-homogeneous system (8) with the degree k for an initial condition $x_0 \in \mathbb{R}^n$, then $y(t) = \Delta_\lambda(x(\lambda^{\frac{k}{\alpha}} t))$ for $\lambda > 0$, and $t \geqslant t_0$ is also a solution of (8) with the initial condition $y_0 = \Delta_\lambda(x_0)$.*

Remark 2.

i – ***The derivative of the dilation vector is given by:***

$$T_\alpha\left(\Delta_\lambda(x)\right) = \left(\lambda^{r_1} T_\alpha x_1, \ldots, \lambda^{r_n} T_\alpha x_n\right)^{\mathrm{T}} = \Delta_\lambda(T_\alpha x).$$

ii – ***If*** *$V : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous of degree k and α-differentiable function, then $T_\alpha V$ is r-homogeneous of degree k. Indeed, we have*

$$V\left(\Delta_\lambda(x)\right) = V\left(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n\right) = \lambda^k V(x), \quad \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n.$$

Hence, it is clear that

$$T_\alpha V(\Delta_\lambda(x)) = T_\alpha V(\Delta_\lambda(x)) = \lambda^k T_\alpha V(x), \quad \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n.$$

Remark 3. *Assume that $x(t) = x(t, t_0, x_0)$ is the unique solution of system (8) for the initial condition (t_0, x_0), then, for $s \in \mathbb{R}$ and $r \in \mathbb{R}_+$ such that $s = t + r$ we have $x(s) = x(s, t_0, x_0) = x(t + r, t_0, x_0) = x(r, t, x(t))$.*

2.3. Existence and uniqueness ([22] but modified)

Consider the system

$$\begin{cases} T_\alpha x(t) = f(t, x(t)), \quad x(t) \in \mathbb{R}^n \\ x(t_0) = x_0, \\ f(t, 0) = 0; \quad t \in (t_0, +\infty) \end{cases} \qquad \alpha \in (0, 1]. \tag{9}$$

2.3.1. Construction of a solution using Euler's method

Let $t_0 < t_1 < \ldots < t_N = t_0 + T$ be the regular subdivision. Denote $h = \dfrac{T}{N+1}$ and construct y_n by induction of y_0 as follows

$$y_{n+1} \simeq y_n + h^\alpha f(t_n, y_n). \tag{10}$$

By connecting the points with the straight lines we note y the approximate solution obtained. We similarly construct the approximate solution on $[t_0 - T, t_0]$ by taking step $h < 0$.

Lemma 6. *Assume $C = [t_0 - T, t_0 + T] \times \overline{B(y_0, r_0)}$ a security cylinder. Let also*

$$T \leqslant \min\left\{T_0, \left(\frac{r_0}{2M}\right)^{\frac{1}{\alpha}} (N+1)\right\}. \tag{11}$$

Any approximate solution y for the system (9) given by Euler's method is contained in the ball $\overline{B(y_0, r_0)}$.

Proof. Let us show by recurrence that for all n $y_n \in \overline{B(y_0, r_0)}$:
for $n = 1$ $y_1 \simeq y_0 + h^\alpha f(t_0, y_0), ||y_1 - y_0|| = ||h^\alpha f(t_0, y_0)|| \leqslant h^\alpha M = \left(\frac{T}{N+1}\right)^\alpha M \leqslant r_0$, and for all $t \in [t_0, t_1]$. So $y(t) \in \overline{B(y_0, r_0)}$.
$\|y(t) - y_0\| = \|(t - t_0)^\alpha f(t_0, y_0)\| \leqslant h^\alpha M = (\frac{T}{N+1})^\alpha M \leqslant \frac{r_0}{2}$, $y([t_0, t_1]) \subset \overline{B(y_0, r_0)}$.
Assume that: $y_{n-1} \in \overline{B(y_0, r_0)}$ and $y([t_0, t_n]) \subset \overline{B(y_0, r_0)}$, $(t_n, y_n) \in C$, for all $t \in [t_0, t_{n+1}]$, $\|y(t) - y_0\| \leqslant \|y(t) - y_n\| + \|y_n - y_0\| \leqslant M|t - t_n|^\alpha + M|t_n - t_0|^\alpha \leqslant 2T^\alpha M \leqslant r_0$
then $y([t_0, t_{n+1}]) \subset \overline{B(y_0, r_0)}$. □

2.3.2. Approximate solutions

Definition 5. *Let $\psi : [a, b] \longrightarrow \mathbb{R}^n$ is a piecewise α-differentiable function (this means that there exists a subdivision $a = a_0 < a_1 < .. < a_N = b$ of $[a, b]$ such that for all n the restriction $y|_{[a_n, a_{n+1}]}$ is α-differentiable; we therefore only assume continuity and the existence of a right and left derivative of ψ at the points a_n).*
We say that ψ is a $\epsilon - approximate$ solution of (9) if:

- **i** $(\forall t \in [a, b])$, $(t, y(t)) \in C$
- **ii** $(\forall n, \forall t \in]a_n, a_{n+1}[)$ $||T_\alpha(f)(t) - f(t, y(t))|| \leqslant \epsilon$. *In other words, ψ is a ϵ-approximate solution if ψ satisfies (9) with $error \leqslant \epsilon$.*

For increase the error, we introduce the continuity module $\omega_f(\delta) = \max\{\|f(x) - f(y)\|\}$, then $\|y_1 - y_2\| + |t_1 - t_2|^\alpha \leqslant \delta\}$. C is compact then f is uniformly continuous on C and $\omega_f(\delta) \longrightarrow 0$.

