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Analysis of the fractional descriptor linear systems by the decomposition into the dynamical and static parts

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A new method of the decomposition of the fractional descriptor linear continuoustime and discrete-time systems into dynamical and static parts is proposed. Conditions for the decomposition of the fractional descriptor linear systems are established and procedures for compositions of the matrices of dynamical and static parts are given. The procedures are illustrated by numerical examples.

Key words: decomposition, descriptor, fractional, linear, procedure, system, dynamical, static, part

1. Introduction

Mathematical fundamentals of the fractional calculus are given in the monographs [6, 8, 11, 12]. The descriptor linear systems have been analyzed in many books [1, 2, 6–8] and the fractional standard and descriptor linear systems in [9, 11, 12, 14–16]. Positive linear systems consisting of n subsystems with different fractional orders have been introduced and investigated in [3, 5–8, 10]. The fractional descriptor discrete-time linear systems have been analyzed by the use of the shuffle algorithm in [9]. The stability of the delayed fractional discrete-time linear system has been investigated in [13, 14]. The decentralized stabilization of descriptor fractional positive discrete-time linear systems with delays have been addressed in [15] and the stabilization of positive descriptor

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fractional continuous-time linear system with two different fractional orders by decentralized controller in [16].

In this paper a method of the decomposition of the fractional linear systems into the dynamical and static parts is proposed.

The paper is organized as follows. In Section 2 the decomposition method for the fractional descriptor linear continuous-time systems is proposed. The extension of this method to the fractional linear discrete-time systems is presented in Section 3. Concluding remarks are given in Section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, I_n – the $n\times n$ identity matrix.

2. Decomposition of the fractional descriptor linear continuous-time systems

Consider the fractional linear system

$$E\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = Ax + Bu, \quad 0 < \alpha < 1, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, $E,\ A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, \quad \dot{x}(\tau) = \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} \tag{1c}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t, \quad \mathrm{Re}(z) > 0 \tag{1d}$$

is the gamma function [6–8, 12].

It is assumed that

$$\det E = 0 \quad \text{and} \quad \det[Es^\alpha - A] \neq 0. \tag{2}$$

In this case equation (1a) has unique solution.

The following elementary operations on real matrices will be used [7]:

1. Multiplication of any i-th row (column) by the number a. This operation will be denoted by $L[i\times a]$ for row operation and by $R[i\times a]$ for column operation.

2. Addition to any i-th row (column) of the j-th row (column) multiplied by any number b. This operation will be denoted by $L[i + j \times b]$ for row operation and by $R[i + j \times b]$ for column operation.
3. $L[i, j]$ for interchange of rows and $R[i, j]$ for interchange of columns.

The elementary operations do not change the rank of the matrices [7].

Using the row elementary operations to the equation (1a) we obtain

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{3a}$$

where $E_1,\ A_1 \in \Re^{r\times n}$, $B_1 \in \Re^{r\times m}$, $A_2 \in \Re^{(n-r)\times n}$, $B_2 \in \Re^{(n-r)\times m}$ and

$$\operatorname{rank} E_1 = \operatorname{rank} E = r < n. \tag{3b}$$

Applying the column elementary operations to the matrices $\begin{bmatrix} E_1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ we obtain

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{4}$$

where $\det E_{11} \neq 0$ and $\det A_{22} \neq 0$ since (2).

From

$$A_{21}x_1 + A_{22}x_2 + B_2u = 0 \tag{5}$$

we have

$$x_2 = -A_{22}^{-1}\left(A_{21}x_1 + B_2u\right) \tag{6a}$$

and

$$\frac{\mathrm{d}^\alpha x_2}{\mathrm{d}t^\alpha} = -A_{22}^{-1}\left(A_{21}\frac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha} + B\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}\right). \tag{6b}$$

