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Optimal schedule for extended basic period approach of economic lot scheduling problem

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Most algorithms for the economic lot scheduling problem (ELSP) following the extended basic period approach consist of two decision levels. On the upper level, the length of the production cycle and the number of lots (frequency) within the cycle for all products are determined. On the lower level, lots are scheduled to level workloads of all periods and ensure a timely start of production. This paper presents a new mixed-integer programming (MIP) model for the scheduling subproblem under the power-of-two policy. This is the first MIP model that exactly determines and minimizes additional inventory holding costs due some lots' premature start of production. It may be solved by a free general-purpose solver within a fraction of a second. Experiments with several problem instances described in the literature confirmed that using the new model within a heuristic algorithm ensures a significant cost reduction for the entire ELSP. Additionally, all optimal schedules for the Bomberger case are presented.

Key words: inventory control, production, scheduling, mixed-integer programming

1. Introduction

The economic lot scheduling problem (ELSP) aims to determine a cyclic production schedule for multiple products with a constant demand rate produced on the same machine. It is an extension of *the economic production quantity* (EPQ) model, which considers only one product and ignores the capacity limit, which, in turn, is an extension of *the economic order quantity* model (EOQ), which assumes zero production time. An extensive comparison of these models is presented in [15].

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The ELSP belongs to the broad class of simultaneous lot-sizing and scheduling problems that describe each production company's critical element of the planning system. It integrates two decision levels. First, lot-sizing determines sizes and completion dates of production orders (lots) to minimize the set-up and inventory (work-in-process) holding costs. Second, machine scheduling determines a detailed production schedule for fixed lots during some fixed time interval, assignment to machines, and sequence of orders to ensure a feasible production plan. In the ELSP, the planning horizon is infinite, demand is assumed to be constant, and the solution is cyclic. In contrast, in dynamic problems, the planning horizon is finite, divided into periods, demand varies from period to period, and planning is executed in a rolling-horizon approach [17].

Integration of lot-sizing and scheduling is impossible when system requirements make scheduling hard to solve, even as a stand-alone problem [22]. Then, schedules are determined after fixing the number and sizes of production lots (orders). Set-up and processing times are usually aggregated [22]; they seldom are modeled separately to enable shorter product cycle times [19] or because set-up times depend on the sequence of products [5].

The ELSP has garnered significant attention in operations research and industrial engineering due to its practical relevance in optimizing production schedules in several branches of the industry with high and stable demand, e.g., in the automotive, metal, food, chemical, and plastic products sector [3, 24].

Recently, many authors have considered various extensions of the ELSP, e.g., batch shipments [2], energy and power management [1, 9], random breakdowns [25], coordination of production with distribution [23], variable production rates [9], or deteriorating items and shortage costs [18]. All the mentioned authors develop specialized heuristic algorithms that do not guarantee the finding an optimal solution because of the lack of mathematical programming models for the entire ELSP.

Several approaches (simplifying assumptions, models) have been proposed for the ELSP (e.g., [15, 24]). In the *common cycle solution* [12], all products have the same cycle, often leading to high total costs.

In the *basic period solution* [4], products may have different cycles. It is possible because the production cycle is divided into periods of the same length, each product is produced once per period at most, and product cycles are integer multiples of the basic period length. To ensure the schedule's feasibility, the basic period is assumed to be long enough to make lots of all products. Most authors use the *power-of-two policy*, i.e., the number of lots within the cycle (order frequency) for each product must be an integer power of two. The basic period solution usually ensures a lower total cost. However, the average utilization of machine is low.

In the *extended basic period solution* [8], the basic period length must be greater than or equal to the average workload per period, and the scheduling of lots becomes a non-trivial optimization problem [7]. Moreover, some author allow a premature start of production before the stocks created by previous lots run out [6, 7, 14, 16], which leads to periods of unequal length. Such earliness of lots may be helpful at the end of a period with a low workload to utilize idle time [16], but it also increases inventory holding costs.