Lemma 7. *If y is approximate solution of Euler method for N points then the error satisfies $\epsilon \leqslant \omega_f((M + 1)h^\alpha)$.*

Proof. To increase $\|T_\alpha(y_p)(t) - f(t, y(t))\|$ for all $t \in [t_0, t_0 + T]$ and y $\epsilon - approximate$ solution associated at the subdivision $t_0 < t_1 < ... < t_N = t_0 + T$ for all $t \in]t_n, t_{n+1}[$ then $\|T_\alpha(y_p)(t) - f(t, y(t))\| = \|f(t_n, y_n)(t) - f(t, y(t))\| \leqslant \omega_f((M + 1)h^\alpha)$. □

2.3.3. Convergence of approximate solutions

Let us consider the following system

$$\begin{cases} T_\alpha y(t) = f(t, y(t)), \quad y(t) \in \mathbb{R}^n, \\ y(t_0) = y_0 \,. \end{cases} \tag{12}$$

Lemma 8. *If y_p solution $\epsilon_p - approximate$ obtained which uniformly converges to y then that exactly converge of (12) problem.*

Proof. Let $\|T_\alpha(y_p)(t) - f(t, y(t))\| \leqslant \frac{1}{p}$ then

$$\|I_\alpha(T_\alpha(y_p)(t) - f(t, y(t)))\| = \|I_\alpha(T_\alpha(y_p)(t)) - I_\alpha(f(t, y(t)))\|$$
$$= \|y_p(t) - y_0 - I_\alpha(f(t, y(t))\| \leqslant \frac{1}{p\alpha}|t - t_0|^\alpha \to 0, p \to \infty$$

and if $\delta_p = sup\|y_p - y\|$ then $\|f(t, (y_p(t)) - f(t, y(t))\| \leqslant \omega_f(\delta_p) \to 0$.
Thus by uniform convergence we can pass to the limit
$\lim_{\varepsilon\to 0}(y_p) = y(t) = y_0 + I_\alpha(s, y(s))(t)$
so y is continuous by uniform convergence and solution for system (12). □

Theorem 2. ***(Ascoli)*** *[22] We assume that E and F are compact metric spaces. Let $\phi_{(p)} : E \longrightarrow F$ be a sequence of k-Lipschitzian maps, where $k \geqslant 0$ is a given constant. Then we can extract from $\phi_{(p_n)}$ a uniformly convergent subsequence $\phi_{(p_n)}$, and the limit is a k-Lipchitzian map.*

2.4. Stability

Definition 6. *The maximal trajectory of the differential system (8) passing through (t_0, x_0) is denoted $\Psi(t, t_0, x_0)$.*
The origin of the system (8) is said to be:

- **i** – *Stable, if for $\varepsilon > 0$, there exists $\delta > 0$ such that the origin of the system (8) satisfies $||\Psi(t, t_0, x_0)|| < \varepsilon$ for all $t > t_0$ when $||x_0|| < \delta$.*
- **ii** – *Uniformly stable if i) holds with $\delta = \delta(\varepsilon)$ independent of t_0.*
- **iii** – *Globally attractive if $\lim_{t\to+\infty} \Psi(t, t_0, x_0) = 0$ for all x_0 and for all $t_0 > 0$.*
- **iv** – *Asymptotically stable if it is stable and for $t_0 \geqslant 0$ there exists a positive constant $\delta = \delta(t_0)$ such that for all x_0, if $\|x_0\| \prec \delta$ then $\lim_{t\to+\infty} \Psi(t, t_0, x_0) = 0$.*
- **v** – *Uniformly asymptotically stable if it is uniformly stable and in addition there exists $\delta \succ 0$ such that for all $\varepsilon \succ 0$ there exists $T = T(\varepsilon) \succ 0$, satisfying:*
  for all t_0, x_0, if $\|x_0\| \succ \delta$, then $\Psi(t, t_0, x_0) \succ \varepsilon$, for all $t \succ T + t_0$.
- **vi** – *Globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon\to+\infty} \delta(\varepsilon) = +\infty$, and, for each η and c two positive reals, there exists $T = T(\eta, c) \succ 0$, such that for all t_0, x_0, if $\|x_0\| \prec \eta$, then $\|\Psi(t, t_0, x_0)\| \prec c$ for all $t \succ T + t_0$.*

Definition 7. *A continuous function $V : \mathcal{V} \to \mathbb{R}_+$ is called a Lyapunov function for the system (8), where $\mathcal{V}$ is a non empty neighborhood of the origin of $\mathbb{R}^n$, if there exist $\varrho_i \in \mathcal{K}_\infty$, $(i = 1, 2)$ such that:*

i – $\varrho_1(\|x\|) \leqslant V(x) \leqslant \varrho_2(\|x\|)$,

ii – *V is α-differentiable for all* $0 < \alpha \leqslant 1$. $\forall\ t > t_0$,

iii – $T_\alpha V(x) \leqslant 0$.

