10.24425/acs.2024.153106

Archives of Control Sciences
Volume 34(LXX), 2024
No. 4*, pages 881–895*

On practical h-observer design for nonlinear non-autonomous dynamical systems with disturbances

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In this paper, a particular form of practical h-observers for piecewise continuous Lipschitz, one-sided piecewise continuous Lipschitz systems and quasi-one-sided piecewise continuous Lipschitz systems is extended to nonlinear non-autonomous dynamical systems with disturbances. With the notion of practical h-stable functions, the obtained state estimates are used for an eventual feedback control, and the practical separation principle is tackled. An example is given to show the applicability of the main result.

Key words: practical h-observer, Lipschitz analysis, Lyapunov method, Practical h-stability

1. Introduction

Several studies have attempted to solve the observer design problem for nonlinear non-autonomous dynamical systems [2, 4, 6, 8, 10, 13, 18]. In the literature, the most tackled class of nonlinear system is the so-called Lipschitz class of systems. In this regard, [17] built a sufficient condition, ensuring the stability of the observers for Lipschitz systems. In practice, Lipschitz systems constitute important real systems, which has motivated the increasing attention in designing observers for Lipschitz systems. However, many existing results work only for the small Lipschitz constant. Therefore, the literature of mathematics [11] built the one-sided Lipschitz continuity for generalized Lipschitz continuity. In the same concept [1] introduced, for the nonlinear systems the quadratic inner-bonudedness, the resolution of drive tractable observer design conditions for

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Received 31.03.2024. Revised 20.11.2024.

one-sided Lipschitz. The one-sided Lipschitz property is a tool that has been used in the mathematics literature, see [9, 12, 19, 20]. However, only very few results for nonlinear one-sided Lipschitz time-varying systems exist in the literature, see Pao [15]. In [16], Pinto introduced the notion of h-stability to obtain results about stability for weakly stable systems under some perturbations. This concept was introduced for differential systems under some perturbations and extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. The authors in ([3–5, 7]) studied the asymptotic behavior of solutions in the practical sense called practical h-stability for nonlinear systems. For instance, the authors in ([3, 5]) introduced the concept of input-to-state practical h-stability (h-ISpS), integral input-to-state practical h-stability (h-iISpS), input-to-state practical partial h-stability (h-ISppS) and integral input-to-state practical partial h-stability (h-iISppS) for nonlinear time-varying systems by using a scalar practical h-stable function on the time-derivative of the Lyapunov function.
In this paper and inspired by ([4, 13]), a particular form of observers is studied for piecewise continuous Lipschitz, one-sided piecewise continuous Lipschitz systems and quasi-one-sided piecewise continuous Lipschitz systems. Then, with the help of the practical h-stable scalar functions, we give sufficient conditions to guarantee the global uniform practical h-stability of the closed-loop systems by using an estimated feedback controller via a global uniform practical h-stable observer for piecewise continuous Lipschitz.

The rest of this article is organized as follows: In Section 2, some notations, definitions and hypotheses are summarized and the system description is given. A practical h-observer design for piecewise continuous Lipschitz non-autonomous systems is given in Section 3. In Section 4, we present a practical h-observer design for one-sided piecewise continuous Lipschitz non-autonomous systems. A practical h-observer design for quasi-one-sided piecewise continuous Lipschitz non-autonomous systems is given in Section 5. Consequently, a practical separation principle is established in Section 6. A numerical example is provided to show the efficiency of the proposed approach in Section 7. Finally, our conclusion is presented in Section 8.

2. Preliminaries

Throughout this paper, we denote by

- $\mathbb{R}_+$ denotes the set of all non-negative real numbers and $\mathbb{R}_+^* =]0, \infty)$.
- $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, with the scalar product $\langle \cdot, \cdot \rangle$.
- $\mathrm{PC}(\mathbb{R}_+, \Delta)$ is the space of Δ-valued piecewise continuous functions.

- $\mathrm{PC}^1(\mathbb{R}_+, \Delta)$ is the space of Δ-valued piecewise continuous differentiable functions.
- $\mathrm{BC}(\mathbb{R}_+, \Delta)$ is the space of Δ-valued bounded functions endowed with the norm $\|\phi\|_\infty = \sup\limits_{t\in I} |\phi(t)|$.
- $\mathrm{C}^1(\Delta, \Omega)$ is the space of continuously differentiable functions from Δ to Ω.

Consider the nonlinear system

$$\begin{cases} \dot{x}(t) = \Phi(t, x(t)), & t \geqslant t_0 \geqslant 0, \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, t_0 is the initial time, x_0 is the initial condition and $\Phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in state and piecewise continuous in time.

The unique solution to equation (1), passing through $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, is denoted by $\phi(\cdot, t_0, x_0)$, which satisfies $\phi(t, t_0, x_0) = x_0$. We present the following definition, which is recently introduced in [7].

Definition 1. *Let $h \in \mathrm{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$. System (1) is globally uniformly practically h-stable if there are $S \geqslant 1$ and $\varrho > 0$, such that for all $x_0 \in \mathbb{R}^n$, all $t_0 \geqslant 0$, and all $t \geqslant t_0$ it holds that*

$$\|\phi(t, t_0, x_0)\| \leqslant S\|x_0\| h(t) h(t_0)^{-1} + \varrho. \tag{2}$$

Definition 2. *[4] Let $h \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star) \cap \mathrm{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$. The practical h-stable function pair is defined by a 2− tuple (χ, ξ) where $\chi, \xi \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R})$ and there exist $\zeta \geqslant 0, \kappa > 0$ such that*

$$\int_{t_0}^{t} \chi(\varsigma) \mathrm{d}\varsigma \leqslant -\ln(h(t_0)) + \ln(h(t)) + \zeta,$$

and

$$\int_{t_0}^{t} \Theta(t, \varsigma) |\xi(\varsigma)| \mathrm{d}\varsigma \leqslant \kappa$$

hold where $\Theta(t, \varsigma) = e^{\int_\varsigma^t \chi(s)\mathrm{d}s}$.

