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Selected Reinsurance Models

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Abstract

In this paper, we investigate non-classical reinsurance models. Two kinds of such models are presented. One is based on dependent binomial distributions and the second on fuzzy numbers. First, we study dependent random variables representing claims using copulas. We investigate the number of claims the reinsurer covers and the total value of covered claims. We present the influence of the degree of dependence and different copulas on the number and the value of the claims covered by the reinsurer. Second, we analyze the case in which the main parameter of the model, the probability that the reinsurer covers the claim, is uncertain. We treat such a parameter as a fuzzy number in this case and combine randomness and fuzziness. We also study the case when the parameter of a copula which describes the degree of dependence is uncertain.

Keywords: reinsurance, copula, generalized binomial distribution, fuzzy numbers

JEL Classification: C10, G22

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1 Introduction

The paper is devoted to selected, non-classical, mathematical reinsurer models. We present two kinds of such models. The first is based on generalized binomial distributions (see, Heilpern 2020) and the second is based on fuzzy numbers. First, we investigate the dependent random variables representing claims. In classical actuarial and financial models, random variables are generally independent. This assumption is very convenient from a mathematical point of view. However, expecting that claims are independent is not actually easy. There occur common external factors: economic, political, and climatic, such as crises, catastrophic events, inflation, recessions, pandemics, or wars. These can affect the variables' dependency.
The dependent structure is characterized by copulas. The influence of the degree of dependence and different copulas on the number and value of claims covered by the reinsurer is presented using simple examples. For this purpose, we use the Clayton, Gumbel, and Spearman copulas. When claims are identically distributed and the copula is exchangeable, the number of claims covered by the reinsurer has a dependent binomial distribution (Heilpern 2020). For large dependencies, it differs significantly from the classical binomial distribution. These investigations enable a better analysis of reinsurance problems, for instance, estimating the total value of claims covered by the reinsurer.
Next, we study the case when the probability that the reinsurer covers the claim is an uncertain parameter. In this situation, we treat such a parameter as a fuzzy number (see Dubois and Prade 1980, Heilpern 1992). We obtain the combination of randomness and fuzziness in this case. In the classical approach, this parameter is estimated based on a random and representative sample. However, these conditions are not always fully met, and we have some doubts as to the accuracy of the value of the obtained parameter. For instance, we can consider the value of this parameter as an uncertain value, a fuzzy number "about p" in this case (see Heilpern 2018; Heilpern 2020, Dębicka et al. 2022).
The paper is structured as follows. Section 2 presents the basic assumptions regarding our models. In section 3 we assume that claims are identically distributed and that the copula describing the dependent structure of the claims is exchangeable. We also study the distribution of the number of claims covered by the reinsurer. The total value of claims covered by the reinsurer is investigated in section 4. Here we analyze a case with a random number of claims. Section 5 discusses the basic definitions and notions connected with fuzzy sets. In section 6, we investigate the number of claims covered by the reinsurer when the probability that the reinsurer covers the claim is imprecise and the claims are dependent. We also study the case of imprecision probabilities and when the parameter of the copula is imprecise.
We adopted and developed methods presented in Heilpern's 2020 paper. He used this model in issues related to credits.

2 Basic assumptions

We will investigate the following excess-of-loss reinsurer contract (see, Daykin 1994; Kolev, Paiva 2005). We consider a portfolio consisting of n claims $X_1, \ldots, X_n$, and retention d. First, we will study the number of claims covered by the reinsurer, i.e., the random variable:

$$K = \sum_{j=1}^{n} I_j$$

where for every $j = 1, \ldots, n, I_j$ is the Bernoulli random variable taking the value 1 if the reinsurer covers the claim, and the value 0 otherwise. In other words

$$I_j = \begin{cases} 0 & X_j \le d \\ 1 & X_j > d \end{cases}.$$

This is the status of the claims.
In classical actuarial models, there is an assumption regarding the independence of the occurring random variables. However, in practice, common external factors – economic, climatic, and political – often influence the investigated risks. These may include fires, floods, tornadoes, earthquakes, economic or political crises, inflation, or wars. In this situation, we assume that the random variables $X_1, \ldots, X_n$ describing the claims may be dependent. We will investigate the homogeneous claims only. Therefore, we assume that the random variables X_i have the same continuous distribution. The random variables I_j are equally distributed, too.
The statuses $I_1, \ldots, I_n$ are a finite sequence of Bernoulli random variables. They represent the results of trials. To denote the probability of "success" when the reinsurer covers the claim X_j in the j th trial I_j we use the symbol p, while for the probability of "defeat" we use $q = 1 - p$, i.e.

$$p = \Pr(I = 1).$$

The probability mass function (p.m.f.) (see Heilpern 2020)

$$f_{\mathbf{I}}(i_1, \ldots, i_n) = \Pr(I_1 = i_1, \ldots, I_n = i_n),$$

where $i_j \in \{0, 1\}$ and $\mathbf{I} = (I_1, \ldots, I_n)$ and the cumulative distribution function (c.d.f.)

$$F_{\mathbf{I}}(i_1, \ldots, i_n) = \Pr(I_1 \le i_1, \ldots, I_n \le i_n)$$

describe the joint distribution of the Bernoulli variables I_j. We will be interested in the point of jump of c.d.f. only, i.e., $i_j \in \{0, 1\}$. The marginal c.d.f. is equal to

$$F_{I_j}(i_j) = \Pr(I_j \le i_j) = v_j = I_j = \begin{cases} 1 & i_j = 1 \\ q & i_j > 0 \end{cases}. \tag{1}$$

The dependent structure of the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ can be described by copula functions. The copula C is the n-dimension c.d.f. on $[0,1]^n$ with uniform marginal distribution (Nelsen 1999). This is the link between the joint c.d.f. $F_{\mathbf{X}}$ and the marginal c.d.f. F_{X_i} and copula C to satisfy the following relation:

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = C_{\mathbf{X}}(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)).$$

We assume that the random variables $X_1, \ldots, X_n$ are continuous. The copula $C_{\mathbf{X}}$ then determines univocally the values of the joint c.d.f. Fx. However, the copula $C_{\mathbf{I}}$ determines univocally the values of the joint c.d.f. $F_{\mathbf{I}}$ in the points of jump i_j only. The marginal c.d.f. F_{X_i} satisfies the condition:

$$F_{X_j}(d) = \Pr(X_j \le d) = \Pr(I_j = 0) = q.$$

Furthermore, the values of copulas $C_{\mathbf{I}}$ and $C_{\mathbf{X}}$ are equal at the point of the jump, i.e.

$$C_{\mathbf{I}}(v_1, \ldots, v_n) = C_{\mathbf{X}}(v_1, \ldots, v_n),$$

where v_j is described by (1). We have

$$C_{\mathbf{I}}(v_1, \ldots, v_n) = \Pr(I_1 \le i_1, \ldots, I_n \le i_n) = \Pr(X_1 \le r_1, \ldots, X_n \le r_n),$$

where

$$r_j = \begin{cases} \infty & i_j = 1 \\ d & i_j = 0 \end{cases}$$

and

$$\Pr(X_1 \le r_1, \ldots, X_n \le r_n) = C_{\mathbf{X}}(v_1, \ldots, v_n),$$

because

$$F_{X_j}(r_j) = \begin{cases} q & r_j = d \\ 1 & r_j = \infty \end{cases}.$$

We can interpret every point of jump $(i_1, \ldots, i_n)$ of c.d.f. $F_{\mathbf{I}}$ as the subset $A \subset \{1, \ldots, n\}$, such that $i_j \in A$ iff $i_j = 1$, and we will use the notation $\mathbf{1}_A = (i_1, \ldots, i_n)$. The number of elements of subset A, denoted by $|A|$, is equal to the number of "1" at the point of jump $(i_1, \ldots, i_n)$.
Let us assume now that the copula $C_{\mathbf{X}}$ is exchangeable, i.e. we obtain

$$C_{\mathbf{X}}(u_1, \ldots, u_n) = C_{\mathbf{X}}\left(u_{\pi(1)}, \ldots, u_{\pi(n)}\right)$$

for any permutation π of set $\{1, \ldots, n\}$. So, the copula $C_{\mathbf{I}}$ is also exchangeable. Then, we obtain

$$F_{\mathbf{I}}(\mathbf{1}_A) = F_{\mathbf{I}}(\mathbf{1}_B) = F_{k,n},$$

when $|A| = |B| = k$. We have (see, Heilpern 2020)

$$F_{k,n} = \Pr(I_{k+1} = 0, \ldots, I_n = 0) = C_{\mathbf{I}}(\underbrace{1, \ldots, 1}_{k}, \underbrace{q, \ldots, q}_{n-k}).$$

The value of p.m.f. is equal to (see Cossette et. al. 2002; Heilpern 2020)

$$f_{k,n} = f_{\mathbf{I}}\left(\mathbf{1}_A\right) = \Pr\left(I_1 = 1, \ldots, I_k = 1, I_{k+1} = 0, \ldots, I_n = 0\right) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} F_{k-j,n} \tag{2}$$

This results from the basic property of the multidimensional cumulative distribution function (see, Cramer 1999 (8.3.3)). The formula (2) allows us to calculate $f_{k,n}$ knowing the copula $C_{\mathbf{I}}$.