Theorem 3. *([26]) Let $x = 0$ be an equilibrium point for system (8). Assume that the (8) has a Lyapunov function V. Then the origin of system (8) is asymptotically stable.*

Remark 4. *If $V : \mathcal{V} \to \mathbb{R}^n$ is α-differentiable, the conformable fractional derivative of order α of V along the solutions of the system (8) is defined by*
$T_\alpha V(x) = \langle \nabla V(x), f(x) \rangle$.
If V is r-homogeneous of degree k_1 and f is r-homogeneous of degree k_2 then $\langle \nabla V, f \rangle$ is r-homogeneous of degree $k_1 + k_2$.

3. Existence and uniqueness of solution

The theorem of Cauchy-Arzela-Peano for conformable fractional systems finds more applications in the field of differential equations, because it gives a result of existence and uniqueness of solution for differential equations. Here we will show a version for conformable fractional nonlinear differential equations.

Lemma 9. *Let $y(t)$ be function with continuous derivative in the interval $[t_0 - T, t_0 + T]$ with values in $\overline{B}(y_0, r_0)$, then $y(t)$ is solution of the system (12) if and only if it satisfies the integral*
$y(t) = y_0 + \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds = I_\alpha(f(., y))(t) + y_0$.

Proof. Necessary condition: if y is solution of the problem (12) then
$T_\alpha y(t) = f(t, y(t))$; we apply Lemma 2 $y(t) - y_0 = I_\alpha(T_\alpha(y))(t) = I_\alpha(f(., y))(t)$ so
$y(t) = I_\alpha(f(., y))(t) + y_0$.
Sufficient condition: with Lemma 1

$$T_\alpha(y(t)) = T_\alpha(I_\alpha(f(., y))(t) + y_0) = T_\alpha(I_\alpha(f(., y))(t)) = f(t, y(t)).$$

3.1. Theorem Cauchy-Arzela-Peano conformable

Theorem 4. *Let $f(t, y)$ is continuous function defined in a security cylinder $C = [t_0 - T, t_0 + T] \times \overline{B}(y_0, r_0)$. Let also $T \leqslant min\{T_0, (\frac{r_0}{2M})^{\frac{1}{\alpha}}(N+1)\}$. Then there exists in a security cylinder a unique and continuous solution $y(t)$ of the problem (12).*

Proof.

i. Let us start showing that f satisfies the Cauchy problem on I equivalent to the integral problem y continuous on I and for all $t \in I, (t, y(t)) \in U$ and $\forall t \in I, y(t) = y_0 + I_\alpha(s, y(s))(t)$. Furthermore there exist a security cylinder: f is defined on an open U, so there exists a cylinder $C_0 = [t_0 - T_0, t_0 + T_0] \times \overline{B}(y_0, r_0) \subset U$. We note that $M = \sup f$ and $T \leqslant \min\{T_0, (\frac{r_0}{2M})^{\frac{1}{\alpha}}(N+1)\}$ and C this cylinder, see Lemma 6.

ii. Approximate solution y for the system (12) given by Euler's method is contained in the ball $\overline{B}(y_0, r_0)$, see Lemma 7.

iii. y solution of Euler method approximates solution for N points so the error satisfies $\epsilon \leqslant \omega_f((M+1)h^\alpha)$, see Lemma 8.

Conclusion

Let $(y_p)_p$ be a sequence of approximate solutions constructed by Euler's method verifying $h \leqslant \frac{T}{p}$ then $\epsilon_p \leqslant \omega_f((M+1)\frac{T^\alpha}{p\alpha})$. y_p are Lipchitzian by construction of ratio $\frac{M}{\alpha}$. By Ascoli, we can extract a sequence which converges uniformly towards y which becomes a solution. □

4. Stabilization Fractional Differential System conformable

4.1. Stability by compacts subsets

Hypothesis 4.1

Let f α-differentiable function. Assume that there exists a family of compact sets D_λ with the following properties:

i – $\bigcap_{\lambda \in \mathbb{R}^+} D_\lambda = \{0\}$.

ii – For all $0 < \lambda_1 < \lambda_2$, $D_{\lambda_1} \subset \overset{\circ}{D}_{\lambda_2}$ where $\overset{\circ}{D}_{\lambda_2}$ is the interior of D_{λ_2}.

iii – For all $x \in \mathbb{R}^n \backslash \{0\}$, there exists $\lambda > 0$ such that $x \in \partial D_\lambda$, where $\partial D_\lambda = \overline{D}_\lambda \backslash \overset{\circ}{D}_\lambda$ and $\overline{D}_\lambda$ is the closure set of $\overline{D}_\lambda$.

iv – For all $\lambda > 0$ and $x_0 \in \partial D_\lambda$, we have $x(t) \in \overset{\circ}{D}_\lambda$ for all $t \in [t_0, t_0 + T]$,where $x(t)$ is the solution of the system (9) starting at $x_0 \in \partial D_\lambda$ at the initial time t_0.

In [1], the author proves the stability of the closed-loop system using given assumptions on a family of compact sets. This result plays an essential role to verify the stabilization of homogeneous systems [7, 8, 13–15, 17]. For homogeneous time-varying systems, this result is proved in [15], Theorem 4. In the case of the conformable fractional derivative, we give the following result.