Substituting (6b) into (4) we obtain

$$\begin{bmatrix} \bar{E}_{11} & 0 \\ \bar{A}_{21} & I_{n-r} \end{bmatrix} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{10} \\ 0 \end{bmatrix} u + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}, \tag{7a}$$

where

$$\begin{aligned} &\bar{E}_{11} = E_{11} - E_{12}A_{22}^{-1}A_{21}, \\ &\bar{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \bar{A}_{21} = A_{22}^{-1}A_{21}, \\ &\bar{B}_{10} = B_1 - A_{12}A_{22}^{-1}B_2, \quad \bar{B}_{11} = E_{12}A_{22}^{-1}B_2, \quad \bar{B}_{21} = -A_{22}^{-1}B_2. \end{aligned} \tag{7b}$$

Lemma 1. *The matrix*

$$\bar{E}_{11} = E_{11} - E_{12}A_{22}^{-1}A_{21}, \tag{8}$$

is nonsingular if the assumptions (2) are satisfied.

Proof. From assumption (2) it follows that

$$\det \begin{bmatrix} E_{11} & E_{12} \\ A_{21} & A_{22} \end{bmatrix} \neq 0. \tag{9}$$

From the equality

$$\begin{bmatrix} I_r & -E_{12}A_{22}^{-1} \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} E_{11} - E_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix} \tag{10}$$

we have $\det \bar{E}_{11} \neq 0$ since (9) holds and $\det A_{22} \neq 0$.

Considering that $\det \bar{E}_{11} \neq 0$ from (7a) we obtain

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_{10} \\ \hat{B}_{20} \end{bmatrix} u + \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix} \frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \tag{11a}$$

where

$$\begin{aligned} &\hat{A}_{11} = \bar{E}_{11}^{-1}\bar{A}_{11}, \quad \hat{A}_{21} = -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{A}_{11}, \\ &\hat{B}_{10} = \bar{E}_{11}^{-1}\bar{B}_{10}, \quad \hat{B}_{20} = -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{B}_{10}, \\ &\hat{B}_{11} = \bar{E}_{11}^{-1}\bar{B}_{11}, \quad \hat{B}_{21} = \bar{B}_{21} - \bar{A}_{21}\bar{E}_{11}^{-1}. \end{aligned} \tag{11b}$$

The equation (11a) describes the dynamical part of the descriptor system and the equation (5) the static part.

Therefore, the following theorem has been proved. □

Theorem 1. *The fractional descriptor system (1) can be decomposed into its dynamical part (11) and its static part (5).*

To compute the matrices of the dynamical and static parts of the fractional descriptor system the following procedure can be used.

Procedure 1.

Step 1. Check the conditions (2).

Step 2. Using elementary column operations to (1a) find (3) and the matrices E_1, A_1, A_2, B_1, B_2.

Step 3. Using (7b) compute the matrices $\bar{E}_{11}, \bar{A}_{11}, \bar{A}_{21}, \bar{B}_{10}, \bar{B}_{11}, \bar{B}_{21}$.

Step 4. Using (11b) compute the matrices $\hat{A}_{11}, \hat{A}_{21}, \hat{B}_{10}, \hat{B}_{20}, \hat{B}_{11}, \hat{B}_{21}$ of the dynamical part. The matrices A_{21}, A_{22}, B_2 of the static part are given (5).

The procedure will be illustrated by the following numerical example.

Example 1. Consider the fractional descriptor linear system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 \\ -2 & -1 & 0 & -1 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 4 & 1 \\ -1 & 0 & -4 & 0 \\ 0 & 2 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}. \tag{12}$$

Using Procedure 1 we obtain

Step 1. In this case the conditions (2) are satisfied since

$$\det E = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 \\ -2 & -1 & 0 & -1 \\ 2 & 0 & 0 & 0 \end{vmatrix} = 0 \tag{13a}$$

and

$$\det[Es^\alpha - A] = \begin{vmatrix} s^\alpha & -1 & -2 & -1 \\ -2s^\alpha - 1 & s^\alpha - 2 & -4 & s^\alpha - 1 \\ -2s^\alpha + 1 & -s^\alpha & 4 & -s^\alpha \\ 2s^\alpha & -2 & -5 & -2 \end{vmatrix} = s^{2\alpha} + 2s^\alpha - 1. \tag{13b}$$