Some researchers focused on the choice of the cycle length and the number of lots within the cycle (frequency) and neglected the scheduling of lots. And yet scheduling in the extended basic period approach is not only a separate decision-making problem but also verifies whether the selected cycle length and frequency are feasible [7].

Therefore, several authors focused on the scheduling subproblem (proof of feasibility). Haessler and Hogue [13] and Elmaghraby [8] proposed scheduling MIP models with equal periods. Davis [7] presented four MIP models for detailed scheduling with equal or unequal periods. He mentioned the cost of the premature start of production but neither modeled nor minimized it, assuming that its "impact seems unlikely to be severe."

Hodgson and Nuttle [14] considered the ELSP with earliness costs. They proposed an LP model with inventory balance constraints to optimize cycle length and lots timing for known assignment of lots to periods and their complete sequence. Cooke et al. [6] proposed a similar MIP model, which optimizes the assignment of lots to periods, but only approximately minimizes the earliness costs.

Holmbom et al. [16] proposed a two-level heuristic for the extended basic period model under the power-of-two policy with unequal periods. On the upper level, an improvement heuristic searches over the set of reasonable lot frequencies, and on the lower level, a myopic priority heuristic determines a detailed schedule and exactly modeled earliness costs. They tried to apply the Cooke et al. [6] model to the scheduling subproblem but its solving time was "very long."

This paper presents a new MIP model for the scheduling subproblem, which exactly models the earliness costs and may be optimized fast enough to replace heuristic scheduling procedures. The presented experimental results prove that the Holmbom et al. [16] heuristic ensures significantly better solutions for the entire ELSP when the scheduling subproblem is solved using the new model.

The rest of the paper is structured as follows. Section 2 describes the ELSP problem and heuristic proposed by Holmbom et al. [16]. Section 3 presents the proposed MIP model. Section 4 contains results for several data sets known from the literature. Section 5 gives the summary.

2. Problem and heuristic

Basic parameters and the decision variables of the considered ELSP problem are presented below:

Basic parameters:

$\mathcal{N}$ – set of products;
D_j – demand rate of product j [product units per day];
R_j – production rate of product j [product units per day];
S_j – set-up time of product j [days];
A_j – set-up cost of product j [monetary value];
h_j – unit holding cost of product j [monetary value per unit of product and day].

Derived parameters:

$\rho_j = D_j/R_j$ – rate ratio of product j, the fraction of cycle time used to produce product j necessary to meet the demand;
$H_j = \frac{1}{2}h_j D_j(1-\rho_j)$ – aggregated holding cost of product j used to simplify the model's description.

Variables:

T – length of the production cycle;
f_j – frequency of product j, i.e., number of product j lots within the entire cycle.

Several products j with demand rate D_j are produced on a single machine at production rate R_j. Both rates are constant and deterministic. The ratio of these rates $\rho_j = D_j/R_j$ gives the fraction of time (production cycle, capacity) that must be devoted to producing the product j to meet the demand. Before each lot (production start) of product j, the machine must be set up, which takes some time S_j and cost A_j, both independent from the sequence of lots. Inventory holding cost is proportional to the stock and unit holding cost h_j.

Consecutive subsections contain a more detailed description of the ELSP and brief characteristics of the Holmbom et al. heuristic [16]. It is a hierarchical algorithm. On the upper level, the lot-sizing heuristic modifies product frequencies and adjusts the cycle length, determining product lot sizes. On the lower level, the scheduling heuristic determines the detailed schedule and lots' earliness (premature start of production).

2.1. Upper level – lot-sizing

The goal of the lot-sizing ELSP subproblem is to determine the frequency f_j for each product j within the whole cycle and cycle length T to minimize the average total cost C per time unit (e.g., [15]):

$$C(f_j, T) = \sum_{j \in \mathcal{N}} \left(A_j \frac{f_j}{T} + H_j \frac{T}{f_j} \right). \tag{1}$$

The first element of this function describes the set-up costs and the second is the inventory holding cost. When the product j frequency f_j and cycle length are fixed, the individual cycle length equals T/f_j, and the lot size is $D_j T/f_j$.