The formula (2) will allow us to determine the distribution of random variable K, the number of claims covered by the reinsurer. In this case the distribution of K is calculated using the following formula

$$\Pr(K = k) = \sum_{|A|=k} f_{\mathbf{I}}\left(\mathbf{1}_A\right) = \binom{n}{k} f_{k,n} = \sum_{j=0}^{k}(-1)^j \frac{n!}{(n-k)!j!(k-j)!} F_{k-j,n}. \tag{3}$$

We can see that the distribution of K depends on the c.d.f. $F_{k,n}$. But if we know the p.m.f. $f_{k,n}$, we can determine this distribution in a simple way. We will frequently use formula (3) later in this work. It allows us to perform specific calculations in the examples presented. We may say that the random variable K has a dependent binomial distribution and denote it as $K \sim \mathrm{DB}\ (n,\ p,\ C_{\mathbf{I}})$. The expected value of K takes the form (see Heilpern 2020)

$$E(K) = \sum_{j=1}^{n} E\left(I_j\right) = np.$$

The covariance of the random variables I_i, and I_j are equal

$$\mathrm{Cov}\left(I_i, I_j\right) = E\left(I_i I_j\right) - E\left(I_i\right) E\left(I_j\right) = f_{2,2} - (1-q)^2 = C_{\mathbf{I}}(q,q) - q^2,$$

because $f_{2,2} = F_{2,2} - 2F_{1,2} + F_{0,2} = C_{\mathbf{I}}(1,1) - 2C_{\mathbf{I}}(1,q) + C_{\mathbf{I}}(q,q) = 1 - 2q + C_{\mathbf{I}}(q,q)$. Thus, the variance of the number of claims covered by the reinsurer is equal

$$\begin{aligned} V(K) &= V\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} V\left(I_j\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathrm{Cov}\left(I_i I_j\right) = \\ &= npq + \left(n^2 - n\right)\left(C_{\mathbf{I}}(q,q) - q^2\right), \end{aligned} \tag{4}$$

because $V\left(I_j\right) = pq$. This variance depends on the probability p that the reinsurer covers the claim, the copula $C_{\mathbf{I}}$, and portfolio size n.

3 Selected cases

When the random variables $X_1, \ldots, X_n$ are independent, then

$$C_{\mathbf{X}}(u_1, \ldots, u_n) = \Pi(u_1, \ldots, u_n) = u_1 \cdot \ldots \cdot u_n.$$

Therefore, in this case, the random variable K has the classical binomial distribution and we get

$$F_{k,n} = q^{n-k}, \quad f_{k,n} = p^k q^{n-k},$$
$$\Pr(K = k) = \binom{n}{k} p^k q^{n-k}, \quad V(K) = npq, \quad \mathrm{Cov}(I_i, I_j) = 0.$$

The comonotonicity, strict positive dependence is done using the copula

$$C_{\mathbf{X}}(u_1, \ldots, u_n) = M(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n).$$

In this case, we obtain

$$F_{k,n} = \begin{cases} q & k < n \\ 1 & k = n \end{cases}, \quad f_{k,n} = \Pr(K = k) = \begin{cases} q & k = 0 \\ 0 & 0 < k < n \\ p & k = n \end{cases}, \quad V(K) = n^2 pq.$$

The copula, which is the convex combination of independency and comonotonicity

$$(1-\rho)\Pi + \rho M,$$

where $0 \leq \rho \leq 1$, was introduced by Joe in (Joe 1997) as family B11. It is also called the Spearman copula (Hürlimann 2004 a, b, Heilpern 2014). The parameter ρ is the Spearman correlation coefficient (Hürlimann 2004 a, b). The random variables $I_1, \ldots, I_n$ are independent with the probability $1-\rho$ and $I_1 = \ldots = I_n$ with probability ρ in this case.
Therefore, we obtain

$$F_{k,n} = C_{\mathbf{X}}(\underbrace{1, \ldots, 1}_{k}, \underbrace{q, \ldots, q}_{n-k}) = (1-\rho)q^{n-k} + \rho q,$$

when $k < n$ and $F_{n,n} = 1$ and

$$f_{k,n} = \begin{cases} (1-\rho)q^n + \rho q & k = 0 \\ (1-\rho)p^k q^{n-k} & 0 < k < n \\ (1-\rho)p^n + \rho p & k = n \end{cases}.$$

Therefore, we can derive the probability $\Pr(K = k)$ using (3). The variance of K is equal

$$V(K) = npq(1 + \rho(n-1)),$$

because $C_{\mathbf{X}}(q,q) - q^2 = \rho pq$ in this case.
The Archimedean copulas take a simple, quasi-additive form, so they are often used in various applications. These copulas are induced by the generator φ, decreasing the convex function satisfying conditions: $\varphi(0) = \infty, \varphi(1) = 0$. They are defined by the following formula (Nelsen 1999):

$$C\left(u_1, \ldots, u_n\right) = \varphi^{-1}\left(\varphi\left(u_1\right) + \ldots + \varphi\left(u_n\right)\right).$$

The c.d.f. $F_{k,n}$ are equal to

$$F_{k,n} = \varphi^{-1}((n-k)\varphi(q))$$

in this case and so, this value depends on the generator φ and the probability q only.
The Archimedean copulas form families characterized by some parameters, which reflect the degree of dependence. These are the relations between the values of the parameters and the Kendal or Spearman correlation coefficient (Nelsen 1999).
Every Archimedean copula C for $n > 2$ reflects nonnegative dependence between the variables I_j, i.e. it satisfies the following inequalities:

$$\Pi\left(u_1, \ldots, u_n\right) \leq C\left(u_1, \ldots, u_n\right) \leq M\left(u_1, \ldots, u_n\right),$$

and every pair (Y_i, Y_j) are equally correlated.
We will use the Clayton copula, calculated by the formula (Nelsen 1999)

$$C_a\left(u_1, \ldots, u_n\right) = \left(u_1^{-a} + \ldots + u_n^{-a} - n + 1\right)^{-1/a},$$

where parameter $a > 0$, and the generator takes the form

$$\varphi(u) = u^{-a} - 1$$

Moreover, the c.d.f is equal to

$$F_{k,n} = \left((n-k)\left(q^{-a} - 1\right) + 1\right)^{-1/a}.$$

The limit value of parameter $a = 0$ corresponds to independence, and $a = \infty$ implies comonotonicity. Knowing the value of the parameter a, we can determine the value of the Kendal correlation coefficient τ using the formula:

$$\tau = \frac{a}{a+2}.$$

Using (4) we obtain the variance of random variable K when the Clayton copula describes the dependent structure:

$$V(K) = npq + \left(n^2 - n\right)\left(\left(2q^{-a} - 1\right)^{-1/a} - q^2\right). \tag{5}$$

The Gumbel copula done by formula

$$C(u_1, \ldots, u_n) = \exp\left(-\left((-\ln u_1)^\theta + \ldots + (-\ln u_n)^\theta\right)^{1/\theta}\right),$$

where $\theta \geq 1$, is another Archimedean copula (Nelsen 1999). When $\theta = 1$ we obtain independence and for $\theta = \infty$ we have comonotonicity. The generator is equal

$$\varphi(u) = (-\ln u)^\theta,$$

and the c.d.f. takes the form

$$F_{k,n} = q^{(n-k)^{1/\theta}}.$$

Kendal coefficient of correlation τ is equal to

$$\tau = 1 - \frac{1}{\theta}$$

and the variance takes the following form

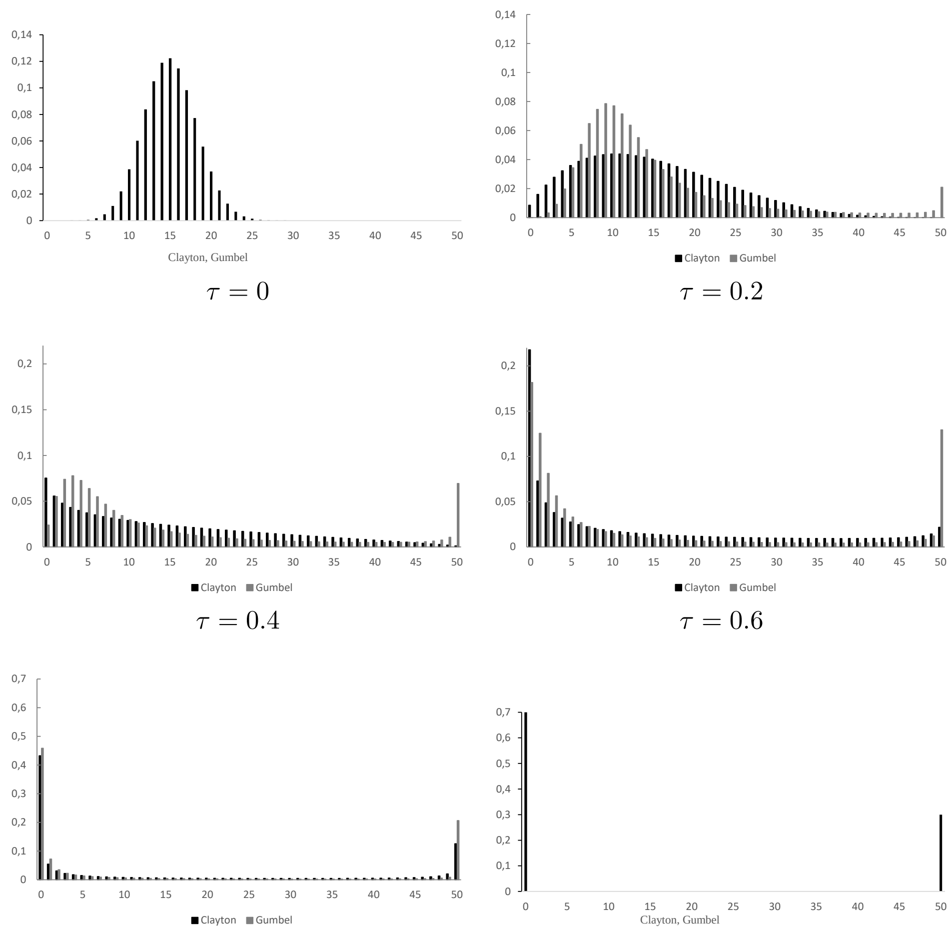
$$V(K) = npq + \left(n^2 - n\right)\left(q^{2^{1/\theta}} - q^2\right).$$