Step 2. Applying the elementary row operation: $L[2+1\times(-1)]$, $L[3+2\times(-1)]$, $L[4+1\times 2]$ to the matrices

$$E,\ A,\ B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 & 1 & 2 & 4 & 1 & 2 & 0 \\ -2 & -1 & 0 & -1 & -1 & 0 & -4 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 & 5 & 2 & 2 & 1 \end{bmatrix} \tag{14}$$

we obtain

$$\begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_1 \\ 0 & 0 & A_{21} & A_{22} & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

Step 3. Using (7b) and (15) we obtain

$$
\begin{aligned}
\bar{E}_{11} &= E_{11} - E_{12}A_{22}^{-1}A_{21} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix},\\
\bar{A}_{11} &= A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - \begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 2\end{bmatrix},\\
\bar{A}_{21} &= A_{22}^{-1}A_{21} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix},\\
\bar{B}_{10} &= B_1 - A_{12}A_{22}^{-1}B_2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & -2\\1 & 0\end{bmatrix},\\
\bar{B}_{11} &= E_{12}A_{22}^{-1}B_2 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\\
\bar{B}_{21} &= -A_{22}^{-1}B_2 = -\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}.
\end{aligned}
\tag{16}
$$

Step 4. Using (11b) and (16) we obtain

$$
\begin{aligned}
\hat{A}_{11} &= \bar{E}_{11}^{-1}\bar{A}_{11} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & -1\\-1 & 2\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & -2\end{bmatrix},\\
\hat{A}_{21} &= -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{A}_{11} = -\begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & -1\\-1 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\-2 & -2\end{bmatrix},\\
\hat{B}_{10} &= \bar{E}_{11}^{-1}\bar{B}_{10} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & -2\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & -2\\1 & -2\end{bmatrix},\\
\hat{B}_{20} &= -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{B}_{10} = -\begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & -2\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\-2 & 4\end{bmatrix},\\
\hat{B}_{11} &= \bar{E}_{11}^{-1}\bar{B}_{11} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\\
\hat{B}_{21} &= \bar{B}_{21} - \bar{A}_{21}\bar{E}_{11}^{-1} = \begin{bmatrix}0 & -1\\-4 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}0 & -1\\-3 & -2\end{bmatrix}.
\end{aligned}
\tag{17}
$$

Therefore, the dynamical part of the fractional descriptor system with (12) is described by the differential equation

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\hat{A}_{11} & 0\\ \hat{A}_{21} & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}\hat{B}_{10}\\ \hat{B}_{20}\end{bmatrix}u + \begin{bmatrix}\hat{B}_{11}\\ \hat{B}_{21}\end{bmatrix}\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}$$
$$= \begin{bmatrix}0 & -1 & 0 & 0\\ -1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ -2 & -2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0 & -2\\ 1 & -2\\ 0 & 0\\ -2 & 4\end{bmatrix}u + \begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & -1\\ -3 & -2\end{bmatrix}\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \tag{18}$$

and the static part by the algebraic equation

$$A_{21}x_1 + A_{22}x_2 + B_2u = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}x_1 + \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}x_2 + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}u = \begin{bmatrix}0\\0\end{bmatrix}. \tag{19}$$

3. Decomposition of the fractional descriptor linear discrete-time systems

In this section the considerations concerning the decomposition of the fractional descriptor linear system into dynamical and static parts will be extended to the fractional descriptor discrete-time systems.

Consider the fractional descriptor discrete-time linear system [6, 8]

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i\,, \quad i \in Z_+ = 0, 1, 2, \ldots, \quad 0 < \alpha < 1, \tag{20}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E,\ A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and

$$\Delta^\alpha x_i = \sum_{j=0}^{i}(-1)^j\binom{\alpha}{j}x_{i-j}\,,$$
$$\binom{\alpha}{j} = \begin{cases}1 & \text{for } j = 0,\\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots\end{cases} \tag{21}$$

is the fractional α-order difference of x_i.