According to the extended basic approach, in a feasible solution, the workload during the entire cycle cannot exceed its length (capacity) (e.g., [15]):

$$\sum_{j \in \mathcal{N}} \left(S_j f_j + \rho_j T \right) \leqslant T. \tag{2}$$

In this paper, all solutions follow the *power-of-two policy*, which means that lot frequencies f_j must be powers of two, $f_j = 1, 2, 4, 8, \ldots$. It simplifies the construction of feasible cyclic schedules and does not increase costs significantly; therefore, it is widely used in practice (e.g., [15, 24]).

The Holmbom lot-sizing heuristic [16] starts from the common cycle solution, with $f_j = 1$ for all products. In the single-product model, the set-up and inventory holding cost ratio is equal to one for the optimal solution. Therefore, the heuristic in each iteration modifies the frequency of a product whose ratio shows the largest deviation from one. A detailed schedule and earliness costs are determined for each new frequency vector. Such search stops in a local optimum, where modification of a single product frequency does not reduce the total cost.

2.2. Lower level – scheduling

The scheduling ELSP subproblem aims to assign lots to periods, fix their order during each period, and determine lot earliness.

In this paper, during each period, lots are processed in the same descending order of frequency f_j (e.g., [16]). This means, that first processed are products which are produced more often. The idle time is allocated at the end of periods. With the power-of-two policy, such a sequence ensures that each lot has the same start time during each period.

The scheduling heuristic proposed in [16] is similar to the *longest processing time* (LPT) priority heuristic successfully used to minimize the length of a schedule (makespan) on identical parallel machines. In the ELSP heuristic, LPT assigns the next longest single activity to the period with the lowest workload.

The LPT heuristic distributes workload evenly over all periods, indirectly minimizing the early starts of lots and earliness costs. For the ELSP, the LPT heuristic has two serious flaws: it assumes that leveling the workload and period length alone minimizes earliness and ignores diversification of unit holding costs.

2.3. Earliness costs

In the basic period approach, each period has the same duration (capacity). In the extended basic period approach, lengths of periods may vary. Therefore, some periods start earlier than necessary to meet demand on time, before the stock drops to zero, enabling high machine utilization but increasing inventory and holding costs. An illustrative example is presented in [16].

In the machine scheduling theory, *earliness* is defined as the positive difference between an activity's due date and completion time. In the ELSP, there are no due dates, but processing of a new lot of some products should start before its stock runs out, which is a kind of due date. Earliness, processing the next lot prematurely, increases inventory and its holding cost. Tardiness (negative earliness), processing the next lot too late, is infeasible because some demand would remain unsatisfied. The lists below present additional parameters and variables used to determine the lot earliness:

Parameters derived from the lot-sizing solution:

F $= \max_{j \in \mathcal{N}} f_j$ – number of periods;
L $= T/F$ – "perfect" length of the basic period.

Variables:

e_{jt} – non-negative earliness of product j lot during period t;
e'_t, e''_{jt}, e'''_j – free, auxiliary earliness variables;
u_t – idle time in period t;
w_t – workload (total processing time) in period t;
$y_{jt} = 1$, if product j is produced during period t; 0 otherwise.

The approach proposed by Holmbom et al. [16] to determine lots earliness is presented below. It has been decomposed here into four steps that may be implemented in a MIP model. In the three initial steps, the preliminary earliness may take negative values.

1. Determine the earliness of periods that is, the difference between the "perfect" and actual start time of each period (may be positive or negative):

$$e'_t = \sum_{s=1}^{t-1} \big(L - (w_s + u_s)\big), \quad t \in \mathcal{T}. \tag{3}$$

2. Determine the earliness of lots:

$$e''_{jt} = \begin{cases} e'_t, & \text{if } y_{jt} = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad j \in \mathcal{N}, \;\; t \in \mathcal{T}. \tag{4}$$

3. Determine the minimal product earliness:

$$e'''_j = \min_{t \in \mathcal{T}:\, y_{jt}=1} e''_{jt}, \qquad j \in \mathcal{N}. \tag{5}$$