Theorem 5. *Consider the nonautonomous system (9) where f satisfies:*

i *– The conditions of Theorem 4.*

ii *– Assume that there exist compact subsets $\{D_\lambda\}_{\lambda\in\mathbb{R}^+}$ such that the conditions (i), (ii), (iii) and (iv) given above are satisfied.*

Then the origin of system $T_\alpha y(t) = f(t, y(t))$ is
a) globally uniformly stable,
b) globally attractive.

Proof.

Step 1. Uniformly stable

Substep 1. There exists $\lambda \succ 0$ such that $D_\lambda \subset B(0,\epsilon)$. Let $\epsilon \succ 0$, by (i) and (ii), there exists $\lambda = \lambda(\epsilon) \succ 0$ such that $D_\lambda \subset B(0,\epsilon)$.

In fact, if we suppose that for all $\lambda \succ 0$, we have $D_\lambda \cap \complement B(0,\epsilon) = \emptyset$, where $\complement B(0,\epsilon) = \mathbb{R}^n \backslash B(0,\epsilon)$, then for all n integer not null, there exists $x_n \in D_\lambda \cap \complement B(0,\epsilon)$ where $\lambda_n = \frac{1}{n}$. But by hypothesis 4.1 (ii), we have $D_{\lambda_n+1} \subset D^\circ_{\lambda_n}$ for all n integer not null. So $x_n \in D_{\lambda_1}$, for all $n > 1$. D_{λ_1} is a compact set, so the sequence $(x_n)_{n\geqslant n_0}$ has a limit point y in D_{λ_1} .

By the fact that for all n, m integers not null such that $n > m$, one has $D_{\lambda_n} \subset D_{\lambda_m}$, we get $x_n \in D_{\lambda_m}$ for all $n > m$. Then the limit point y is in D_{λ_m} for all $m \in \mathbb{N}^*$. So $y \in \cup_{m\in\mathbb{N}^*} D_{\lambda_m}$.

On the other hand, by the hypothesis 4.1 (i), we have $\cap_{\lambda\in\mathbb{R}^+} D_\lambda = \{0\}$, which implies that $\cap_{n\in\mathbb{N}^*} D_{\lambda_n} = 0$, then the limit point $y = 0$. But using the fact that for all $n \in \mathbb{N}^*$, $x_n \in D_{\lambda_n} \cap \complement B(0,\epsilon)$, we obtain $\|x_n\| \geqslant \epsilon$ which gives $\|y\| \geqslant \epsilon$ and this is impossible.

So we can conclude that there exists $\lambda > 0$ such that $D_\lambda \subset B(0,\epsilon)$.

Substep 2. Origin is uniformly stable

Now, by hypothesis 4.1 (i) and (ii), we have D_λ and $\overset{\circ}{D}_\lambda$ being non empty subsets. Since $0 \in \overset{\circ}{D}_\lambda$ and $\overset{\circ}{D}_\lambda$ is an open set, there exists $\delta = \delta(\epsilon) > 0$ such that $B(0,\delta) \subset \overset{\circ}{D}_\lambda$. Let $x \in B(0,\delta)$, by hypothesis 4.1 (iii) there exists $\lambda x > 0$ such that $x \in \partial D_{\lambda x}$. We can easily prove that $\lambda x < \lambda$ and $D_\lambda x \subset \overset{\circ}{D}_\lambda$.

By hypothesis 4.1 (iv), $x \in \partial D_{\lambda x} \subset \overset{\circ}{D}\lambda$ implies $\psi(t, t_0, x) \in \overset{\circ}{D}_\lambda$ for all t_0, for all $t > t_0$. Thus $\|\psi(t, t_0, x)\| \prec \epsilon$ for all x, satisfying $\|x\| \prec \delta$. So, we conclude that the origin is uniformly stable.

Step 2. Globally uniformly stable.

We prove that the origin is globally uniformly stable.

Substep 3. There exists $\lambda \succ 0$ such that $B(0, A) \subset \overset{\circ}{D}_\lambda$.

Now, we prove that $\lim_{\epsilon\to+\infty} \delta(\epsilon) = +\infty$.

Let $A \succ 0$, there exists $\lambda \succ 0$ such that $B(0, A) \subset D_\lambda$. In fact, we prove that there exists $\lambda \succ 0$ such that $\overline{B(0, A)} \subset \overset{\circ}{D}_\lambda$. Suppose that for all $\lambda \succ 0$, $\overline{B(0, A)} \cap \complement \overset{\circ}{D}_\lambda \neq \emptyset$. We get for all integers n, $\overline{B(0, A)} \cap \complement \overset{\circ}{D}_n \neq \emptyset$. Let $x_n \in \overline{B(0, A)} \cap \complement \overset{\circ}{D}_n$, for all integers n. Let m be a fixed integer, we have $\forall n \succ m$ $\overset{\circ}{D}_m \subset \overset{\circ}{D}_n \Longrightarrow \complement \overset{\circ}{D}_n \subset \complement \overset{\circ}{D}_m$. Now $x_n \in \overline{B(0, A)} \cap \complement \overset{\circ}{D}_n$, implies $x_n \in \complement \overset{\circ}{D}_m$ $\forall n \succ m$. However (x_n) is bounded, then it has a limit point which we denote by z, $z \in \complement. \overset{\circ}{D}_m$ $\forall m$, signifies that $z \in \cap_m \complement \overset{\circ}{D}_m = \complement(\cup_m \overset{\circ}{D}_m) = \emptyset$.