Let's denote the set of practical h-stable functions by $\mathcal{P}h\mathcal{SFP}$.

Definition 3. *We define the Dini derivative or the upper right-hand generalized derivative of a function $\mathcal{V}(t, x)$ along solutions of (1) by:*

$$D^+\mathcal{V}(t, x) = \limsup_{\hbar \to 0^+} \left\{ \frac{1}{\hbar} [\mathcal{V}(t + \hbar, x + \hbar\Phi(t, x)) - \mathcal{V}(t, x)] \right\}, \tag{3}$$

with $\mathcal{V}(t,x)$ satisfies the Lipschitz condition with respect the variable x uniformly in t, i.e.;

$$|\mathcal{V}(t,\widetilde{x}) - \mathcal{V}(t,\widetilde{y})| \leqslant C|\widetilde{x} - \widetilde{y}|.$$

If $\mathcal{V}(t,x)$ has a continuous partial derivative with respect to the first variable, then along the solution of (1) we get

$$D^+\mathcal{V}(t,x) = \frac{\partial \mathcal{V}}{\partial x}(t,x)\Phi(t,x) + \frac{\partial \mathcal{V}}{\partial t}(t,x).$$

Lemma 1 (see [21]).
Let $\vartheta, \psi \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R})$ and $\varpi \in \mathrm{PC}^1(\mathbb{R}_+,\mathbb{R})$, such that $\forall t \geqslant t_0$,

$$D^+\varpi(t) \leqslant \vartheta(t)\varphi(t) + \psi(t).$$

Then, $\forall t \geqslant t_0$, we have

$$\varpi(t) \leqslant \varpi(t_0)e^{\int_{t_0}^{t}\vartheta(\tau)\mathrm{d}\tau} + \int_{t_0}^{t} e^{\int_{s}^{t}\vartheta(\tau)\mathrm{d}\tau}\psi(s)\mathrm{d}s.$$

Let's consider the following theorem.

Theorem 1 (see [4]). *Consider the decreasing function $h \in \mathrm{C}^1(\mathbb{R}_+,\mathbb{R}_+^\star)$ and there exist $\mathcal{V} \in \mathrm{C}^1(\mathbb{R}_+ \times \mathbb{R}^n,\mathbb{R}_+)$, $c_1 > 0$, $c_2 \geqslant 0$, $a \geqslant 0$, and $\chi, \xi \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R}) \cap \mathcal{P}h\mathcal{SFP}$, such that for all $t \geqslant 0$ and all $x \in \mathbb{R}^n$,*

$$c_1\|x\|^2 \leqslant \mathcal{V}(t,x) \leqslant c_2\|x\|^2 + a,$$

$$D^+\mathcal{V}(t,x) \leqslant \chi(t)\mathcal{V}(t,x) + \xi(t),$$

then, system (1) is globally uniformly practically $h^{\frac{1}{2}}$-stable.

We consider the following non-autonomous control system with disturbances:

$$\begin{cases} \dot{x}(t) = \mathcal{B}(t)u(t) + \mathcal{A}(t)x(t) + \mathcal{D}(t)d(t) + \Xi(t,x(t),u(t)), \\ y(t) = C(t)x(t), \end{cases} \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$ and $d(t) \in \mathbb{R}^p$ represent the state, the input, the output and the measurable and locally essentially bounded disturbance for the above system. $\mathcal{A}(t) \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R}^{n\times n})$, $\mathcal{B}(t) \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R}^{n\times m})$, $C(t) \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R}^{q\times n})$, $\mathcal{D}(t) \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R}^{m\times p})$ and the function $\Xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in state and piecewise continuous in time with $\Xi(t,0,u) = 0$ for all $t \geqslant 0$.

3. Practical h-observer design for piecewise continuous Lipschitz non-autonomous systems

We shall suppose the following hypothesis.

(C_1) There is a function $\mu \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R}_+)$ such that for all $x_1, x_2 \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\|\Xi(t, x_1, u) - \Xi(t, x_2, u)\| \leqslant \mu(t)\|x_1 - x_2\|. \tag{5}$$

(C_2) There are $\mathcal{L}(t) \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R}^{m\times n}) \cap \mathrm{BC}(\mathbb{R}_+, \mathbb{R}^{m\times n})$, $\mathcal{P}(t) = \mathcal{P}^T(t) \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, two constants $p_2 > p_1 > 0$ and $\nu \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R})$, such that for all $t \in \mathbb{R}_+$, we have

$$\dot{\mathcal{P}}(t) + \mathcal{P}(t)\mathcal{A}_{\mathcal{L}}(t) + \mathcal{A}_{\mathcal{L}}^T(t)\mathcal{P}(t) \leqslant \nu(t)\mathcal{P}(t), \tag{6}$$

$$p_1 I \leqslant \mathcal{P}(t) \leqslant p_2 I,$$

where $A_{\mathcal{L}}(t) = \mathcal{L}(t)C(t) + \mathcal{A}(t)$, $\mathcal{A}_{\mathcal{L}}^T$ the transpose of the matrix $A_{\mathcal{L}}^T$ and $\mathcal{I}$ is the identity matrix.

Consider the system (4) having some state variables not available for direct measurement. Under assumption (C_2), one proposes the following observer, which estimates the state.

$$\begin{cases} \dot{\hat{x}}(t) = \mathcal{B}(t)u(t) + \mathcal{A}(t)\hat{x}(t) + \Xi(t, \hat{x}(t), u(t)) \\ \qquad\quad -\mathcal{L}(t)(\hat{y}(t) - y(t)), \quad t \geqslant t_0, \\ \hat{y}(t) = C(t)\hat{x}(t). \end{cases} \tag{7}$$

Now, we present the following theorem that ensures the practical h-stability of the proposed observer with the Lipschitz condition (5).