4. Determine the final non-negative lot earliness (“early start times” in [16]):

$$e_{jt} = e''_{jt} - e'''_j, \qquad j \in \mathcal{N}, \;\; t \in \mathcal{T}:\, y_{jt} = 1. \tag{6}$$

Table 1 presents an example which explains this approach. Table 1a shows the products A, B, and C schedule during four periods. When processing time $P_{jt} = 0$, there is no lot for product j in period t. The cycle length is $T = 28$, and the “perfect” basic period length $L = 28/4 = 7$. The start time of period 1 is assumed to be 0. The “perfect” periods start equal $\sum_{s=1}^{t-1} L$ and the actual are $\sum_{s=1}^{t-1}(w_s + u_s)$.

Table 1: Example of determining earliness

a) Schedule, workload, and period earliness

	P_{jt}	1	2	3	4
Schedule	A	4	0	0	0
	B	2	2	2	2
	C	0	6	0	6
Idle time	u_t	1	0	3	0
Workload	$w_t + u_t$	7	8	5	8
Perfect period start		0	7	14	21
Actual period start		0	7	15	20
	e'_t	0	0	−1	1

b) Lot earliness

e''_{jt}	1	2	3	4	e'''_j
A	0	∗	∗	∗	0
B	0	0	−1	1	−1
C	∗	0	∗	1	0

c) Non-negative lot earliness

e_{jt}	1	2	3	4
A	0	∗	∗	∗
B	1	1	0	2
C	∗	0	∗	1

∗ – no production lot

Table 1b presents the lot earliness. For the lot of product B in period 3, it is negative, which suggests that the schedule is infeasible (the lot is late). However, this is a cyclic solution; therefore, we do not have to measure time from the beginning of period 1. If we set the start of period 1 at time −1, that is, one time-unit earlier, the actual start times will be $(-1, 6, 14, 19)$, and the product B earliness $(1, 1, 0, 2)$, which is positive and feasible. This is equivalent to subtracting −1 from product earliness e'''_{jt}, leading to the final earliness e_{jt} presented in Table 1c.

3. Scheduling MIP model

The lists below present additional derivative parameters and variables used in the MIP model.

Derived parameters:

$\mathcal{T} = \{1, \ldots, F\}$ – set of periods;

$\tau_j = F/f_j$ – number of periods between lots of product j;

$\mathcal{S}_j = \big\{(s,t) \mid s = 1, \ldots, \tau_j,\ \phi = 1, \ldots, f_j,\ t = s + (\phi - 1)\,\tau_j\big\}$ – set of period pairs (s,t), where t is one of the periods during which product j is processed if its first lot is in period s;

J – any product with $f_J = 1$, where $J \in \mathcal{N}$;

$P_j = S_j + \rho_j T/f_j$ – processing time of product j lot;

$U = T - \sum_{j\in\mathcal{N}} P_j f_j$ – total idle time.

Variables:

$z_{jt} = 1$, if t is the first period during which product j is produced; 0 otherwise.

For example, if the number of periods $F = 8$ and product j frequency $f_j = 4$, it must be produced in every other period ($\tau_j = 2$), the first lot must be in periods 1 or 2, and the set $\mathcal{S}_j = \big\{(1,1),\ (1,3),\ (1,5),\ (1,7),\ (2,2),\ (2,4),\ (2,6),\ (2,8)\big\}$.

Below is the MIP model, which can replace the scheduling procedure used in the Holmbom heuristic [16] on the lower level of their approach. It minimizes the exactly modeled earliness costs and indirectly levels the workload of periods. Solving this model ensures an optimal schedule for given product frequencies and cycle length:

minimize

$$\sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{N}} \frac{h_j D_j}{f_j} e_{jt} \tag{7.1}$$

subject to

$$\sum_{t=1}^{\tau_j} z_{jt} = 1, \qquad j \in \mathcal{N} \tag{7.2}$$

$$z_{js} = y_{jt}, \qquad j \in \mathcal{N},\ (s,t) \in \mathcal{S}_j \tag{7.3}$$