Substep 4. Globally uniformly stable

D_λ is a compact set, so there exists $B \succ 0$ such that $D_\lambda \subset B(0, B)$. For all $t_0 \succ 0$ and $x \in B(0, A)$, $t \succ t_0$ implies $\psi(t, t_0, x) \in D_\lambda \subset B(0, B)$.

So for all $\epsilon \geqslant B$ there exists $\delta(\epsilon) = A$ such that for all t_0, for all $x \in B(0, \delta(\epsilon))$ one has $\psi(t, t_0, x) \in B(0, \epsilon)$ for all $t \succ t_0$. Finally the origin is globally uniformly stable.

Step 3. Origin is globally attractive

Substep 5. Origin is attractive

In this step, we prove that the origin is globally attractive. Let $x \in \mathbb{R}^n$, we show that $\lim_{t\to+\infty} \psi(t, t_0, x) = 0$. By hypothesis 4.1 (iii), there exists $\lambda \succ 0$ such that $x \in \partial D_\lambda$ and for all t $\geqslant t_0$, there exists $\lambda_t \succ 0$ such that $\psi(t, t_0, x) \in \partial D_{\lambda_t}$. Moreover by hypothesis 4.1 (iv), we have for all $t \geqslant t_0, \psi(t, t_0, x) \in \overset{\circ}{D}_\lambda$.

Consider the map defined by $f(t) = \lambda_t$. By the fact that for all $s, t \in \mathbb{R}^+$, we have $\psi(s, t, \psi(t, t_0, x)) = \psi(s, t_0, x)$ and by hypothesis (ii), we can deduce that f is decreasing.

Then $\lim_{t\to+\infty} f(t)$ exists, which we denote by λ_0. This limit satisfies $\lambda_0 \prec f(t)$ for all t $\geqslant t_0$. Let us show that $\lambda_0 = 0$. Suppose that $\lambda_0 \neq 0$.

Let (t_n) be a sequence which tends towards $+\infty$ as n tends to $+\infty$, then $\lim_{t\to+\infty} f(t_n) = \lambda_0$. The sequence $(\psi(t_n, t_0, x))$ satisfies

$\psi(t_n, t_0, x) \in \overset{\circ}{D}_\lambda$ for all $n \geqslant n_0$, where n_0 is an integer satisfying $t_n \geqslant t_0$ for all $n \geqslant n_0$. The sequence $(\psi(t_n, t_0, x))$ is bounded, then there exists a convergent subsequence.

Let (s_n) a subsequence of (t_n) such that $\lim_{n\to+\infty} \psi(s_n, t_0, x) = y$.

We have that for all n, there exists λ_{s_n}, such that $\psi(s_n, t_0, x) \in \partial D_{\lambda_{s_n}}$ and for all $n \geqslant n_0, \lambda_0 \prec \lambda_{s_n}$. So $\psi(s_n, t_0, x) \notin \overset{\circ}{D}_\lambda$ for all $n \geqslant n_0$, and $\lim_{n\to+\infty} \psi(s_n, t_0, x) = y \in \complement\, \overset{\circ}{D}_{\lambda_0} = \mathbb{R}^n \setminus \overset{\circ}{D}_{\lambda_0}$. Thus $y \neq 0$, then there exists $\lambda_y \succ 0$, such that $y \in \partial D_{\lambda y}$. We can deduce that $\lambda_y \geqslant \lambda_0$. If $\lambda_y \neq \lambda_0$, then $\lambda_y \succ \lambda_0$.

We know that $\lim_{n\to+\infty} \lambda_{s_n} = \lambda_0$, then there exists $N \in \mathbb{N}$, such that for all $n \geqslant N, \lambda_0 \prec \lambda_{s_n} \prec \lambda_0 + \frac{\lambda_y - \lambda_0}{2}$. This implies $\lambda_{s_n} \prec \frac{\lambda_y + \lambda_0}{2}$ and $\lambda_{s_n} \subset \overset{\circ}{D}_{\frac{\lambda_y+\lambda_0}{2}}$ for all $n \geqslant N$. So $\psi(t_n, t_0, x) \in D_{\lambda_{s_n}} \subset \overset{\circ}{D}_{\frac{\lambda_y+\lambda_0}{2}} \subset D_{\frac{\lambda_y+\lambda_0}{2}}$. Then $\lim_{n\to+\infty} \psi(s_n, t_0, x) = y \in D_{\frac{\lambda_y+\lambda_0}{2}} \subset \overset{\circ}{D}_{\lambda y}$ and this is impossible. We deduce that $\lambda_0 = \lambda_y$. For all $s \succ 0, \psi(s, 0, y) \in \overset{\circ}{D}_{\lambda_0}$.
By the fact that the sequence (v_n) defined by $v_n = s_n + s$ tends towards $+\infty$ when n tends towards $+\infty$, it follows that $(\psi(v_n, t_0, x))$ is convergent and $\lim_{n\to+\infty} \psi(v_n, t_0, x) = \lim_{n\to+\infty} \psi(s, 0, \psi(s_n, t_0, x)) = \psi(s, 0, y) \in \partial D_{\lambda_0}$. But $\psi(s, t_0, y) \in \overset{\circ}{D}_{\lambda_0}$ and consequently, $y = 0$ and $\lambda_0 = 0$.