Theorem 2. *Consider the decreasing function $h \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ and there exist $\mathcal{V} \in \mathrm{C}^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$. Under conditions (C_1) and (C_2), the system (7) is a global uniform $h^{\frac{1}{2}}$-stable observer for the system (1) if $\left(\dfrac{\nu(t)}{2} + \dfrac{p_2}{p_1}\mu(t), \|D(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$.*

Proof. Let's consider the error equation represented by $\epsilon = x - \hat{x}$,

$$\dot{\epsilon}(t) = (\mathcal{L}(t)C(t) + \mathcal{A}(t))\epsilon(t) + \Delta\Xi + \mathcal{D}(t)d(t), \tag{8}$$

where $\Delta\Xi = \Xi(t, x, u) - \Xi(t, \hat{x}, u)$. Choose

$$\mathcal{V}(t, \epsilon) = \langle \mathcal{P}(t)\epsilon, \epsilon \rangle.$$

The derivative of $\mathcal{V}$ along the trajectories of system (8) is given by:

$$\begin{aligned}\dot{\mathcal{V}}(t,\epsilon) &= \langle \mathcal{P}(t)\dot{\epsilon},\ \epsilon\rangle + \langle \mathcal{P}(t)\epsilon,\ \dot{\epsilon}\rangle + \langle \dot{\mathcal{P}}(t)\epsilon,\ \epsilon\rangle \\ &= \langle \dot{\mathcal{P}}(t)\epsilon,\ \epsilon\rangle + \langle \mathcal{P}(t)(\mathcal{A}_{\mathcal{L}}(t)\epsilon + \Delta\Xi + \mathcal{D}(t)d),\ \epsilon\rangle \\ &\quad + \langle \mathcal{P}(t)\epsilon,\ (\mathcal{A}_{\mathcal{L}}(t)\epsilon + \Delta\Xi + \mathcal{D}(t)d)\rangle.\end{aligned}$$

By applying the equation (6) and using Cauchy-Schwartz, we obtain

$$\begin{aligned}\dot{\mathcal{V}}(t,\epsilon) &\leqslant \nu(t)\mathcal{V}(t,\epsilon) + 2\|\mathcal{P}(t)\|\|\Delta\Xi\|\|\epsilon\| + 2\|\mathcal{P}(t)\|\|\mathcal{D}(t)\|\|d\|_\infty\|\epsilon\| \\ &\leqslant (\nu(t) + \frac{2p_2}{p_1}\mu(t))\mathcal{V}(t,\epsilon) + 2\frac{p_2}{\sqrt{p_1}}\|\mathcal{D}(t)\|\|d\|_\infty\mathcal{V}(t,\epsilon)^{\frac{1}{2}}.\end{aligned}$$

Set $\ell(t) = \mathcal{V}(t,\epsilon)^{\frac{1}{2}}$. The Dini derivative of ℓ verifies

$$D^+\ell(t) \leqslant \left[\frac{\nu(t)}{2} + \frac{p_2}{p_1}\mu(t)\right]\ell(t) + \frac{p_2}{\sqrt{p_1}}\|\|\mathcal{D}(t)\|\|d\|_\infty.$$

Thus, by using Theorem 1, the system (8) is globally uniformly practically $h^{\frac{1}{2}}$-stable if $\left(\frac{\nu(t)}{2} + \frac{p_2}{p_1}\mu(t), \|D(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$. Hence, the system (7) is a global uniform practical $h^{\frac{1}{2}}$-stable observer for the system (4). □

Remark 1. *If $\mu(t) = c$, (constant), then it is easy to verify that the system (7) is a global uniform practical $h^{\frac{1}{2}}$-stable observer for the system (4) if* $\left(\frac{\nu(t)}{2} + \frac{p_2}{p_1}c, \|\mathcal{D}(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$.

4. Practical h-observer design for one-sided piecewise continuous Lipschitz non-autonomous systems

The goal is to design a practical h-observer for one-sided piecewise continuous Lipschitz non-autonomous systems. The advantage gained through this approach is that the broad family on nonlinear systems includes the piecewise continuous Lipschitz systems as a special case.

First, consider the one-sided piecewise continuous Lipschitz class of systems [15] by the following assumption:

(C_3) The function $\Xi(t,x,u)$ is one-sided Lipshitz in $\mathbb{R}^n$ with a Lipschiz function $\gamma \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R})$, that is

$$\begin{aligned}\langle \mathcal{P}(t)\Xi(t,x,u) - \mathcal{P}(t)\Xi(t,\hat{x},u),\ x-\hat{x}\rangle \leqslant \gamma(t)\|x-\hat{x}\|^2, \\ \forall x,\hat{x} \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^m,\end{aligned} \tag{9}$$

where $\mathcal{P}(t) \in \mathrm{C}^1(\mathbb{R}_+,\mathbb{R}^{n\times n})$ is the known matrix given by (C_2).

Remark 2. *If Ξ is Lipshitz continuous in x which satisfies hypothesis (C_1), then (5) is satisfied since*

$$\begin{aligned}\langle \mathcal{P}(t)\Xi(t,x,u)-\mathcal{P}(t)\Xi(t,\hat{x},u),\ x-\hat{x}\rangle &\leqslant \|\mathcal{P}(t)\|\|\Xi(t,x,u)-\Xi(t,\hat{x},u)\|\|x-\hat{x}\|\\ &\leqslant p_2\gamma(t)\|x-\hat{x}\|^2,\end{aligned}$$

where p_2 is a positive constant given in (C_2). In this case $p_2\gamma(t)$ must be non-negative on $\mathbb{R}_+$.

Note that, unlike the piecewise continuous Lipschitz function which must be positive, the one-sided Lipschitz function $\gamma(t)$ may be positive or even negative.

We shall now illustrate an example that satisfies one-sided piecewise continuous Lipschitz but not piecewise-continuous Lipschitz systems.