Example 1. *Now, we will analyze a portfolio of* $n = 50$ *claims. Let us assume that the probability that the reinsurer covers the claim is equal to* $p = 0.3$*, and the dependent structure of* $X_1, \ldots, X_n$ *is described by the Clayton and Gumbel copulas. The distribution of* K *for these copulas and different values of the Kendal* τ *coefficient of correlation:* $0, 0.2, 0.4, 0.6, 0.8$*, and 1 is presented in Figure 1. The corresponding values of the parameter* a *describing the Clayton family take the values: 0 (independency),* $0.5, 1.33, 3, 8$*, and* ∞ *(comonotonicity), and for Gumbel copulas: 1, 1.2, 1.67, 2.5, 5 and* ∞*.*

We can see that an increase in the value of parameter a (growth in the degree of dependence) affects the shape of the graph of the p.m.f. of the random variable K. It changes from the classical unimodal distribution to a distribution focusing on the two points 0 and 1 only through right-sided asymmetric U-shape distributions.
For independence, the distribution of the number of claims covered by the reinsurer is symmetric and unimodally condensed around the expected value equal to $\mathrm{E}(K) = 15$. For the Clayton family and smaller dependencies, the graph of p.m.f. becomes more expanded. Then, the mass of probability moves to the left side. The lack of covered claims obtains the highest probability value, and the next number of covered claims becomes less probable. When the Kendall τ is greater than 0.5 , the graphs become U-shape. The extreme values (no covered claims or all claims are covered) are the most probable in this case. The first event is more probable. For strict dependence, we obtain a two-point distribution. However, this case occurs very rarely. In practice we usually obtain small dependences.

For the Gumbel family, when the Kendall τ is smaller than 0.5 the graphs of the p.m.f. of random variable K have two local extremes, one near $k = 0$ and the second for $k = 50$. But for the greater values of the Kendall τ we obtain a similar situation as for the Clayton family. We can see, that the distribution of the random variable K, the number of the covered claims, depends significantly on the choice of the copula. Differences occur especially for small values of the degree of dependence, a case usually occurring in practice.

Figure 1: Distribution of the random variable K for different degrees of dependence Clayton and Gumbel copula



$\tau = 0$ $\tau = 0.2$

$\tau = 0.4$ $\tau = 0.6$

$\tau = 0.8$ $\tau = 1$

The variance of the number of claims K covered by the reinsurer is equal to

$$V(K) = 10.5 + 2450\left(\left(2 \cdot 0.7^{-a} - 1\right)^{-1/a} - 0.49\right)$$

for Clayton's family whereas, the following formula we obtain for Gumbel copulas

$$V(K) = 10.5 + 2450\left(0.7^{2^{1/\theta}} - 0.49\right).$$

We use the formula (4) in this case. The values of this variance for the different values of the degree of dependence are included in Table 1.
When the dependent structure of $X_1, \ldots, X_n$ is described by the Spearman copula, we face a more complicated situation.

Example 2. *(continuation of Example 1) Let us assume that the dependent structure of $X_1, \ldots, X_n$ is described by the Spearman copula. Figure 2 includes the graphs of the distribution of random variable K for the values of parameter ρ, i.e. Spearman's coefficient of correlation. We investigated the distribution of K when the dependent structure is characterized by Clayton and Gumbel copulas for the Kendal coefficient of correlation equal to* 0.2, 0.4, 0.6, *and* 0.8. *The relation between the Kendal τ and Spearman ρ coefficients of correlation is done by the formula*

$$\tau = \frac{\rho(\rho + 2)}{3}$$

for Spearman copula case (Nelsen 1999). Thus, the coefficient ρ is equal to 0.2649, 0.4832, 0.6733 *and* 0.8439 *in this case.*

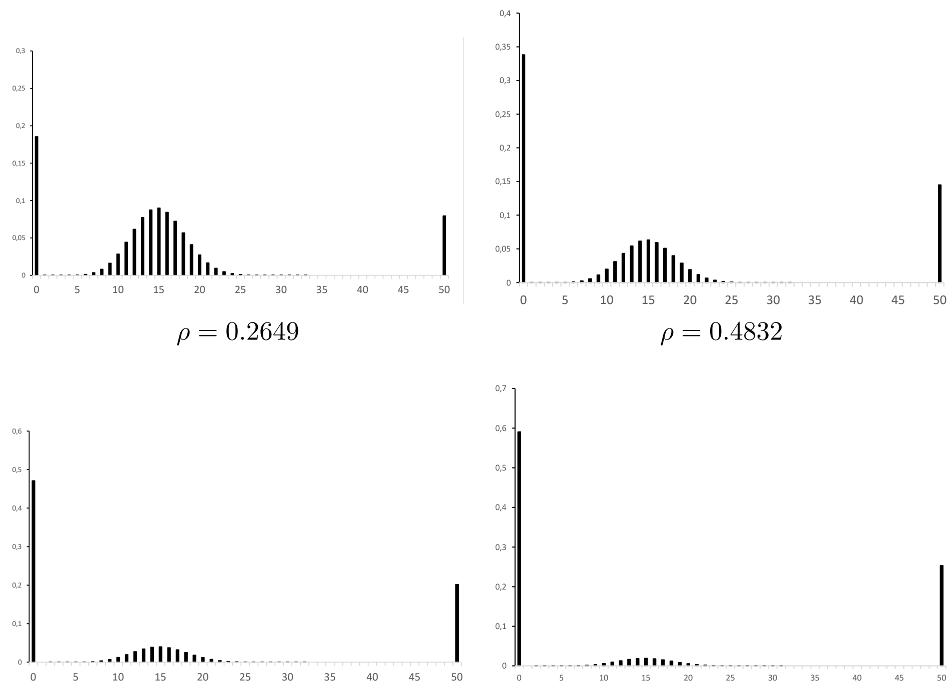
The graphs of p.m.f. for random variable K consist of three fragments, the external parts connected with $\Pr(K = 0), \Pr(K = 50)$, and the middle unimodal part centered around the expected values of the random variable K. This part plays a decreasing role as the dependency increases. The variance of K is included in Table 1.

Table 1: The values of $V(K)$

τ	0	0,2	0,4	0,6	0,8	1
Clayton	10.50	77.22	157.81	259.29	388.42	525.00
Gumbel	10.50	107.66	236.85	340.28	436.41	525.00
Spearman	10.50	146.79	259.11	356.91	444,67	525.00

The variance of the random variable K grows significantly with increasing dependency for these copulas. We can see that the variance for the Gumbel and Spearman families is greater than for the Clayton family. In contrast, the variance for the Gumbel family is comparable to the variance for Spearman copulas.

Figure 2: Distribution of the random variable K for different degrees of dependence – Spearman copula



$\rho = 0.2649$ $\rho = 0.4832$

$\rho = 0.6733$ $\rho = 0.8439$

4 Random number of claims

4.1 Number of claims covered by the reinsurer

Now, we can treat the number of claims as the random variable N. Then, the number of claims covered by the reinsurer is a random sum

$$K = \sum_{j=1}^{N} I_j$$

in this case and we obtain

$$\Pr(K = k) = \sum_{n=k}^{\infty} \Pr(N = n) \Pr(K = k \mid N = n) = \sum_{n=k}^{\infty} \Pr(N = n) \Pr(K_n = k), \tag{6}$$

where $k = 0, 1, 2, \ldots, K_n = I_1 + \ldots + I_n$ and $K_0 = 0$.
When the random variables I_i are independent, then the random variables K_n, for $n \geq 2$, as the sum of the Bernoulli distributed random variables, have a binomial distribution, i.e. $K_n \sim \mathrm{B}(n, p)$. If the dependent structure of I_i is described by the exchangeable copula C, then the probability $P(K_n = k)$ is calculated by the formula (3). The expected value of K is equal to $E(K) = E(N)p$.