Substep 6. Origin is globally attractive
Let $\epsilon \succ 0$. In the step 1, we have proved that there exists $\mu \succ 0$ such that $D_\mu \subset B(0, \epsilon)$. By the fact that $\lim_{t\to+\infty} \lambda_t = 0$, there exists $T \succ 0$ such that for all $t \succ T$, one has $\lambda_t \prec \mu$ which implies $D_{\lambda_t} \subset \overset{\circ}{D}_\mu$. So for all $t \succ T$, $\|\psi(t, t_0, x)\| \prec \epsilon$. Finally, the origin is globally attractive.

4.2. Stabilization of homogeneous systems with multiple input

We are interested in the stabilization of homogeneous multi-linear systems. We provide sufficient conditions for the multi-linear system to be stabilizable by a homogeneous feedback of degree zero. The stabilizing feedback is explicitly given.

We are concerned with homogeneous multi-linear systems of the type

$$\begin{cases} T_\alpha x = Mx + \sum_{i=1}^{l} u_i N_i x \\ x \in \mathbb{R}^n, u_i \in \mathbb{R}, M, N_i \in M_{n\times n}(\mathbb{R}) \ for\ all\ 0 \leqslant i \leqslant l. \end{cases} \tag{13}$$

Definition 8. *It is said that the system (13) is stabilizable if there exists a family of control law* $(u_i(x))_{0\leqslant i\leqslant l}$ *such that for the closed-loop system:*

$$T_\alpha x = Mx + \sum_{i=1}^{l} u_i(x) N_i x \tag{14}$$

the origin is a globally asymptotically stable equilibrium point.

Definition 9. *A function* $u : \mathbb{R}^n \to \mathbb{R}$ *is said to be positively homogeneous of degree* k *if* $u(\lambda x) = \lambda^k u(x)$ *for all* $x \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^+$.

Theorem 6. *Assuming there exist* $b \in \mathbb{R}^n$, $F_i \in M_{1\times n}(\mathbb{R})$, *and* $P \in M_{n\times n}(\mathbb{R})$ *as a symmetric, positive definite matrix such that the following two properties are satisfied:*

(i) $N_i^T P + PN_i = 0$ *for all* $0 \leqslant i \leqslant l$,

(ii) $(M + N_i b F_i)^T P + P(M + N_i b F_i) < 0$ *for all* $0 \leqslant i \leqslant l$.

Then the system (4.2) is stabilizable by the positively homogeneous feedback $u(x) = (u_1(x), \ldots, u_l(x))$ *of degree* 0 *where:*

$$u_i(x) = \begin{cases} \dfrac{x^T(Pb\cdot F_i)x + F_i x\sqrt{\left(x^T Pb\right)^2 + \left(\beta - b^T Pb\right)\left(x^T Px\right)}}{\left(x^T Px\right)} & if\ x \neq 0, \\ 0 & otherwise, \end{cases}$$

where β *is a positive constant satisfying* $\beta \geqslant b^T Pb$.

Proof. Consider the quadratic form $V(x) = (x-b)^T P(x-b)$. The conformable derivative of V along the trajectories of the closed-loop system

$$T_\alpha x = Mx + \sum_{i=1}^{l} F_i x N_i x \tag{15}$$

is given by

$$\begin{aligned} T_\alpha V(x) &= \langle x, (M^T P + PM)x\rangle - 2\langle x, M^T Pb\rangle \\ &\quad - \sum_{i=1}^{l}(F_i x)\langle x, N_i^T Pb\rangle + \sum_{i=1}^{l}(F_i x)\langle x, P(N_i b)\rangle, \end{aligned}$$

where $\langle , \rangle$ denotes the usual inner product. By using the associativity of matrix multiplication and the facts that

$$Pb = b^T P \quad \text{and} \quad (N_i^T P)b = -b^T(N_i^T P),$$

it follows that

$$\begin{aligned} T_\alpha V(x) &= \left\langle x, (M^{\mathrm{T}}P + PM)x \right\rangle - 2\left\langle x, M^{\mathrm{T}}Pb \right\rangle \\ &\quad + \sum_{i=1}^{l} \left\langle x, (F_i^{\mathrm{T}} b^T N_i^{\mathrm{T}} P)x \right\rangle + \sum_{i=1}^{l} \langle x, P(N_i b)Fx \rangle \\ &= \sum_{i=1}^{l} \left\langle x, (M + NbF_i)^T P + P(M + N_i bF_i)x \right\rangle - 2\langle x, M^T Pb\rangle. \end{aligned}$$

By assumption (ii),

$$\langle x, (M + N_i bF_i)^T P + P(M + N_i bF_i)x\rangle < 0, \quad x \neq 0$$

therefore,

$$\exists \alpha > 0 \text{ such that } ||x|| \geqslant \alpha \implies T_\alpha V(x) < 0.$$