Example 1. A simple example of a one-sided piecewise continuous Lipschitz function which indeed is not piecewise continuous Lipschitz is $\Xi(t,x) = -\mathrm{sgn}(x)\gamma(t)\sqrt{|x|}$, where $\gamma \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R}), x \in \mathbb{R}$ and 'sgn' denote the sign function.

Theorem 3. *Consider the decreasing function $h \in \mathrm{C}^1(\mathbb{R}_+,\mathbb{R}_+^\star)$. Under conditions (C_1) and (C_3), the system (7) is a global uniform $h^{\frac{1}{2}}$-stable observer for the system (1) if $\left(\dfrac{\nu(t)}{2}+\dfrac{|\gamma(t)|}{p_1}, \|D(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$.*

Proof. Consider

$$\mathcal{V}(t,\epsilon) = \langle \mathcal{P}(t)\epsilon,\epsilon\rangle,$$

then,

$$\begin{aligned}\dot{\mathcal{V}}(t,\epsilon) &= \langle \mathcal{P}(t)\dot{\epsilon},\epsilon\rangle + \langle \mathcal{P}(t)\epsilon,\dot{\epsilon}\rangle + \langle \dot{\mathcal{P}}(t)\epsilon,\epsilon\rangle\\ &= \langle \dot{\mathcal{P}}(t)\epsilon,\epsilon\rangle + \langle \mathcal{P}(t)(\mathcal{A}_{\mathcal{L}}(t)\epsilon+\Delta\Xi+\mathcal{D}(t)d),\epsilon\rangle\\ &\quad + \langle \mathcal{P}(t)\epsilon,(\mathcal{A}_{\mathcal{L}}(t)\epsilon+\Delta\Xi+\mathcal{D}(t)d)\rangle.\end{aligned}$$

Using (9), we have

$$\begin{aligned}\dot{\mathcal{V}}(t,\epsilon) &\leqslant \nu(t)\mathcal{V}(t,\epsilon)+2\gamma(t)\|\epsilon\|^2+2\|\mathcal{P}(t)\|\mathcal{D}(t)\|\|d\|_\infty\|\epsilon\|\\ &\leqslant (\nu(t)+\frac{2|\gamma(t)|}{p_1})\mathcal{V}(t,\epsilon)+2\frac{p_2}{\sqrt{p_1}}\|\mathcal{D}(t)\|\|d\|_\infty\mathcal{V}(t,\epsilon)^{\frac{1}{2}}.\end{aligned}$$

Let $\rho(t) = \sqrt{\mathcal{V}(t,e)}$. The Dini derivative of ρ satisfies

$$D^+\rho(t) \leqslant \left[\frac{\nu(t)}{2}+\frac{|\gamma(t)|}{p_1}\right]\rho(t)+\frac{p_2}{\sqrt{p_1}}\|\|\mathcal{D}(t)\|\|d\|_\infty.$$

Thus, by using Theorem 1, the system (8) is globally uniformly practically $h^{\frac{1}{2}}$-stable if $\left(\frac{\nu(t)}{2}+\frac{|\gamma(t)|}{p_1}, \|\mathcal{D}(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$. Consequently, the system (7) is a global uniform practical $h^{\frac{1}{2}}$-stable observer for the system (4). □

5. Practical h-observer design for quasi-one-sided piecewise continuous Lipschitz non-autonomous systems

A quasi-one-sided piecewise continuous Lipschitz condition is introduced instead of the piecewise continuous Lipschitz condition (5) and the one-sided piecewise continuous Lipschitz condition (9).

(C_4) $\Xi(t,x,u)$ is quasi-one-sided piecewise continuous Lipschitz in $\mathbb{R}^n$ with a one-sided Lipschitz constant matrix and function $\delta \in \mathrm{PC}(\mathbb{R}_+,\mathbb{R})$, that is

$$\begin{aligned}\langle \mathcal{P}(t)\Xi(t,x,u)-\mathcal{P}(t)\Xi(t,\hat{x},u), x-\hat{x}\rangle &\leqslant \delta(t)(x-\hat{x})^T M(x-\hat{x}),\\ &\forall x,\hat{x}\in\mathbb{R}^n,\quad \forall u\in\mathbb{R}^m,\end{aligned} \tag{10}$$

where $\mathcal{P}(t) \in \mathrm{C}^1(\mathbb{R}_+,\mathbb{R}^{n\times n})$ is the known matrix given by (C_2) and M is a real symmetric matrix.

Theorem 4. *Consider the decreasing function $h \in \mathrm{C}^1(\mathbb{R}_+,\mathbb{R}_+^\star)$. Under conditions (C_1) and (C_4), the system (7) is a global uniform $h^{\frac{1}{2}}$-stable observer for the system (1) if $\left(\frac{\nu(t)}{2}+\frac{\lambda_{\max}(M)|\delta(t)|}{p_1}, \|\mathcal{D}(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$.*

Proof. Let $\epsilon(t)=x(t)-\hat{x}(t)$, $\mathcal{V}(t,\epsilon)=\langle\mathcal{P}(t)\epsilon,\epsilon\rangle$ and similarly to the proof of theorem 3, we find

$$\dot{\mathcal{V}}(t,\epsilon) \leqslant \nu(t)\mathcal{V}(t,\epsilon)+2\langle\mathcal{P}(t)\Delta\Xi,\epsilon\rangle+2p_2\|\mathcal{D}(t)\|\|d\|_\infty.$$

Then using (10), we have

$$\dot{\mathcal{V}}(t,\epsilon) \leqslant [\nu(t)+\frac{2\lambda_{\max}(M)|\delta(t)|}{p_1}]\mathcal{V}(t,\epsilon)+2\frac{p_2}{\sqrt{p_1}}\|\mathcal{D}(t)\|\|d\|_\infty\sqrt{\mathcal{V}(t,\epsilon)}.$$