Substitution of (21) into (20) yields

$$x_{i+1} = A_\alpha x_i - \sum_{j=2}^{i+1}c_j x_{i-j+1}\,, \quad i \in Z_+\,, \tag{22a}$$

where

$$A_\alpha = A + I_n\alpha. \tag{22b}$$

It is assumed that the matrices E and A of the system (20) satisfy the conditions

$$\det E = 0 \quad \text{and} \quad \det[Ez - A] \neq 0. \tag{23}$$

Using the elementary row operations to the equation (20) we obtain

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \Delta^\alpha x_i = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x_i + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, \quad i = 0, 1, \dots \tag{24a}$$

where $E_1,\ A_1 \in \Re^{r\times n}$, $B_1 \in \Re^{r\times m}$, $A_2 \in \Re^{(n-r)\times n}$, $B_2 \in \Re^{(n-r)\times m}$ and

$$\operatorname{rank} E_1 = \operatorname{rank} E = r < n. \tag{24b}$$

Applying the elementary column operations to the matrices $\begin{bmatrix} E_1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ we obtain

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \Delta^\alpha \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, \quad i = 0, 1, \dots \tag{25}$$

where $\det E_{11} \neq 0$ and $\det A_{22} \neq 0$ by the assumptions (23).

From (25) we have

$$A_{21}x_{1i} + A_{22}x_{2i} + B_2u_i = 0, \quad i = 0, 1, \dots, \tag{26}$$

$$x_{2i} = -A_{22}^{-1}(A_{21}x_{1i} + B_2u_i), \quad i = 0, 1, \dots \tag{27}$$

and

$$\Delta^\alpha x_{2i} = -A_{22}^{-1}(A_{21}\Delta^\alpha x_{1i} + B\Delta^\alpha u_i), \quad i = 0, 1, \dots \tag{28}$$

Substituting (28) into (25) we obtain

$$\begin{bmatrix} \bar{E}_{11} & 0 \\ \bar{A}_{21} & I_{n-r} \end{bmatrix} \Delta^\alpha \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} \bar{B}_{10} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \Delta^\alpha u_i, \quad i = 0, 1, \dots, \tag{29a}$$

where

$$\begin{aligned} &\bar{E}_{11} = E_{11} - E_{12}A_{22}^{-1}A_{21}, \\ &\bar{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \bar{A}_{21} = A_{22}^{-1}A_{21}, \\ &\bar{B}_{10} = B_1 - A_{12}A_{22}^{-1}B_2, \quad \bar{B}_{11} = E_{12}A_{22}^{-1}B_2, \quad \bar{B}_{21} = -A_{22}^{-1}B_2. \end{aligned} \tag{29b}$$

Lemma 2. *The matrix $\bar{E}_{11}$ defined by (29b) is nonsingular if the assumption (23) is satisfied.*

Proof. Proof is similar to the proof of Lemma 1.

Taking into account the $\det E = 0$ from (29a) we obtain

$$\Delta^\alpha \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & 0 \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} \hat{B}_{10} \\ \hat{B}_{20} \end{bmatrix} u_i + \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix} \Delta^\alpha u_i, \quad i = 0, 1, \dots, \tag{30a}$$

where

$$\begin{aligned}\hat{A}_{11} &= \bar{E}_{11}^{-1}\bar{A}_{11}, & \hat{A}_{21} &= -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{A}_{11},\\ \hat{B}_{10} &= \bar{E}_{11}^{-1}\bar{B}_{10}, & \hat{B}_{20} &= -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{B}_{10},\\ \hat{B}_{11} &= \bar{E}_{11}^{-1}\bar{B}_{11}, & \hat{B}_{21} &= \bar{B}_{21} - \bar{A}_{21}\bar{E}_{11}^{-1}.\end{aligned} \tag{30b}$$

Equations (30a) describes the dynamical part of the descriptor system and the equation (26) its static part of the system.

Therefore, the following theorem has been proved. □

Theorem 2. *The fractional descriptor linear discrete-time system (20) can be decomposed into its dynamical part (30a) and its static part (26).*

To compute the matrices of the dynamical and static parts of the fractional descriptor discrete-time system Procedure 2.1 (with some evident modifications) can be used.