Example 3. *Let us assume that the number of claims is determined by the Poisson distributed random variable N with the parameter $\lambda = 50$, the probability that the reinsurer covers the claim is equal to $p = 0.3$, and the dependent structure is described by the Clayton and Gumbel copulas. We investigate six cases: independence, $\tau = 0.2, 0.4, 0.6, 0.8$, and comonotonicity. The distributions of the random variable K, the number of claims covered by the reinsurer, are presented in Figure 3 for different values of the Kendal correlation coefficient τ. For $\tau \geq 0.6$ the values of the probabilities $\Pr(T = 0)$ are included in Table 2. These values are much greater than those of other probabilities and they are not shown on the chart. The expected value of random variable K is equal to $E(K) = 15$.*

For lower values of the correlation coefficient τ we obtain a situation similar to that in Example 1. However, we get the local extrema for $k = 50$, the expected number of claims, for the greater values of degrees of dependence, especially in the case of Clayton copula.

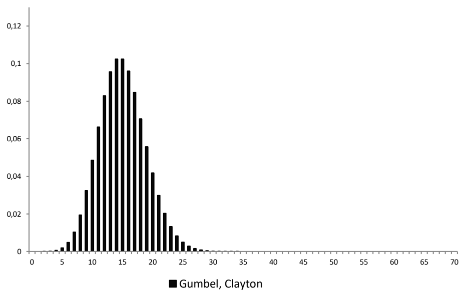
Table 2: The probability $\Pr(K = 0)$

τ	Gumbel	Clayton
0.6	0.183	0.219
0.8	0.459	0.433
1	0.700	0.700

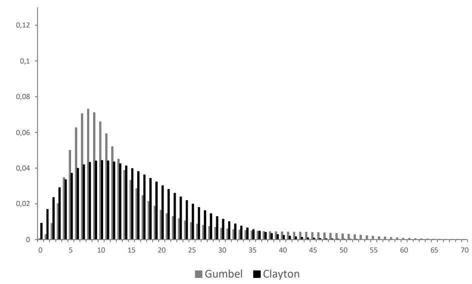
Example 4. *(continuation of Example 3) We assume that the dependent structure is described by the Spearman copula. The distributions of the random variable K are presented in Figure 4. We investigate six cases: independence, Spearman coefficient of correlation ρ equal to $0.2649, 0.4832, 0.6733, 0.8439$, and comonotonicity. The expected value of K is equal to $E(K) = 15$. Table 3 contains the values of the probability $\Pr(K = 0)$.*

For a smaller dependence, close to independence, the graph is unimodal with the maximum near the expected value of random variable K. For a greater dependence, the maximum is achievable for $k = 0$. It increases with a rise in the degree of dependence. The graphs also have two local extrema, one in the expected value of random variable K and the second in the expected value of random number of claims

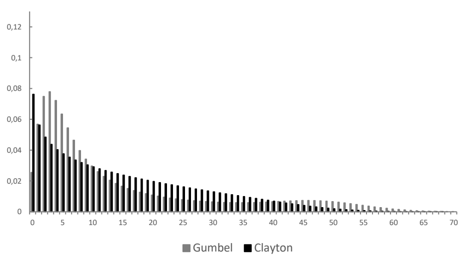
Figure 3: Distribution of the random variable K for the random number of claims – Clayton and Gumbel copulas



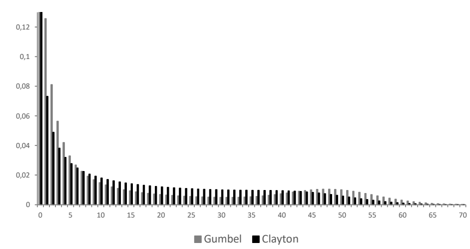
$\tau = 0$



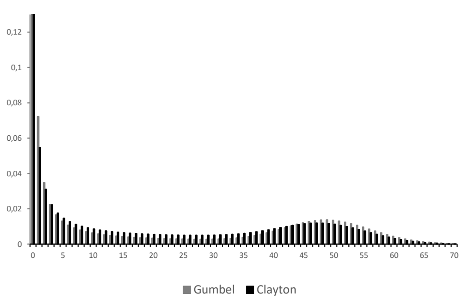
$\tau = 0.2$



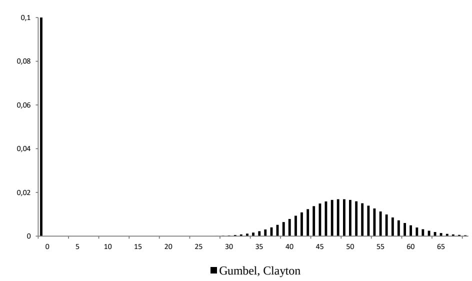
$\tau = 0.4$



$\tau = 0.6$



$\tau = 0.8$



$\tau = 1$

N equal to 50. The value of the second local extreme increases as the degree of dependence rises.

Table 3: The probability $\Pr(K=0)$

ρ	Spearman
0.2	0.185
0.4	0.338
0.6	0.471
0.8	0.591
1	0.700

We can observe, that for smaller values of the degree of dependence, the distributions of random variable K for the Gumbel copula are different than Clayton copula. But for the greater value of the degree of dependence, we obtain a similar situation. The graph of the distribution of K, the number of the covered claims, for Spearman copula, is significantly different from the Gumbel and Clayton cases. Also, the probability, that no claims will be covered by the reinsurer is greater in this case. It seems that the distribution of the random variable K significantly depends on the choice of the copulas.

The values of the variance $V(K)$ are included in the Table 4.

Table 4: The values of $V(K)$

τ	0	0,2	0,4	0,6	0,8	1
Clayton	15	133.57	245.97	351.51	449.60	534
Gumbel	15	83.08	165.60	268.87	400.64	534
Spearman	15	120	225	330	435	534

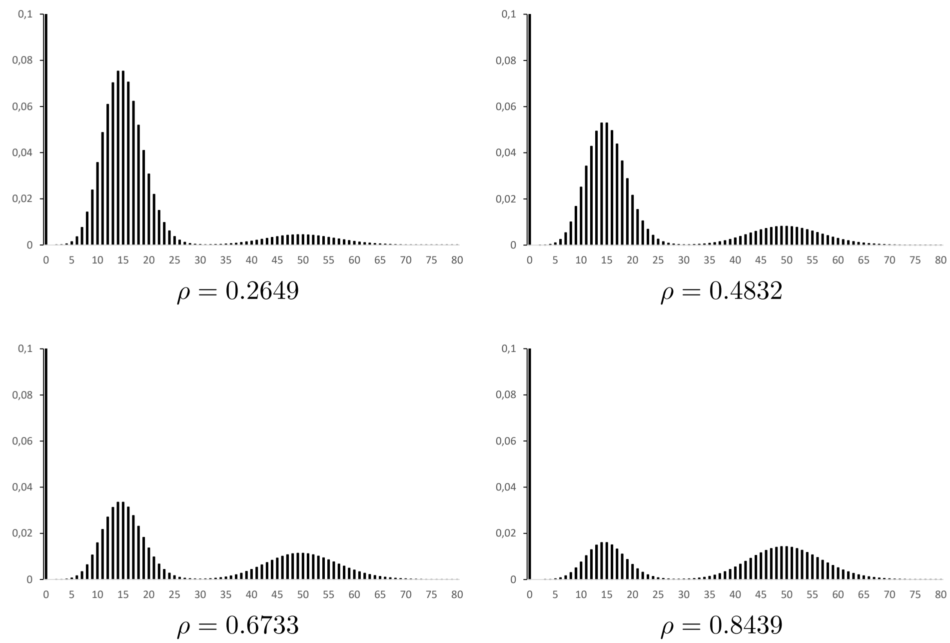
Similarly to Example 2, the variance $V(K)$ grows significantly with the increasing dependency for these copulas. We can see that the values of variances for the Clayton families are the greatest and the variances for the Gumbel copulas are the smallest. These values of the variances depend significantly on the choice of the copula.

4.2 Total value of claims

So far we have dealt with a number of policies (the number of successes) K. Now we will study the total value of claims covered by the reinsurer. First, we assume that the number of claims is fixed, equal to n.