Let $\zeta(t)=\sqrt{\mathcal{V}(t,\epsilon)}$. Thus,

$$D^+\zeta(t) \leqslant \left[\frac{\nu(t)}{2}+\frac{\lambda_{\max}(M)|\delta(t)|}{p_1}\right]\zeta(t)+\frac{p_2}{\sqrt{p_1}}\|\mathcal{D}(t)\|\|d\|_\infty.$$

Applying Theorem 1, we obtain that the system (8) is globally uniformly practically $h^{\frac{1}{2}}$-stable if $\left(\frac{\nu(t)}{2}+\frac{\lambda_{\max}(M)|\delta(t)|}{p_1}, \|\mathcal{D}(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$. Therefore, the system (7) is a global uniform practical $h^{\frac{1}{2}}$-stable observer for the system (4). □

6. Practical separation Principle

Here, we investigate the separation principle problem for a class of nonlinear non-autonomous systems of the form (4). First, assume that (4) satisfies the following condition required for stabilization purposes.

(C_5) There exist $\mathcal{K}(t) \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R}^{m\times n}) \cap \mathrm{BC}(\mathbb{R}_+, \mathbb{R}^{m\times n})$, $\widetilde{\mathcal{P}}(t) = \widetilde{\mathcal{P}}^T(t) \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $\widetilde{p}_2 > \widetilde{p}_1 > 0$ and $\omega \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that $\forall t \in \mathbb{R}_+$, we have

$$\dot{\widetilde{\mathcal{P}}}(t) + \mathcal{A}_{\mathcal{K}}^T(t)\widetilde{\mathcal{P}}(t) + \widetilde{\mathcal{P}}(t)\mathcal{A}_{\mathcal{K}}(t) \leqslant \omega(t)\widetilde{\mathcal{P}}(t), \tag{11}$$

$$\widetilde{p}_1 I \leqslant \widetilde{\mathcal{P}}(t) \leqslant \widetilde{p}_2 I,$$

where $\mathcal{A}_{\mathcal{K}}(t) = \mathcal{B}(t)\mathcal{K}(t) + \mathcal{A}(t)$.

Based on these conditions, we show the following lemma.

Lemma 2. *Consider the decreasing function $h \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$. Under conditions (C_1) and (C_5), the system (4) in closed-loop with the linear feedback $u(t,x) = \mathcal{K}(t)x$ is globally uniformly practically $h^{\frac{1}{2}}$-stable if $\left(\frac{\omega(t)}{2} + \frac{\widetilde{P}_2}{\widetilde{P}_1}\mu(t), \|D(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$.*

Proof. Let

$$\mathcal{W}(t,x) = \langle \widetilde{\mathcal{P}}(t)x, x\rangle.$$

The derivative of $\mathcal{W}$ along the trajectories of system (4) is given by

$$\begin{aligned}
\dot{\mathcal{W}}(t,x) &= \langle \dot{\widetilde{\mathcal{P}}}(t)x, x\rangle + \langle \widetilde{\mathcal{P}}(t)\dot{x}, x\rangle + \langle \widetilde{\mathcal{P}}(t)x, \dot{x}\rangle \\
&= \langle \dot{\widetilde{\mathcal{P}}}(t)x, x\rangle + \langle \widetilde{\mathcal{P}}(t)\mathcal{A}_{\mathcal{K}}(t)x + \Xi(t,x,u), x\rangle + \langle \widetilde{\mathcal{P}}(t)x, \mathcal{A}_{\mathcal{K}}(t)x + \Xi(t,x,u)\rangle \\
&\leqslant \omega(t)\mathcal{W}(t,x) + 2\|\widetilde{\mathcal{P}}(t)\|\mu(t)\|x\|^2 + 2\widetilde{\mathcal{P}}_2\|\mathcal{D}(t)\|\|d\|_\infty\|x\| \\
&\leqslant \left(\omega(t) + \frac{2\widetilde{\mathcal{P}}_2}{\widetilde{\mathcal{P}}_1}\mu(t)\right)\mathcal{W}(t,x) + \frac{2\widetilde{\mathcal{P}}_2}{\sqrt{\widetilde{\mathcal{P}}_1}}\|\mathcal{D}(t)\|\|d\|_\infty\mathcal{W}(t,x)^{\frac{1}{2}}.
\end{aligned}$$

Let $\pi(t) = \mathcal{W}(t,x)^{\frac{1}{2}}$. Then,

$$D^+\pi(t) \leqslant \left(\frac{\omega(t)}{2} + \frac{\widetilde{P}_2}{\widetilde{P}_1}\mu(t)\right)\pi(t) + \frac{\widetilde{P}_2}{\sqrt{\widetilde{P}_1}}\|D(t)\|\|d\|_\infty.$$

Thus, by using Theorem 1, the system (4) in closed-loop with the feedback $u(t,x) = \mathcal{K}(t)x$ is globally uniformly practically $h^{\frac{1}{2}}$-stable if $\left(\frac{\omega(t)}{2} + \frac{\widetilde{P}_2}{\widetilde{P}_1}\mu(t), \|D(t)\|\right) \in \mathcal{P}h\mathcal{SFP}$. $\square$

Now, consider system (4) controlled by the feedback law $u(t,\hat{x}) = \mathcal{K}(t)\hat{x}$ estimated by the observer (7). Then, we provide the following theorem.