$$\sum_{j\in\mathcal{N}} P_j y_{jt} = w_t, \qquad t \in \mathcal{T} \tag{7.4}$$

$$\sum_{t\in\mathcal{T}} u_t = U, \tag{7.5}$$

$$z_{J1} = 1 \tag{7.6}$$

$$\sum_{s=1}^{t-1}\big(L - (w_s + u_s)\big) = e'_t, \qquad t \in \mathcal{T} \tag{7.7}$$

$$e'_t - L(1 - y_{jt}) \leqslant e''_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T} \tag{7.8}$$

$$e'_t + L(1 - y_{jt}) \geqslant e''_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T} \tag{7.9}$$

$$e''_{jt} + L(1 - y_{jt}) \geqslant e'''_j, \qquad j \in \mathcal{N},\ t \in \mathcal{T} \tag{7.10}$$

$$e''_{jt} - e'''_j - L(1 - y_{jt}) \leqslant e_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T} \tag{7.11}$$

$$e_{jt}, u_t, w_t \geqslant 0, \qquad j \in \mathcal{N},\ t \in \mathcal{T} \tag{7.12}$$

$$y_{jt}, z_{jt} \in \{0, 1\}, \quad j \in \mathcal{N},\ t \in \mathcal{T}. \tag{7.13}$$

The objective function (7.1) is the average per time unit of the earliness costs of all lots. e_{jt} is the earliness of product j in period t, and $D_j e_{jt}$ is the additional inventory, held only over one "perfect" product cycle T/f_j. To determine the average cost per time unit the inventory must be multiplied by the unit holding cost h_j and divided by the cycle length T.

The first part of the model (7.2)–(7.5) describes the assignment of lots to periods. Equation (7.2) forces the unique start of a series of product lots. Constraint (7.3) directly determines the value of the variable y_{jt} for known f_j and z_{jt}. Therefore, y_{jt} does not have to be declared as a binary variable and is only an auxiliary variable; it could be replaced in the whole model by z_{js} using (7.3). The considered model is, however, easy to solve, does not require any tightening, and thanks to y_{jt}, it is easier to build and understand. Equality (7.4) determines the total workload during period t, where $P_j y_{jt}$ is the time that the machine spends on item j in this period. Equality (7.5) allocates idle time to periods.

Each ELSP schedule is cyclic; therefore, it is not essential which period in the cycle is listed as the first; one may roll the periods "in a cycle" without any changes to the feasibility and quality of solutions. Constraint (7.6) reduces the number of symmetrical solutions by assigning to the first period an arbitrary product with frequency equal to one.

The second part of the model (7.7)-(7.11) describes the earliness. Constraint (7.7) determines the difference between perfect and actual period length. The pair of constraints (7.8) and (7.9) explicitly describes condition $(y_{jt} = 1) \Rightarrow (e''_{jt} = e'_t)$ with the help of the indicator (big M) constraints. Both constraints (7.8) and (7.9) are essential for the model because variables e''_{jt} may take on both positive and negative values. e''_{jt} values for $y_{jt} = 0$ are irrelevant because they are omitted in all other constraints. Thanks to (7.10), variable e'''_{jt} equals e''_{jt} only if $y_{jt} = 1$. The variables e_{jt} are minimized and, therefore, do not have to be constrained from above. Due to (7.11), the variables e'''_{jt} are maximized and do not have to be constrained from below.

Constraints (7.12)–(7.13) define the domain of all variables. The y_{jt} variables do not have to be declared binary because of (7.3). The auxiliary variables, e'_t, e''_{jt}, and e'''_j, may take negative values.

4. Problem instances and solutions

The results of computations with two approaches are presented below. The first is the author's implementation of the Holmbom lot-sizing and scheduling heuristic [16]. The second one uses the same approach to solve the lot-sizing subproblem and the proposed MIP model (7) to solve the scheduling subproblem.

The MIP models were solved with the help of two solvers, the commercial Gurobi solver (v. 11.0.3) [11] and the free GLPK solver (v. 4.65) [10], on an Intel Core i9-7900X processor with 3.3 GHz clock speed, 16 GB RAM, 10 physical cores, 20 logical processors, using up to 20 threads. Both solvers with the standard settings determined optimal solutions for all solved scheduling problem instances in a fraction of a second.