Let $Z_j = \max(X_j - d, 0)$, where $j = 1, \ldots, n$, is the value of the j th claim covered

Figure 4: Distribution of the random variable K for the random number of claims – Spearman copula



by the reinsurer. The c.d.f. of Z_j is equal to

$$F_{Z_j}(x) = F_{X_j}(x + d)$$

for $x \geq 0$ and the random variable Z_j has the atom in 0 equals $F_{X_j}(d)$. The expected value of this random variable is equal to

$$E(Z_j) = \int_d^\infty \left(1 - F_{X_j}(x)\right) dx.$$

The total value of the reinsured claims is the random variable

$$T_n = \sum_{j=1}^{n} I_j Z_j,$$

the main object of our interest. The c.d.f of the total value of covered claims is equal to

$$F_{T_n}(x) = \Pr\left(\sum_{j=1}^{n} I_j Z_j \le x\right) = \sum_{k=0}^{n} \Pr\left(\sum_{j=1}^{n} I_j Z_j \le x \mid K = k\right) \Pr(K = k) =$$
$$= \Pr(K = 0) + \sum_{k=1}^{n} \Pr\left(S_k \le x\right) \binom{n}{k} f_{k,n},$$

where

$$S_k = \sum_{j=1}^{k} Z_j = \sum_{j=1}^{k} X_j - kd. \tag{7}$$

When the random variables Z_j are independent, then the c.d.f. of sum S_k is the k th convolution:

$$F_{S_k}(x) = F_Z^{*k}(x).$$

The expected value of the total sum T_n of the values of reinsured claims is equal to $E\left(T_n\right) = npE\left(Z_j\right)$.
Now we can explore the total value of covered claims T when the number of claims is random. We have

$$T = \sum_{j=1}^{N} I_j Z_j$$

in this case, where the random variable N represents the number of claims. The c.d.f. of a random variable T takes the form

$$F_T(x) = \Pr\left(\sum_{j=1}^{N} I_j Z_j \le x\right) = \sum_{n=0}^{\infty} \Pr\left(\sum_{j=1}^{n} I_j Z_j \le x\right) \Pr(N = n) =$$
$$= \sum_{n=0}^{\infty} F_{T_n}(x) \Pr(N = n).$$

Example 5. *Let us assume that the values of the claims X_i have exponential distribution with the expected value $E\left(X_i\right) = 3$ and retention is equal to $d = 3.6$, so $q = 1 - \exp\left(-\frac{3.6}{3}\right) = 0,6988$. We also assume that the dependent structure of $X_1, \ldots, X_n$ is described by the Spearman copula and that the random number of claims N has Poisson distribution with the parameter $\lambda = 50$. The reinsured claims Z_i, of course, have the same dependent structure. We obtain that the sums $X_1 + \ldots + X_k$ have a gamma distribution $\mathrm{Ga}(k, 3)$ for the independent case. When the random variables X_j are comonotonic, then these sums have exponential distribution with the expected value $3k$. Figure 5 included the graphs of p.m.f. for the global sum T for different values of the parameters ρ of the Spearman family without values of the probabilities $\Pr(T = 0)$. They are presented in Table 4. The expected value $E(T) = 13.6080$, because $E\left(Z_j\right) = 0.9036$.*

The graph of the distribution of random variable T, the total value of covered claims, is increasing for $T = 0$ and decreasing in other cases for each degree of dependence. As the degree of dependence increases, the graph becomes more flattened and stretched. The values of probability that the total value of claims covered by the reinsurer is equal to zero, $\Pr(T = 0)$ are greater than 0.79 . These values are much bigger than those of other probabilities.

Figure 5: Distribution of the total value of the covered claims T for different degrees of dependence – random number of claims

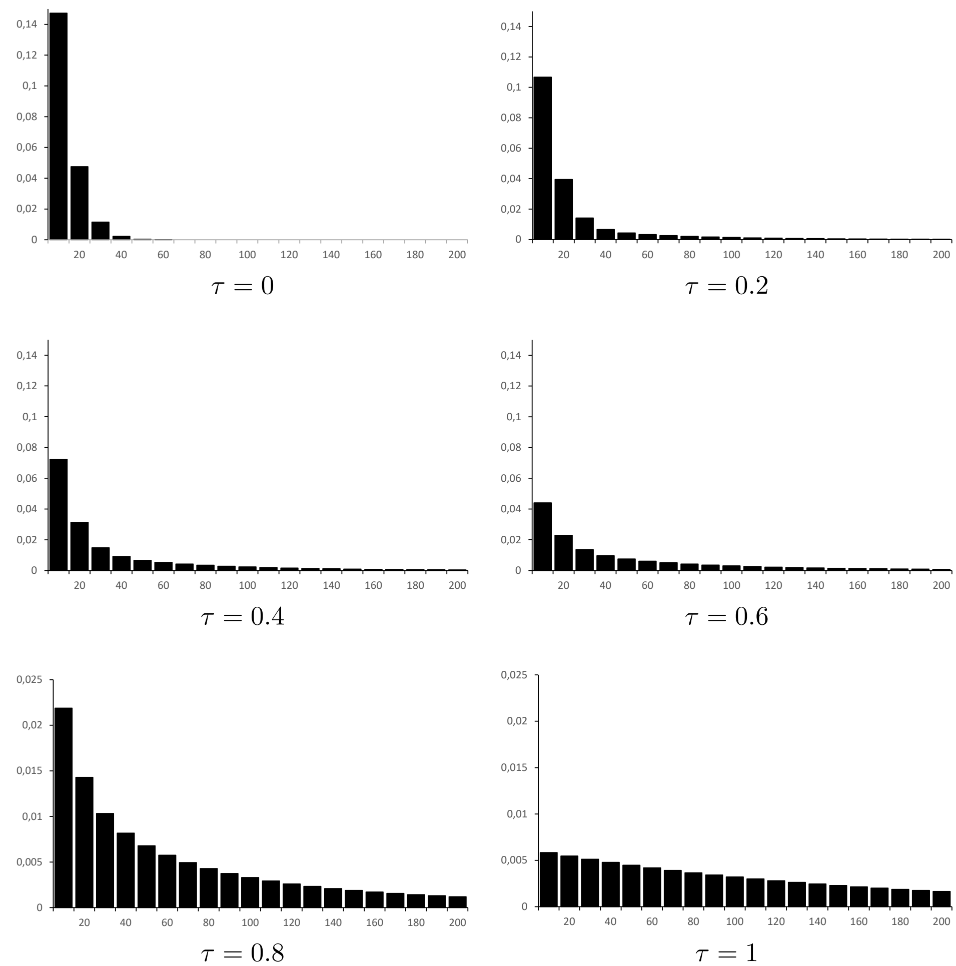


Table 5: The probability $\Pr(K = 0)$

ρ	Spearman
0	0.790
0.2	0.810
0.4	0.832
0.6	0.855
0.8	0.881
1	0.909

5 Fuzziness

Now we will investigate a case when the probability that the reinsurer covers the claim p is imprecisely determined. For instance, we obtain information that it is equal to "about 0.3". For this purpose, we use fuzzy sets.

First, we should recall some definitions and notions connected with fuzzy sets (see e.g. Zadeh 1965; Dubois, Prade 1980). Fuzzy set A, defined in the space Z is described by its membership function $\mu_A : Z \to [0, 1]$, a generalization of the characteristic function of the crisp, nonfuzzy set. Every fuzzy set is univocally characterized by its α-cuts, the crisp sets $A_\alpha = \{z \in Z : \mu_A(z) \geq \alpha\}$, where $0 < \alpha \leq 1$. The cut A_1 is the core of A, and A_0, the closure of set $\{z \in Z \mid \mu_\mathrm{A}(z) > 0\}$, is the support of fuzzy set A.

A fuzzy number, the main concept used in our paper, is a fuzzy subset of the real line R. Its every α-cut A_α is the compact interval $[A^L_\alpha, A^U_\alpha]$ (Dubois, Prade 1980). The trapezoidal fuzzy number $A = (a, b, c, d)$ has a linear membership function at the intervals $[a, b]$ and $[c, d]$. The interval $[a, d]$ is the support of the fuzzy number while $[b, c]$, when the membership function takes the value one, is its core. The triangular fuzzy number $A = (a, b, d)$ has a one-element core, i.e. $b = c$. We can order fuzzy numbers $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ as follows

$$A \leq B \Leftrightarrow a_i \leq b_i$$

for $i = 1, 2, 3$.

The image $f(A)$ of a fuzzy subset of Z, where $f : Z \to R$, has the following membership function

$$\mu_{f(A)}(x) = \sup_{f(z)=x} \mu_A(z).$$

This is the so-called extension principle (Zadeh 1975). The extension principle allows us to define the arithmetic operations * on fuzzy numbers. The membership function of the fuzzy number $A * B$ is equal to

$$\mu_{A*B}(z) = \sup_{x*y=z} \{\min\{\mu_A(x), \mu_B(y)\}\}.$$

Therefore, we can observe that the borders of α-cuts of arithmetic operation $A*B$ are defined by the borders of A and B, e.g. $(A+B)_\alpha^L = A_\alpha^L + B_\alpha^L$, $\quad (A+B)_\alpha^U = A_\alpha^U + B_\alpha^U$ (Dubois, Prade 1980).
We can define the mean value of the fuzzy number A using the following formula, see e.g. (Campos, Gonzalez 1989; Heilpern 1992)

$$MV(A) = \frac{1}{2}\int_0^1 \left(A_\alpha^L + A_\alpha^U\right) d\alpha.$$

For trapezoidal fuzzy number $A = (a, b, c, d)$, we obtain the following formulas

$$A_\alpha^L = (b-a)\alpha + a, \quad A_\alpha^U = (c-d)\alpha + d, \quad MV(A) = \frac{a+b+c+d}{4}.$$

We can define the spread of the fuzzy number in the following way

$$S(A) = \int_0^1 \left(A_\alpha^U - A_\alpha^L\right) d\alpha.$$

We interpret $S(A)$ as the measure of imprecision of fuzzy number A. For $A = (a, b, c, d)$, we have $S(A) = ((d-a) + (c-b))/2$.