Theorem 5. *Consider the decreasing function $h \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ such that the function $h(t) = \hat{h}(t)e^{\eta t} \in \mathrm{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$ with $\eta > 0$ and the function $\|\mathcal{D}(\cdot)\|\hat{h}^{-1} \in L^q([0,\infty))$ with $q > 1$. Under conditions (C_1), (C_2) and (C_4), the system*

$$\begin{cases} \dot{\hat{x}}(t) = \mathcal{B}(t)u(t,\hat{x}) + \mathcal{A}(t)\hat{x}(t) + \Xi(t,\hat{x},u(t,\hat{x})) + \mathcal{L}(t)C(t)\epsilon(t), \\ \dot{\epsilon}(t) = \big(\mathcal{L}(t)C(t) + \mathcal{A}(t)\big)\epsilon(t) + \Xi(t,\hat{x}(t),u(t,\hat{x})) \\ \qquad\quad -\Xi(t,\hat{x}(t)-\epsilon,u(t,\hat{x})) + \mathcal{D}(t)d(t) \end{cases} \tag{12}$$

is globally uniformly practically $h^{\frac{1}{2}}$-stable if $\left(\frac{1}{2}(\varphi(t) + \lambda\mu(t)), \|D(\cdot)\|\right) \in \mathcal{P}h\mathcal{SFP}$, where $\varphi(t) = \max(w(t), v(t))$, $\lambda = \max\left(\frac{2p_2}{p_1}, \frac{2\widetilde{\mathcal{P}}_2}{\widetilde{\mathcal{P}}_1}\right)$, with $u(t,\hat{x}) = \mathcal{K}(t)\hat{x}$.

Proof. Set

$$\mathcal{Z}(t,\hat{x},\epsilon) = \mathcal{W}(t,\hat{\epsilon}) + \mathcal{V}(t,\epsilon),$$

where $\mathcal{W}(t,\hat{x}) = \langle\widetilde{\mathcal{P}}(t)\hat{x},\hat{x}\rangle$ and $\mathcal{V}(t,\epsilon) = \langle\mathcal{P}(t)\epsilon,\epsilon\rangle$. The derivative of $\mathcal{Z}$ along the trajectories of system (12) satisfies

$$\begin{aligned} \dot{\mathcal{Z}}(t,\hat{x},\epsilon) \leqslant{}& (\omega(t) + \frac{2\widetilde{\mathcal{P}}_2}{\widetilde{\mathcal{P}}_1}\mu(t))\mathcal{W}(t,\hat{x}) + \left(\nu(t) + \frac{2p_2}{p_1}\mu(t)\right)\|\mathcal{V}(t,\epsilon) \\ &+ 2\frac{p_2}{\sqrt{p_1}}\|\mathcal{D}(t)\|\|d\|_\infty\sqrt{\mathcal{V}(t,\epsilon)} + 2\|\widetilde{\mathcal{P}}(t)\|\|\mathcal{L}(t)C(t)\epsilon(t)\|\|\hat{x}(t)\|. \end{aligned}$$

For any $\zeta \in \mathbb{R}_+^\star$, by applying Young's inequality

$$\|\epsilon(t)\|\|\hat{x}(t)\| \leqslant \frac{1}{2\epsilon}\|\epsilon(t)\|^2 + \frac{\epsilon}{2}\|\hat{x}(t)\|^2.$$

Considering that for all $\lambda_1, \lambda_2 \in \mathbb{R}_+$, $\sqrt{\lambda_1} + \sqrt{\lambda_2} \leqslant 2\sqrt{\lambda_1 + \lambda_2}$, we state

$$\dot{\mathcal{Z}}(t,\hat{x},\epsilon) \leqslant (\varphi(t) + \lambda\mu(t) + \varsigma)\mathcal{Z}(t,\hat{x},\epsilon) + 2\varrho\|\mathcal{D}(t)\|\sqrt{Z(t,\hat{x},\epsilon)},$$

where $\varsigma = \max\left(\frac{\widetilde{P}_2\|LC\|_\infty}{\zeta P_1}, \frac{\widetilde{P}_2\zeta\|LC\|_\infty}{\widetilde{P}_1}\right)$ and $\varrho = \left(\frac{p_2}{\sqrt{p_1}}\|d\|_\infty\right)$.

Let

$$\varpi(t) = \sqrt{\mathcal{Z}(t,\hat{x},\epsilon)}.$$

Then,

$$D^+\varpi(t) \leqslant \bar{\chi}(t)\varpi(t) + \widetilde{\xi}(t),$$

where $\widetilde{\chi}(t) = \frac{1}{2}(\varphi(t) + \lambda\mu(t) + \varsigma)$ and $\widetilde{\xi}(t) = \varrho\|\mathcal{D}(t)\|$. Since, $\dfrac{1}{2}(\varphi(t) + \lambda\mu(t)), \|\mathcal{D}(t)\| \in \mathcal{P}h\mathcal{SFP}$, then there exists $k > 0$, such that

$$\begin{aligned}
\int_{t_0}^{t} \widetilde{\chi}(\tau)\,\mathrm{d}\tau &\leqslant \int_{t_0}^{t} \hat{h}'(\tau)\hat{h}^{-1}(\tau)\,\mathrm{d}\tau + \frac{\varsigma}{2}(t - t_0) + k \\
&\leqslant \ln(\hat{h}(t)) - \ln(\hat{h}(t_0)) + \ln\left(e^{\frac{\varsigma}{2}t}\right) - \ln\left(e^{\frac{\varsigma}{2}t_0}\right) + k \\
&\leqslant -\ln\left(\hat{h}(t_0)e^{\frac{\varsigma}{2}t_0}\right) + \ln\left(\hat{h}(t)e^{\frac{\varsigma}{2}t}\right) + k \\
&\leqslant -\ln(h(t_0))\ln(h(t)) + k.
\end{aligned}$$

Now, let's

$$\Theta(t, \tau) = e^{\int_{\tau}^{t} \widetilde{\chi}(s)\,\mathrm{d}s}.$$

We have,

$$\int_{t_0}^{t} \Theta(t, \tau)|\widetilde{\xi}(\tau)|\,\mathrm{d}\tau = e^{k} \int_{t_0}^{t} \varrho\widetilde{h}(t)e^{\frac{\varsigma}{2}t}\hat{h}^{-1}(\tau)e^{-\frac{\varsigma}{2}\tau}\|\mathcal{D}(\tau)\|\,\mathrm{d}\tau.$$