Now consider the domain $D = \{x \in \mathbb{R}^n; \quad V(x) \leqslant \beta\}$, where β is a strictly positive number. For sufficiently large β, 0 is inside D, and we have $T_\alpha V(x) < 0 \quad \forall x \in \partial D$. Thus, the trajectories of the closed-loop system (15) do not leave the bounded domain D. It remains to show that the solutions converge to 0. For this, introduce the positively homogeneous function k defined by

$$\kappa(x) = \frac{x^{\mathrm{T}}Pb + \sqrt{(x^T Pb)^2 + (\beta - b^{\mathrm{T}}Pb)(x^{\mathrm{T}}Px)}}{x^{\mathrm{T}}Px}$$

where β is a positive constant satisfying $\beta \geqslant b^T Pb$. By construction, the homogeneous feedback of degree 0 is given by

$$\begin{aligned} u_i(x) &= \kappa(x)F_i x \\ &= \begin{cases} \dfrac{-x^{\mathrm{T}}(Pb \cdot F_i)x + \sqrt{(x^{\mathrm{T}}Pb)^2 + (\beta - b^{\mathrm{T}}Pb)(x^{\mathrm{T}}Px)}F_i x}{(x^{\mathrm{T}}Px)} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This implies that the closed-loop system

$$T_\alpha x = X(x) = Mx + \sum_{i=1}^{l} u_i(x)N_i x \tag{16}$$

is homogeneous of degree 1. Note that the function κ satisfies

$$\kappa(x)x \in \partial D \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad k(x) = 1 \quad \forall x \in \partial D,$$

so systems (15) and (16) coincide on ∂D.

Now consider the domain D_λ, the image of D under the homotype centered at 0 with ratio $\lambda > 0$, defined by $D_\lambda = \{x \in \mathbb{R}^n; \quad V\left(\frac{x}{\lambda}\right) \leqslant \beta\}$. It is clear that $\{D_\lambda\}_{\lambda \in \mathbb{R}^+}$ is a family of compact sets satisfying the conditions (i), (ii), and (iii) of Theorem 5. It remains to show that for any $\lambda \in \mathbb{R}^+$ and for any $x \in \partial D\lambda$, the vector field $X(x)$ is inward pointing with respect to the domain D_λ. Since the vector field $X(x)$ is homogeneous of degree 1 and inward pointing with respect to the domain D, it suffices to prove that the tangent spaces to ∂D and ∂D_λ at x^0 and λx^0 respectively are parallel.

If we define $T_y(S)$ as the tangent space to the surface S at the point y and X_y as an element of $T_y(S)$ (tangent vector at y), then for any function $V\alpha$-differentiable, we have $X_{x^0}(V) = \sum_{i=1}^{n} \frac{\partial^\alpha V}{\partial x_i^\alpha}(x^0)(x_i - x_i^0)$ and

$$X_{\lambda x^0}(V) = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{\partial^\alpha V}{\partial x_i^\alpha}(x^0)(x_i - \lambda x_i^0).$$

Clearly, the tangent vectors X_{x^0} and $X_{\lambda x^0}$ are parallel. Consequently, the vector field $X(x)$ is inward pointing with respect to the bounded domain D_λ, $(\forall \lambda \in \mathbb{R}^+)$.

By applying the results of Theorem 5, it follows that the closed-loop system (13) has the origin as a globally asymptotically stable equilibrium point. □

5. Numerical example

Let us consider the following system

$$\begin{aligned} T_{0.98}x_1(t) &= -x_1(t) + x_2(t) + x_3(t) + ux_2(t) \\ T_{0.98}x_2(t) &= x_1(t) + 4x_2(t) + 2x_3(t) + u(-x_1(t) + x_3(t)). \\ T_{0.98}x_3(t) &= -x_1(t) - x_2(t) - 2x_3(t) - u(t)x_2(t) \end{aligned} \tag{17}$$

System (17) can be rewritten as the following form

$$T_{0.98}x(t) = Mx(t) + uNx(t)$$

with

$$M = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 4 & 2 \\ -1 & -1 & -2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let $P = \mathrm{Id}_{\mathbf{R}^3}$, $b = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^{\mathrm{T}}$ and $F = \begin{pmatrix} 2 & 5 & 1 \end{pmatrix}$.

5.1. Existence and uniqueness

We assume that $f(t, x(t)) = Mx(t) + uNx(t)$ which is continuous function on $\mathbb{R}$. We can easily verify f the assumptions of Theorems 12, if we choose $r_0 > 0$ and $t_0 > 0$ and $T_0 > 0$ and y_0. We deduce the problem (17) has a unique solution in $C = [t_0 - T, t_0 + T] \times \overline{B}(y_0, r_0)$.

5.2. Stabilization

We use Matlab to find the eigenvalues in the case the control $u = 0$.
The spectre of M, spec(M)={λ_1 =-1.3811187- 0.6962072i; $\lambda_2 = -1.3811187 -$ 0.6962072i; $\lambda_3 = 3.7622375 + 0.i;$}.

We note that the real part of $\lambda_3 \succ 0$. Then the linear system(17) is global instable.