6 Imprecision parameters

6.1 Fuzzy probability

Let us now assume that the random variables I_i describing the status of claims may be dependent. Therefore, the random variable K, the number of claims covered by the insurer, has the dependent binomial distribution $\mathrm{DB}(n, p, C_{\mathbf{I}})$. We will suppose that we cannot make a valid estimation of the main parameter p, the probability that the reinsurer covers the claim, and we only know the imprecision value of such a parameter. We can treat such imprecision value as the fuzzy number P in this case. This fuzzy number induces the fuzzy subset $\mathbf{K}$ on the family of dependent binomial random variables. We can designate its membership function

$$\mu_{\mathbf{K}}(K) = \mu_P(p),$$

where $K \sim \mathrm{DB}(n, p, C_{\mathbf{I}})$, using the extension principle. The sample size n, the number of claims in our case, is fixed. For instance, if we know only that the probability p

is equal to "about p_0", then we can treat such information as the triangular fuzzy number $P = (p_1, p_0, p_2)$. The fuzzy set $\mathbf{K}$ has the following α-cut:

$$\mathbf{K}_\alpha = \{K \sim \mathrm{DB}\,(n, p, C_{\mathbf{I}}) : p \in P_\alpha\}\,.$$

We can also define the expected value $E(\mathbf{K})$ and the variance $V(\mathbf{K})$ of $\mathbf{K}$ using the extension principle. These are fuzzy subsets of the real line. Let K_1 and K_2 be the dependent binomial random variables, such that $E(K_1) = m = np$ and $V(K_2) = s = g(p)$, e.g. $g(p) = np(1-p)(1+\rho(n-1))$ for Spearman copula, then

$$\mu_{E(\mathbf{K})}(m) = \mu_{\mathbf{K}}(K_1) = \mu_P\left(\frac{m}{n}\right),$$
$$\mu_{V(\mathbf{K})}(s) = \mu_{\mathbf{K}}(K_2) = \sup_{\{p:g(p)=s\}} \mu_P(p).$$

Their α-cuts take the following form

$$E(\mathbf{K})_\alpha = \left\{m : \mu_{E(\mathbf{K})}(m) \geq \alpha\right\} = \left\{m : \mu_P\left(\frac{m}{n}\right) \geq \alpha\right\} = \{m = np : p \in P_\alpha\}\,.$$

Similarly, we obtain

$$V(\mathbf{K})_\alpha = \{s = g(p) : p \in P_\alpha\}\,.$$

Example 6. *Now, we assume that we have $n = 50$ claims and the dependent structure of $\mathbf{X}$ is described by some copula C_ρ. We consider Clayton, Gumbel and Spearman copulas. We obtain information that the probability that the claim is covered by the reinsurer is equal to "about 0.3". We can treat this as the triangular fuzzy number $P = (0.25, 0.3, 0.4)$. The value of the membership function of $\mathbf{K}$ at $K \sim \mathrm{DB}\,(n, p, C_\rho)$ is equal to*

$$\mu_{\mathbf{K}}(K) = \mu_P(p) = \begin{cases} 20p - 5 & 0.25 \leq p \leq 0.3 \\ -10p + 4 & 0.3 < p \leq 0.4 \\ 0 & \textit{otherwise} \end{cases}.$$

For instance, if $K \sim \mathrm{DB}\,(n, 0.28, C_\rho)$, then $\mu_{\mathbf{K}}(K) = 0.6$. The expected value of $\mathbf{K}$ is a triangular fuzzy number (12.5, 15, 20). We can interpret $E(\mathbf{K})$ as "about 15". Its mean value is equal to 15.625.

However, the variance is not a triangular fuzzy number. When the dependent structure is described by Spearman copula its α-cuts are equal to

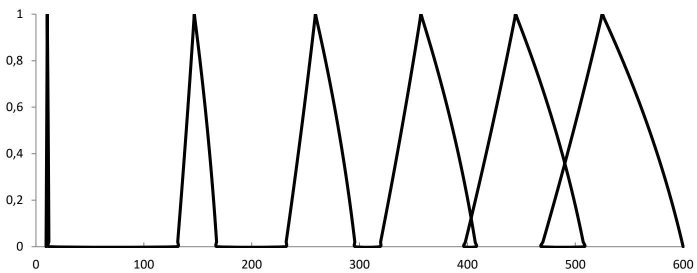
$$V_\rho(\mathbf{K})_\alpha = \left[\left(-0.125\alpha^2 + 1.25\alpha + 9.375\right)(1+49\rho), \left(-0.5\alpha^2 - \alpha + 12\right)(1+49\rho)\right]$$

for $\rho \in [0,1]$. Because, the function $g(p) = np(1-p)(1+\rho(n-1))$ is increasing on $[0.25, 0.4]$,

$$\begin{aligned} P_\alpha &= \left[p_\alpha^L, p_\alpha^U\right] = [0.05\alpha + 0.25, 0.4 - 0.1\alpha], np_\alpha^L\left(1 - p_\alpha^L\right) = \\ &= 50(0.05\alpha + 0.25)(0.75 - 0.05\alpha) = -0.125\alpha^2 + 1.25\alpha + 9.375 \end{aligned}$$

and $np^U_\alpha\left(1-p^U_\alpha\right) = -0.5\alpha^2-\alpha+12$. The graph of the membership functions of variance $V_\rho(\mathbf{K})$ for values of Spearman coefficient of correlation ρ equal to 0, 0.2649, 0.4832, 0.6733, 0.8439, and 1, that correspond to values of Kendal coefficients τ equal to 0, 0.2, 0.4, 0.6, 0.8 and 1 respectively, are presented in Figure 6.

Figure 6: Graph of the membership functions of $V_\rho(\mathbf{K})$ for different values of coefficient τ and Spearman copula



The function $g(p)$ representing variance is increasing on $[0.25, 0.4]$ for Clayton and Gumbel copulas, too. So, we obtain the α-cuts of the variance $V_\tau(\mathbf{K})$ in a similar way to the Spearman copula. We can approximate the fuzzy variance $V_\tau(\mathbf{K})$ using the triangular fuzzy numbers. Table 6 contains the variances $V_\tau(\mathbf{K})$ treated as the triangular fuzzy numbers, for different values of Kendal coefficient of correlation τ and for different copulas. We also present the mean values and the spreads of them. We can see, that as the degree of dependence increases, and the variance rises, too. The spread is getting bigger, so the imprecision of these fuzzy sets is increased. Only in the case of the Clayton copula the imprecision is greatest when τ is equal to 0.8. Similarly, we can determine the fuzzy probability $\Pr(\mathbf{K}\in B)$, where B is a crisp event. Let $f(p)=\Pr(K_p\in B)$, where $K_p\sim \mathrm{DB}(n,p,C_\mathbf{I})$, then

$$\begin{aligned}\mu_{\Pr(\mathbf{K}\in B)}(q) &= \sup_{f(p)=q}\mu_P(p),\\ \Pr(\mathbf{K}\in B)_\alpha &= \left\{\Pr(K_p\in B): K_p\sim \mathrm{DB}(n,p,C_\mathbf{I}), p\in P_\alpha\right\}.\end{aligned}$$

Example 7. *(continuation of Example 6) Now, we compute the probability that there are ten covered claims, i.e.* $\Pr(\mathbf{K}=10)$. *Let* $C_\mathbf{I}$ *be Spearman copula with parameter* $\rho, K\sim \mathrm{DB}(n,p,C_\rho)$ *and* $f_\rho(p)=\Pr(K=10)$. *The function* $f_\rho(p)=\binom{50}{10}(1-\rho)p^{10}(1-p)^{40}$, *where* $\rho<1$, *is decreasing on the interval* $[0.25, 0.4]$. *So, the* α*-cuts of such a fuzzy probability are equal to*

$$\Pr(\mathbf{K}=10)_\alpha = \left[f_\rho(0.4-0.1\alpha), f_\rho(0.05\alpha+0.25)\right].$$

The graph of the membership function of these fuzzy sets for $\tau=0, 0.2, 0.4, 0.6$, *and 0.8 is presented in Figure 7.*

Table 6: The parameters of fuzzy variances $V_\tau(\mathbf{K})$ for different values of the coefficient τ and different copulas Clayton copula

τ	triangular fuzzy numbers	mean	spread
Clayton copula			
0	$(9.38, 10.5, 12)$	10.646	1.375
0.2	$(45.55, 58.24, 82.08)$	61.179	18.561
0.4	$(96.83, 123.71, 172.63)$	129.640	38.642
0.6	$(172.41, 215.86, 290.38)$	224.598	60.484
0.8	$(291.78, 351.13, 442.46)$	361.116	78.006
Gumbel copula			
0.2	$(88.20, 96.21, 103.56)$	100.820	8.361
0.4	$(176.50, 193.54, 210.77)$	174.743	18.485
0.6	$(271.49, 299.76, 331.55)$	302.344	32.036
0.8	$(370.03, 411.51, 462.70)$	413.938	48.949
Spearman copula			
0.265	$(131.06, 146.79, 167.76)$	148.830	19.223
0.483	$(231.35, 259.11, 296.12)$	262.705	33.931
0.673	$(318.67, 356.91, 407.91)$	361.870	46.739
0.844	$(397.04, 444.69, 508.21)$	450.863	58.233
1	$(468.75, 525, 600)$	532.292	68.750

We can see, that as the degree of dependence increases, the mean value and the spread of the fuzzy probability that there are ten covered claims, decreases. Therefore, the imprecision of such fuzzy sets decreases, too. For monotonicity $\rho = 1$ we obtain crisp probability $P(\mathbf{K} = 10) = 0$.