Example 2. Consider the fractional descriptor linear system (20) with the matrices

$$E = \begin{bmatrix}1&2&2\\0&2&1\\1&0&1\end{bmatrix}, \quad A = \begin{bmatrix}1&1&1\\1&0&1\\0&2&1\end{bmatrix}, \quad B = \begin{bmatrix}1&1\\1&0\\1&2\end{bmatrix}. \tag{31}$$

Using Procedure 2.1 we obtain

Step 1. The conditions (23) are satisfied since

$$\det E = \begin{vmatrix}1&2&2\\0&2&1\\1&0&1\end{vmatrix} = 0 \tag{32a}$$

and

$$\det[Ez - A] = \begin{vmatrix}z-1&2z-1&2z-1\\-1&2z&z-1\\z&-2&z-1\end{vmatrix} = z^2 - 1 \tag{32b}$$

Step 2. Applying the elementary row operations: $L[1+2\times(-1)]$, $L[3+1\times(-1)]$ to the matrices

$$E,\ A,\ B = \begin{bmatrix}1&2&2&1&1&1&1&1\\0&2&1&1&0&1&1&0\\1&0&1&0&2&1&1&2\end{bmatrix} \tag{33}$$

we obtain

$$\begin{bmatrix}E_{11}&E_{12}&A_{11}&A_{12}&B_1\\0&0&A_{21}&A_{22}&B_2\end{bmatrix} = \begin{bmatrix}1&0&1&0&1&0&0&1\\0&2&1&1&0&1&1&0\\0&0&0&0&1&1&0&1\end{bmatrix}. \tag{34}$$

Step 3. Using (29b) and (34) we obtain

$$\begin{aligned}
&\bar{E}_{11} = E_{11} - E_{12}A_{22}^{-1}A_{21} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \\
&\bar{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad \bar{A}_{21} = A_{22}^{-1}A_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
&\bar{B}_{10} = B_1 - A_{12}A_{22}^{-1}B_2 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \bar{B}_{11} = E_{12}A_{22}^{-1}B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \\
&\bar{B}_{21} = -A_{22}^{-1}B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\end{aligned} \tag{35}$$

Step 4. Using (30b) and (35) we obtain

$$\begin{aligned}
&\hat{A}_{11} = \bar{E}_{11}^{-1}\bar{A}_{11} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \hat{A}_{21} = -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{A}_{11} = \begin{bmatrix} 1 & 0 \end{bmatrix}], \\
&\hat{B}_{10} = \bar{E}_{11}^{-1}\bar{B}_{10} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{B}_{20} = -\bar{A}_{21}\bar{E}_{11}^{-1}\bar{B}_{10} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \\
&\hat{B}_{11} = \bar{E}_{11}^{-1}\bar{B}_{11} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \hat{B}_{21} = \bar{B}_{21} - \bar{A}_{21}\bar{E}_{11}^{-1} = \begin{bmatrix} 0 & 0 \end{bmatrix}.
\end{aligned} \tag{36}$$

Therefore, the dynamical part of the fractional discrete-time system with (31) is described by the differential equation

$$\Delta^\alpha \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} u_i + \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Delta^\alpha u_i, \quad i = 0, 1, \ldots \tag{37}$$

and the static part by the algebraic equation

$$\begin{bmatrix} 0 & 1 \end{bmatrix} x_{1i} + x_{2i} + \begin{bmatrix} 0 & 1 \end{bmatrix} u_i = 0, \quad i = 0, 1, \ldots \tag{38}$$

4. Concluding remarks

A new method of the decomposition of the fractional descriptor linear continuous-time and discrete-time systems into dynamical and static parts has been proposed. Conditions for the decomposition of the fractional descriptor linear systems are established and procedures for decompositions of the matrices into the dynamical and static parts have been given. The procedures have been illustrated by numerical examples. Open problems are extensions of these considerations to the fractional different orders linear systems and to 2-D standard and fractional orders discrete-time linear systems.

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