4.1. Holmbom instances

Nilsson et al. [20] presented a problem instance with five products, here numbered 0, which Holmbom et al. [16] used to prepare ten other instances, numbered 1 to 10. A total of eleven cases are considered in this section.

To solve a single problem instance (for the best frequency vector), the Gurobi solver needed to explore, on average, 0.9 nodes (33.7 simplex iterations), which lasted about 0.007 seconds. The GLPK solver needed to explore 7.8 nodes (143.2 simplex iterations), which lasted about 0.03 seconds (8 times 0.0 and 3 times 0.1 seconds).

Table 2 presents the results. In eight cases, the optimal solutions provided with the help of the MIP model have the same frequencies and total cost as the schedules provided by the original Holmbom et al. heuristic [16] and, therefore, are omitted from the table.

In three cases, the MIP model significantly improved the solutions. The cycle length and frequencies remain the same; only the assignment to periods and allocation of idle time are different. Table 2 shows the total cost reduction and,

Table 2: Results for problem instances from [16]

Instance	Total cost		Improvement	First period	
	Heuristic	MIP		Heuristic	MIP
2	278.78	275.38	1.2%	[1, 1, 1, 2, 3]	[1, 1, 1, 2, 3]
7	762.60	736.10	3.5%	[1, 1, 1, 4, 2]	[1, 1, 1, 2, 4]
10	156.32	153.89	1.6%	[1, 1, 3, 1, 2]	[1, 1, 3, 1, 2]

for each of the five products, the *first period* of the cycle in which that product is processed.

Case 2 differs from case 0 only by demand pattern, while the average ratio of demand and production rates remains the same. Cases 7 and 10 have a much higher ratio of the set-up to inventory holding costs. However, the number of cases is too small to make general conclusions.

Cyclic schedules provided by the ELSP are intended to be used many times over weeks or months. If this one solution, which the company will use for a long time, is burdened with a significant error, then the company's costs will be significantly overstated. So, in the case of the ELSP, a worst case is a more appropriate criterion for assessing the quality of solutions than an average. Decrement of total cost by 1.2%, 3.5%, and 1.6% in three of eleven cases is therefore significant.

To avoid differences caused only by the symmetry of solutions prevented in the MIP model by equality (7.6), the heuristic schedules were rolled through the cycle to minimize the difference with the MIP solutions. The rolling approach is simple: increase each period number by one and make the last period the first.

In Table 2, for the 2nd and 10th instances, the first period for each product is the same in both solutions, heuristic and MIP, but for the 7th instance, the solutions differ for the two last products. An additional experiment was conducted to clarify whether this difference is significant, whether these schedules are symmetrical (equivalent), and whether the total costs differ only due to the different distributions of idle time. The MIP model was solved again with the first period of the last product fixed to the value from the heuristic solution. For such a problem, the total cost equals 749.04, 1.77% better than for the heuristic and 1.75% worse than the optimal solution. Therefore, both the assignment of lots to periods and the allocation of idle time impact the earliness costs.

Table 3 presents the idle time distribution among periods for the 2nd instance. For this case, the heuristic and MIP schedules are identical regarding all other variables. The heuristic seeks to level the workload; therefore, the idle time is uniformly spread over periods with the lowest workload. To minimize the earliness costs, the MIP model assigns most of the idle time to the 4th period, which ensures a cost reduction by 1.2%.

Table 3: Workload w_t and idle time u_t in solutions for the 2nd instance

	Value	Periods				Total
		1	2	3	4	
	w_t	6.741	6.378	6.932	6.378	26.428
Heuristic schedule	u_t	0.	0.146	0.	0.146	0.291
	$w_t + u_t$	6.741	6.523	6.932	6.523	26.719
MIP schedule	u_t	0.	0.050	0.	0.241	0.291
	$w_t + u_t$	6.741	6.428	6.932	6.619	26.719