If the dependent structure of $\mathbf{X}$ is characterized by Clayton copula then the function $f_\tau(p) = \Pr(K_p = 10)$ is decreasing on $[0,25; 0,4]$, too. We computed the values of the functions $f_\tau(p)$ using (3). The graphs of these functions for the different values of Kendall coefficient τ are presented in Figure 8.

The graphs of the membership functions $\Pr(\mathbf{K} = 10)$ for different values of degree of dependence are presented in Figure 9. We see that the mean value of fuzzy numbers $\Pr(\mathbf{K} = 10)$ and their imprecision (spread) decreases, when the degree of dependence increases, too. This case is different, thought. The mean values are greater in this case as for Clayton copula, but the spreads are smaller. We have less imprecision here. The shapes of the graph are different. Unlike in the case of the Spearman copula, the fuzzy sets $\Pr(\mathbf{K} = 10)$ are disjoint and approximately linear.

Figure 7: Graph of membership functions $\Pr(\mathbf{K} = 10)$ for different values of τ and Spearman copula

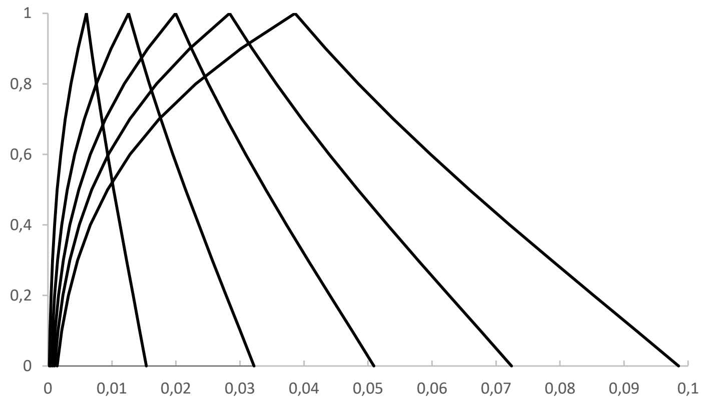


Figure 8: Graph of the functions $f_\tau(p)$ for different values of τ and Clayton copula

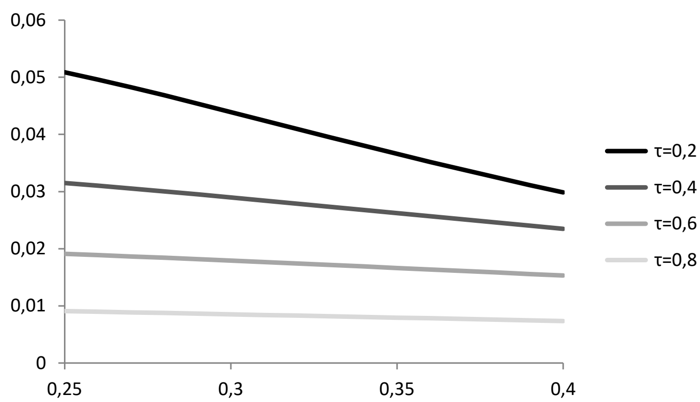
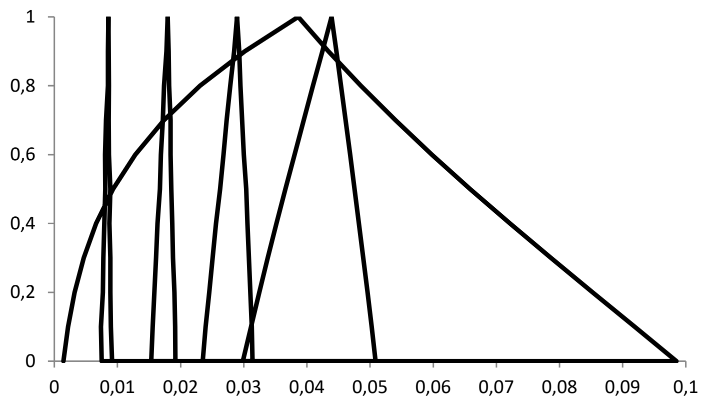


Figure 9: Graph of membership functions $\Pr(\mathbf{K}=10)$ for different values of τ and Clayton copula



6.2 Imprecision copula parameter

The parameter a of copula C_a, which determines the dependent structure of $\mathbf{X}$ is defined by the Kendall coefficient τ by the relation $a = l(\tau)$. For Clayton copula, we obtain $l(\tau) = \frac{2\tau}{1-\tau}$, for Gumbel copula we have $l(\tau) = \frac{1}{1-\tau}$ and for Spearman copula $l(\tau) = \sqrt{1+3\tau} - 1$. However, in many cases, we cannot estimate this coefficient exactly. In this situation, we only know the imprecision value of such a coefficient and treat it as the fuzzy number T. Thus, the coefficient a is also imprecisely defined. We can also consider it as fuzzy number A. Using the extension principle, we obtain the following membership function of A

$$\mu_A(a) = \mu_T\left(l^{-1}(a)\right).$$

Let us consider a family of copulas indexes by parameter a, e.g. Clayton family. The fuzzy number A induces the fuzzy subset $\mathbf{C}$ of such a family using the formula

$$\mu_{\mathrm{C}}\left(C_a\right) = \mu_{\mathrm{A}}(a).$$

Every copula C_a generates the random variable K_a with the dependence binomial distribution $\mathrm{DB}\left(n, p, C_a\right)$. We obtain the fuzzy subset of such random variables $\mathbf{K}_A$ with the membership function

$$\mu_{K_A}\left(K_a\right) = \mu_A(a).$$

The expected value of fuzzy random variable $\mathbf{K}_A$ is a crisp number. The expected value $E\left(\mathbf{K}_A\right) = np$ because we have $E\left(K_a\right) = np$ for every a, but its variance $V\left(\mathbf{K}_A\right)$

is a fuzzy number. Its membership function takes the form

$$\mu_{V(\mathbf{K}_A)}(s) = \sup_{\{a:v(a)=s\}} \mu_A(a),$$

where (see (4))

$$v(a) = npq + \left(n^2 - n\right)\left(C_a(q,q) - q^2\right).$$

In this case, we can define the fuzzy probability $\Pr(\mathbf{K}_A \in B)$, where B is a crisp event. Let $f(a) = \Pr(K_a \in B)$, where $K_a \sim \mathrm{DB}(n,p,C_a)$, then

$$\mu_{\Pr(\mathrm{K}_A \in B)}(q) = \sup_{\{a:f(a)=q\}} \mu_A(a)$$

and its α-cut is equal

$$\Pr(\mathbf{K}_A \in B)_\alpha = \{\Pr(K_a \in B) : a \in A_\alpha\}.$$

Example 8. *Let $n = 50$ and the probability that the claim is covered by the reinsurer be equal to 0.3. We obtain information that the Kendall coefficient of correlation is equal to "about 0.15".*

We can treat such information as the triangular fuzzy number $T = (0.1, 0.15, 0.2)$.
Now, we assume that the dependent structure of $\mathbf{X}$ is described by Clayton copula C_a. The fuzzy number A, the fuzzy subset of the parameters, has the following membership function:

$$\mu_A(a) = \mu_T\left(l^{-1}(a)\right) = \begin{cases} \frac{20a}{a+2} - 2 & \frac{2}{9} \le a \le \frac{6}{17} \\ \frac{-20a}{a+2} + 4 & \frac{6}{17} < a \le 0.5 \\ 0 & \text{otherwise} \end{cases}.$$

because $l^{-1}(a) = \frac{a}{a+2}$ for Clayton copula. We can approximate it using the triangular fuzzy number $(0.222, 0.353, 0.5)$. The expected value of $E(\mathbf{K}_A) = 15$.
In this case, we have

$$v(a) = 10.5 + 2450\left(\left(2 \cdot 0.7^{-a} - 1\right)^{-1/a} - 0.49\right).$$

This function increases, and then

$$\mu_{V(\mathbf{K}_A)}(s) = \mu_A\left(v^{-1}(s)\right).$$

Let the dependent structure of $\mathbf{X}$ be described by Gumbel copula C_θ. Then $l^{-1}(a) = 1 - \frac{1}{a}$ and

$$\mu_A(a) = \begin{cases} 18 - \frac{20}{a} & 1\frac{1}{9} \le a \le 1\frac{3}{17} \\ 4 + \frac{20}{a} & 1\frac{3}{17} < a \le 1.2 \\ 0 & \text{otherwise} \end{cases}.$$

The triangular fuzzy number $(1.111, 1.176, 1.2)$ approximates the fuzzy number A. The variance function is equal to

$$v(a) = 10.5 + 2450\left(0.7^{2^{1/a}} - 0.49\right)$$

in this case. It is an increasing function, too.
For Spearman copula we have $l^{-1}(\rho) = \rho(\rho+2)/3$ and

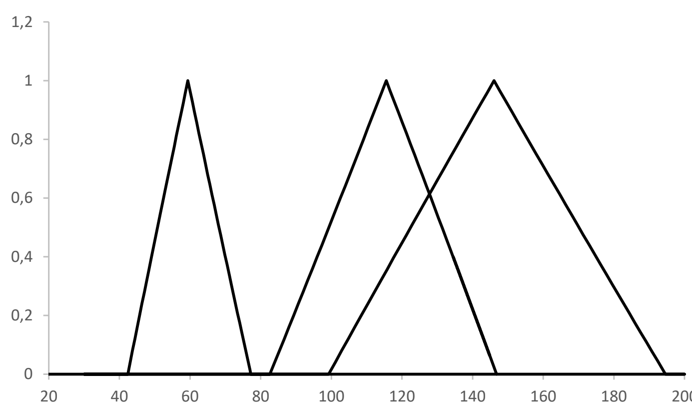
$$\mu_A(\rho) = \begin{cases} \frac{20}{3}\rho(\rho+2) - 2 & \sqrt{1.3} - 1 \le \rho \le \sqrt{1.45} - 1 \\ -\frac{20}{3}\rho(\rho+2) + 4 & \sqrt{1.45} - 1 < \rho \le \sqrt{1.6} - 1 \\ 0 & \text{otherwise} \end{cases} .$$

We can approximately treat it as a triangular fuzzy number $(0.140, 0.213, 0.265)$. We also obtain that

$$v(\rho) = 10.5(1 + 49\rho)$$

for Spearman copula. It is, of course, an increasing function. The graphs of the membership function of $V(\mathbf{K}_A)$ for these copulas are presented in Figure 10.