It follows that

$$\int_{t_0}^{t} \Theta(t, \tau)|\widetilde{\xi}(\tau)|\,\mathrm{d}\tau \leqslant e^{k}\|h\|_{\infty} \int_{t_0}^{t} \varrho\hat{h}^{-1}(\tau)e^{-\frac{\varsigma}{2}\tau}\|\mathcal{D}(\tau)\|\,\mathrm{d}\tau,$$

Hence, by applying the fact that $\hat{h}^{-1}\|\mathcal{D}(\cdot)\| \in L^{q}([0, \infty))$ one obtains

$$\int_{t_0}^{t} \Theta(t, \tau)|\widetilde{\xi}(\tau)\,\mathrm{d}\tau \leqslant \frac{2\varrho e^{k}(q-1)\|h\|_{\infty}}{\varsigma q}\|\mathcal{D}(\cdot)\|\widetilde{h}^{-1}\|_{q}\,.$$

Thus, $(\bar{\chi}, \widetilde{\xi}) \in\in \mathcal{P}h\mathcal{SFP}$. Consequently, the cascaded system (12) is globally uniformly practically $h^{\frac{1}{2}}$-stable. □

7. Example

Consider a class of control systems in the form (4) with

$$\mathcal{A}(t)=\begin{pmatrix}-\dfrac{2+t}{2(1+t)} & 0\\ 0 & -\dfrac{2+t}{2(1+t)}+\dfrac{1}{2(1+t^2)}\end{pmatrix},\quad \mathcal{B}(t)=\begin{pmatrix}\dfrac{1}{2(1+t^2)}\\ 0\end{pmatrix},$$

$$\mathcal{D}(t)=\begin{pmatrix}e^{-t}\\ 0\end{pmatrix},\quad d(t)=\arctan(t),$$

$$C(t)=\begin{pmatrix}1 & 0\end{pmatrix},\quad \Xi(t,x,u)=\begin{pmatrix}e^{-t}\cos(u)\sin(x_2)\\ \sin(x_1)\end{pmatrix}.$$

Condition (C_5) is satisfied with

$$\mathcal{K}(t)=\begin{pmatrix}1 & 0\end{pmatrix},\quad \widetilde{\mathcal{P}}(t)=I$$

and

$$\omega(t)=\frac{1}{1+t^2}-\frac{2+t}{1+t}.$$

Moreover, condition (C_1) is verified with $\mu(t)=e^{-t}$.

On the other hand, it is easy to verify that condition (C_2) is satisfied with

$$\mathcal{L}(t)=\begin{pmatrix}\dfrac{1}{1+t^2}\\ 0\end{pmatrix},\quad \mathcal{P}(t)=I$$

and

$$\nu(t)=\frac{2}{1+t^2}-\frac{2+t}{1+t}.$$

It is easy to see that $\left(\dfrac{|\omega(t)|}{2}+\lambda\mu(t)+\dfrac{|\nu(t)|}{2},\ \|D(t)\|\right)\in\mathcal{P}\hat{h}\mathcal{S}\mathcal{F}\mathcal{P}$ with $\hat{h}(t)=\dfrac{e^{-t}}{1+t}$. Then, by applying Theorem 5, the cascaded system (12) is globally uniformly practically $h^{\frac{1}{2}}$-stable with $h(t)=\dfrac{1}{1+t}$.

By using the initial conditions $(x_1(0),x_2(0))=(1,1)$ and $(\hat{x}_1(0),\hat{x}_2(0))=(1,1)$. Simulation results for the estimated and actual states are shown in Figures 1 and 2.