The reader should be aware that the above results, obtained with the author's implementation of the heuristic differ from the results reported in [16] for two instances. For the 7th instance Holmbom et al. [16] provide the best frequency $f_j = [1, 4, 2, 4, 2]$, cost-optimal cycle $T_c = 104.046$, and total cost $C = 1268.70$. For the same frequency, the cost-optimal cycle obtained by the author is much shorter $T_c = 65.354$. The new implementation of the heuristic also delivered another frequency $f_j = [1, 4, 8, 2, 2]$, cost-optimal cycle $T_c = 64.053$, and total cost $C = 762.60$, which is 39.9% better. For the 10th instance, Holmbom et al. [16] present total cost $C = 153.54$, and the new code gives $C = 156.32$, 1.8% worse, although the frequencies and cycle time are the same.

4.2. Bomberger case

Bomberger [4] established a standard problem instance based on real-world data from a stamping facility, frequently used to compare the quality of solutions provided by various ELSP algorithms. In the original paper, there are some mistakes (the data does not match the solution); therefore, the results below are based on the data version presented in [15].

The Holmbom heuristic [16] cannot determine the optimal solution for the Bomberger case – it provides total cost equal to 32.21, which is 0.4% worse than the minimal cost 32.07 already described in the literature (e.g., [13]). The approach with the MIP model finds the optimum.

For the last frequency vector, the Gurobi solver needed to explore only 1 node (107 simplex iterations), which lasted about 0.02 seconds. The GLPK solver needed to explore 7 nodes (357 simplex iterations), which lasted about 0.1 seconds.

Moreover, the Gurobi MIP solver allows for determining all optimal solutions. Table 4 presents three optimal schedules determined with the valid inequality (7.6) eliminating symmetrical (equivalent) solutions. The frequencies of products are the same in all three solutions, and the lengths of all periods are equal to 23.425; therefore, they are simple basic period solutions.

Table 4: First periods of products in three optimal schedules for the Bomberger case

	Solution	Products									
		1	2	3	4	5	6	7	8	9	10
Frequency	all	1	4	4	8	4	2	1	8	4	4
First period	1	1	2	2	1	2	3	5	1	1	2
	2	1	2	1	1	1	3	5	1	2	1
	3	1	1	1	1	2	3	5	1	2	1

Table 5 presents the allocation of idle time in these solutions. It should be noted that there may be many optimal idle time allocations for the same integer solution.

Table 5: Idle times in three optimal solutions for the Bomberger case

Solution	Periods							
	1	2	3	4	5	6	7	8
1	0.34	1.26	1.46	1.26	0.09	1.26	1.46	1.26
2	1.11	0.50	2.23	0.50	0.86	0.50	2.23	0.50
3	1.02	0.59	2.14	0.59	0.76	0.59	2.14	0.59

5. Summary

This paper proposes a new *mixed-integer programming* (MIP) model for the *economic lot-scheduling problem* (ELSP) following the *extended basic period approach* under the power-of-two policy. It determines the detailed schedule of lots and the additional inventory holding costs due to premature start of lots (earliness costs) for given product frequencies and cycle time length.

The presented model ensures optimal schedule, and its solving time is shorter than a second; therefore, it may replace scheduling heuristics. For several problem instances from the literature, the heuristic method using the new scheduling MIP model provides significantly better solutions than the pure heuristic approach [16]. Analysis of solutions proves that idle time evenly distributed over periods with the lowest workload does not ensure optimal solutions, and both the assignment of lots to periods and the allocation of idle time impact the cost of premature start of lots.

In future research, the MIP model presented in this paper may be extended to optimize the entire ELSP problem following the extended basic approach. Two problems need to be solved to achieve this goal. First, variables from the lot-sizing model are used in the scheduling model to define sets and indexes of variables, parameters, and constraints. The second, more significant difficulty is the non-linear character of the objective function. Solving mixed-integer non-linear problems is challenging; therefore, the objective function must be simplified (approximated) to enable effective and efficient optimization.

Data availability statement

The data set with all problem instances and solutions is openly available from the repository RODBUK of the AGH University (agh.rodbuk.pl) at DOI: 10.58032/AGH/T7ZXDH.

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