Figure 10: Graphs of membership function $V(\mathbf{K}_A)$ for different copulas



The shape of these graphs is almost linear and so we can approximate fuzzy variable $V(\mathbf{K}_A)$ using the triangular fuzzy number for these copulas. Then, we obtain for Clayton copula triangular fuzzy number $(43.376, 59.395, 77.217)$, for Spearman copula $(82.620, 115.540, 146.797)$, and $(99.266, 147.800, 194.597)$ for Gumber copula. The fuzzy variance $V(\mathbf{K}_A)$, when the dependent structure is described by Gumbel copula, is characterized by the greatest imprecision. Because, the spreads of such fuzzy numbers are equal, respectively, to: 16.92, 32.09, 47.67. For Clayton copula, we obtain the least imprecision. When we compare these fuzzy variances, we get the same relations.

Now, we derive the fuzzy probability $\Pr(\mathbf{K}_A = 10)$ when the dependent structure is characterized by Clayton copula. The function $f(a) = \Pr(K_a = 10)$ is decreasing (see Figure 10) on [2/9, 0.5]. We computed the values of the functions $g(a)$ using (3). The membership function of the fuzzy probability takes the following form

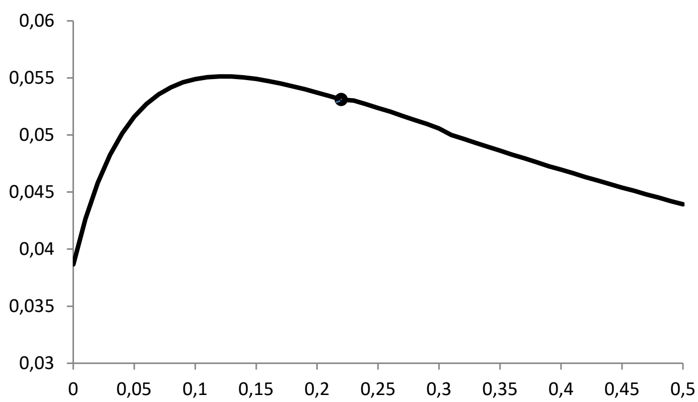
$$\mu_{\Pr(\ \mathrm{K}=10)}(q) = \begin{cases} \mu_A\left(f^{-1}(q)\right) & f(0.5) \le q \le f\left(\frac{2}{9}\right) \\ 0 & \text{otherwise} \end{cases}.$$

The graph of such a function is presented in Figure 12.
For Spearman copula, we obtain, that

$$f(\rho) = \binom{50}{10}(1-\rho)p^{10}(1-p)^{40},$$

where $\rho < 1$. It is a decreasing function over the entire domain in this case. The graph of the membership function of $\Pr(\mathbf{K}_A = 10)$ is presented in Figure 11.

Figure 11: Graph of the function $f(a)$ for Clayton copula

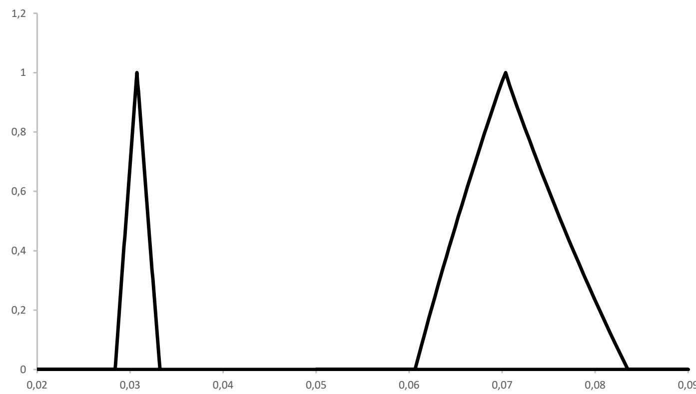


We can approximate these fuzzy numbers using the triangular fuzzy numbers $(0.061, 0.071, 0.085)$ for Clayton copula and $(0.028, 0.031, 0.033)$ for Spearman copula. The shapes of these graphs are almost linear. Therefore the fuzzy probability that the claims will be covered is greater for Clayton copula. The imprecision is greater in this case, too.

7 Conclusions

The paper is devoted to non-classical, mathematical reinsurance models. Firstly, we allow for the dependence of random variables representing the claims. The dependent structure is described by the copula. The paper examined a number of losses covered

Figure 12: Graph of membership functions $\Pr(\mathbf{K}_A = 10)$ for Clayton and Spearman copulas



by the reinsurer, the random variable K. The Clayton, Gumbel, and Spearman copulas were assumed to describe the dependent structure. Two cases were considered. The first one is when there is a constant number of claims, and the second one is when the number of claims is random and it is described by a random variable with a Poisson distribution. The form of the distribution is mainly influenced by two factors: the degree of dependence and the choice of the copula. When the number of claims is fixed, the shape of the graph of the p.m.f. of this random variable changes as the degree of dependence increases, from unimodal distribution, through a right-sided asymmetry to a U-shape distribution. The distribution of the K variable is significantly different for the Spearman's copula. The probabilistic mass is concentrated at the ends of the domain and the local extremum. However, for the Clayton and Gumbel copulas, we obtain similar distributions, especially for higher degrees of dependence. When we have a random number of claims, for the Clayton and Gumbel copulas, when the degrees of dependence are small, we obtain a similar situation as in case of a constant number of damages. However, for larger degrees of dependence, local extremes occur. In case of the Spearman's copula, we have a distribution of the K variable centered at the beginning of the domain and two local extremes. For all the copulas, the probability $\Pr(K = 0)$ increases as the degree of dependence increases. For large dependencies, it reaches 0.7, the highest in case of the Spearman's copula. The probability that the reinsurer will not cover any loss then prevails.

A random variable T representing the total value of the covered loss was also examined in case of a random number of claims. It was assumed that the claims have an exponential distribution and the case was considered when the relationship structure was described by the Spearman's copula. In this situation, the distributions were more regular than before, with extreme right-sided asymmetry increasing as the degree of dependence increased. The probability that the value of covered damages will be positive $\Pr(T > 0)$ decreases as the dependency inreases, up to a value less than 0.1.

Later, we considered cases when some parameters in the model are imprecisely defined. We examined a situation when the probability that the reinsurer will cover the claim is imprecisely determined and the case of an imprecise copula parameter. This parameter was treated as fuzzy triangular number. In both cases, we were interested in the number of claims covered by the reinsurer $\mathbf{K}$ and the probability $\Pr(\mathbf{K} = 10)$. The dependency structure was described using three different copulas: Clayton, Gumbel, and Spearman. Whenever the probability that the reinsurer will cover the claim is imprecise, an increase in the degree of dependence causes a rise in the variance $V(\mathbf{K})$, the mean, and the spread, i.e. the imprecision. In case of the Spearman copula, the variation is the largest, while for the Clayton copula the smallest. However, as the degree of dependence increases, the probabilities $\Pr(\mathbf{K} = 10)$ and spreads decrease. In this case, the choice of a copula is important. The fuzzy probabilities for the Spearman copula are clearly different from those obtained for the Clayton copula. The appropriate choice of a copula is also important when the copula parameters are imprecise. We obtain different variances $V(\mathbf{K}_A)$, probabilities $\Pr(\mathbf{K} = 10)$ and their spreads.

The paper presents two approaches to the reinsurance model. The first one, based on copulas, considers the case where claims may be dependent. However, in the second one, imprecisely defined parameters are studied using fuzzy sets. Both methods complement each other, they cannot be compared, they concern different issues. However, they can be combined, as was done in the second part of the article where we considered a combination of randomness and fuzziness.

In future research, we would like to consider more copulas, e.g. Frank, AMH, FGM. Moreover, in case of fuzzy parameters, we want to examine the total value of the covered claims and investigate cases when the parameter of the copula and the probability that the reinsurer covers the claim are imprecise. Furthermore, we plan to use the energy and entropy measures as the measure of imprecision.

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