Now let the control $u \neq 0$. The system (17) is stabilizable by the following feedback control

$$u(x) = -\frac{x_3 + (x_3^2 - 2(x_1^2 + x_2^2 + x_3^2))^{\frac{1}{2}}(2x_1 + 5x_2 + x_3)}{x_1^2 + x_2^2 + x_3^2}$$

$$T_\alpha V(x) = \left\langle x, \left((M + NbF)^{\mathrm{T}} P + P(M + NbF)\right) x \right\rangle = -2\left(x_1^2 + x_2^2 + 2x_3^2\right).$$

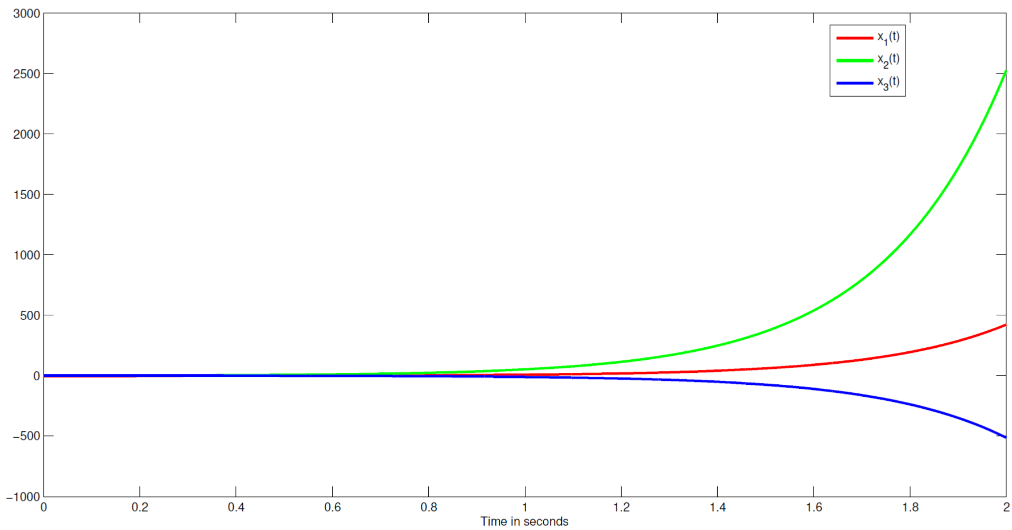

Figure 1: Evolution of the state $x_1(t), x_2(t)$ and $x_3(t)$ of Example 17, without feedback with initial conditions $x_1(0) = -2, \quad x_2(0) = 2$ and $x_3(t) = 1.2$.

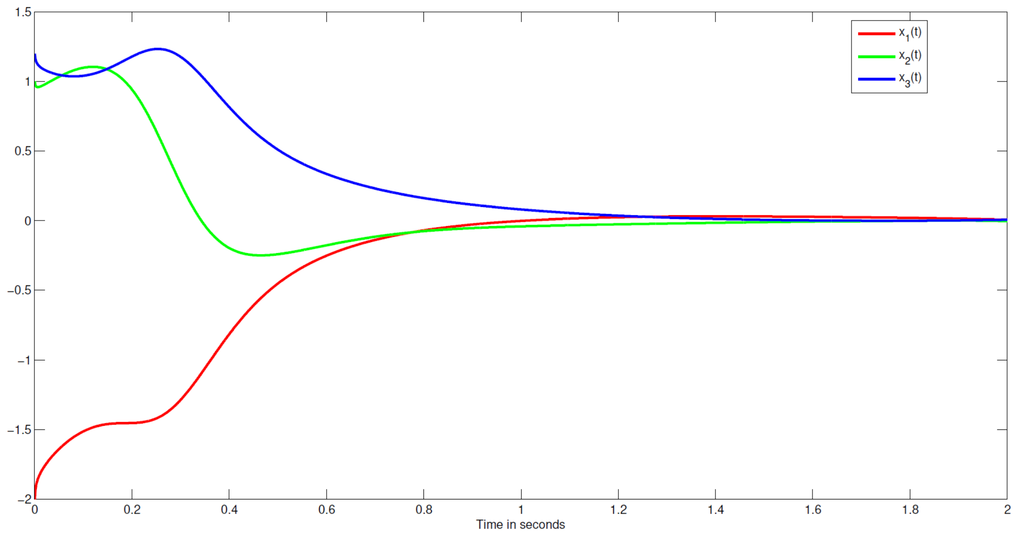

Figure 2: Evolution of the state $x_1(t)$, $x_2(t)$ and $x_3(t)$ of Example 17 with initial conditions $x_1(0) = -2, \ \ x_2(0) = 2$ and $x_3(t) = 1.2$.

The numerical solution to the system (17) is shown in Fig. 2 for some suitable value of fractional order $\alpha = 0.98$, it indicates that the zero solution is asymptotically stable.

6. Conclusion

In our work, we present existence and uniqueness theorems for sequential linear conformable fractional differential equations. It has been found that results obtained from this work is analogous to the results obtained from the ordinary case and the notion of stability and stabilization of homogeneous systems has been introduced in the classical derivative, and there are many interesting results for these systems. In this paper, we study the stability of homogeneous conformable fractional order systems and provide the stabilization of an affine control system via an explicit homogeneous feedback control with the requirement that a control Lyapunov function exists and satisfying a homogeneous condition and satisfies a homogeneous condition. and with the method of a set of compacts.

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