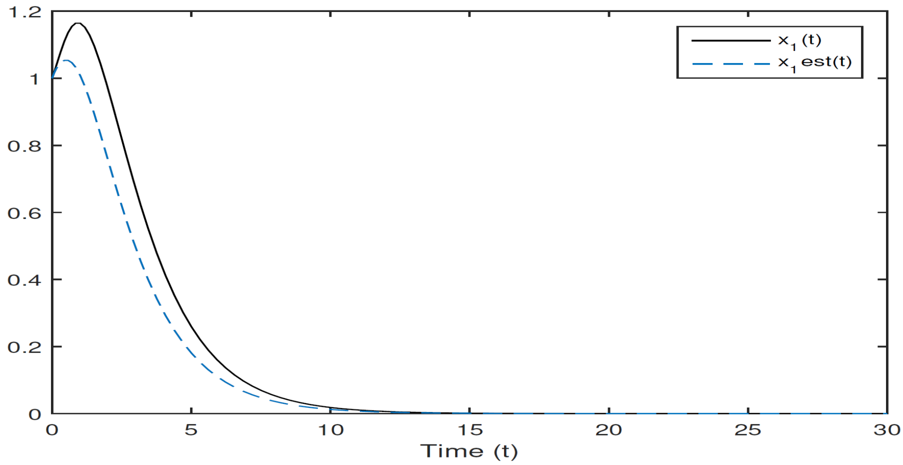

Figure 1: x_1 and its estimated $\widehat{x}_1$

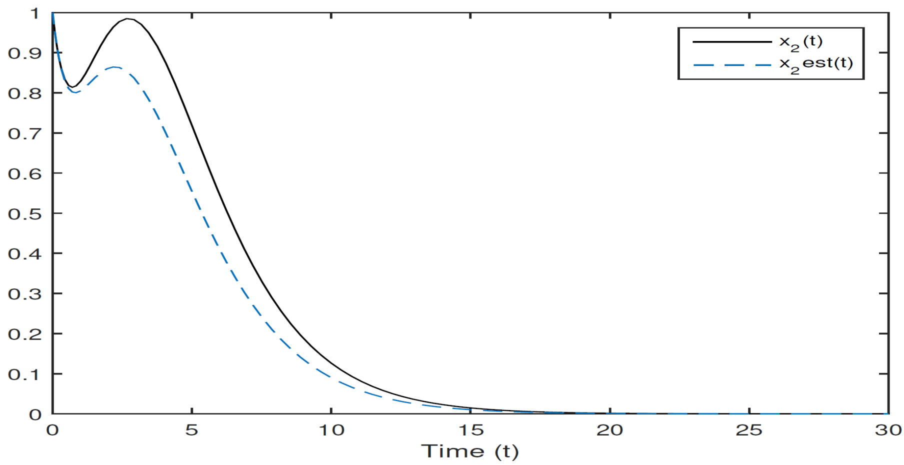

Figure 2: x_2 and its estimated $\widehat{x}_2$

8. Conclusions

In this paper, a new way to design the observers for nonlinear non-autonomous dynamical systems with disturbances is presented. It concerns the cases of nonlinearity that either meets a piecewise continuous Lipschitz condition, one-sided piecewise continuous Lipschitz or simply quasi-one-sided piecewise continuous Lipschitz. Some results are obtained, and the observer can therefore be designed under some sufficient conditions with the help of the notion of practical h-stable functions. Furthermore, an illustrative example is given.

References

[1] M. Abbaszadeh and H.J. Marquez: Nonlinear observer design for one-sided Lipschitz systems. *Proceedings of the American Control Conference*, Baltimore, USA, (2010), 5284-5289. DOI: 10.1109/ACC.2010.5530715

[2] I. Akrouti and N. Echi: Global practical exponential stabilization for one-sided Lipschitz systems with time delay. *Journal of Systems Science and Complexity*, **35**(3), (2022), 2029-2045. DOI: 10.1007/s11424-022-1061-4

[3] H. Damak, N. Hadj Taieb and M.A. Hammami: On input-to-state practical h-stability for nonlinear time-varying systems. *Mediterranean Journal of Mathematics*, **19**(6), (2022), 1-19. DOI: 10.1007/s00009-022-02179-z

[4] H. Damak N. Hadj Taieb and M.A. Hammami: A practical separation principle for nonlinear non autonomous systems. *International Journal of Control*, **96**(1), (2023), 214–222. DOI: 10.1080/00207179.2021.1986640

[5] H. Damak N. Hadj Taieb and M.A. Hammami: Input-to-state practical partial h-stability of nonlinear non-autonomous systems. *Circuits, Systems, and Signal Processing*, **42**(7), (2023), 3854-3872. DOI: 10.1007/s00034-023-02313-6

[6] H. Damak, I. Ellouze and M.A. Hammami: A separation principle of a class of time-varying nonlinear systems. *Nonlinear Dynamics and Systems Theory*, **13**(2), (2013), 133-143.

[7] H. Damak, M.A. Hammami and A. Kicha: A converse theorem on practical h-stability of nonlinear systems. *Mediterranean Journal of Mathematics*, **17**(3), (2020), 1-18. DOI: 10.1007/s00009-020-01518-2

[8] H. Damak and M.A. Hammami: Stabilization by an estimated state controller of nonlinear time-varying systems. *Analysis*, **43** (2023), 15-30. DOI: 10.1515/anly-2022-1050

[9] F. Delmote, M.A. Hammami and N.E.H. Rettab: On the state estimation for nonlinear continuous-time fuzzy systems. *Archives of Control Sciences*, **32** (2022), 57-72. DOI: 10.24425/acs.2022.140864

[10] I. Ellouze, M.A. Hammami and J.C. Vivalda: A separation principle for linear impulsive systems. *European Journal of Control*, **20**(3), (2014), 105-110. DOI: 10.1016/j.ejcon.2014.02.001

[11] G-D. Hu: Observers for one-sided Lipschitz non-linear systems. *IMA Journal of Mathematical Control and Information*, **23**(4), (2006), 395-401. DOI: 10.1093/imamci/dni068

[12] J. Huang, L. Yu L. and M.J. Shi: Adaptive observer design for quasi-one-sided Lipschitz nonlinear systems. *Proceedings of 2017 Chinese Intelligent Systems Conference*, **1** (2017), 13-22. DOI: 10.1007/978-981-10-6496-8_2

[13] A. Jmal, O. Naifar, A. Ben Makhlouf, N. Derbel and M.A. Hammami: On observers design for nonlinear Caputo fractional-order systems. *Asian Journal of Control*, **20**(9), (2018), 1-8. DOI: 10.1002/asjc.1645

[14] H. Khalil: *Nonlinear Systems*. Third ed., Prentice-Hall, Englewood Cliffs, NJ, 2002.

[15] C.V. Pao: On stability of non-linear differential systems. *International Journal of Non-Linear Mechanics*, **8**, (1973), 219-238.

[16] M. Pinto: Perturbations of asymptotically stable differential equations. *Analysis*, **4** (1984), 161-175. DOI: 10.1524/anly.1984.4.12.161

[17] R. Rajamani: Observers for Lipschitz nonlinear systems. *IEEE Transactions on Automatic Control*, **43**(3), (1998), 397-401. DOI: 10.1109/9.661604

[18] X. Fu, Y. Kang, P. Li and P. Yu: Control for a class of stochastic mechanical systems based on the discrete-time approximate observer. *Journal of Systems Science and Complexity*, **32** (2019), 526-541. DOI: 10.1007/s11424-018-7296-4

[19] W. Zhang, H.-S. Su, Y. Liang and Z.-Z. Han: Nonlinear observer design for one-sided Lipschitz systems: A linear matrix inequality approach. *IET Control Theory and Applications*, **6**(9), (2012), 1297-1303. DOI: 10.1049/iet-cta.2011.0386

[20] W. Zhang, H.-S. Su, Y. Liang and Z.-Z. Han: LMI-based observer design for one-sided Lipschitz nonlinear systems. *Proceedings of the 30th Chinese Control Conference*, Yantai, China, (2011), 256-260.

[21] B. Zhou: Stability analysis of non-linear time-varying systems by Lyapunov functions with indefinite derivatives. *IET Control Theory and Applications*, **11**(9), (2017), 1434-1442. DOI: 10.1049/iet-cta.2016.1538