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On symmetric dual models associated with multiple cost control problems

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In this paper, we build a symmetric dual model associated with a family of multiple cost control problems. More concretely, under weaker generalized convexity hypotheses than those formulated in previous research works (for classical multiobjective variational problems), we establish and extend the framework to controlled variational models, and, therefore, the derived results become significantly stronger and more generous than those presented so far in the specialized literature.

Key words: symmetric dual models; multiple cost variational control problems; properly efficient point.

1. Introduction

Because of the increasingly accentuated randomness of the medium, initial data more often include inaccuracies. For modeling various processes in industry and economics to make decisions, some complete information regarding the parameters and variables involved is not always attainable. Therefore, a suitable and adequate framework of uncertainty is a necessity to build the model, or new techniques must be adapted to obtain efficient solutions in a given sense. In the realm of optimization problems grappling with uncertainty, robust optimization, fuzzy optimization, and interval-valued optimization stand out as burgeoning branches within the vast landscape of mathematics. Recent developments in this

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regard can be read in the following research works: Treanţă and Saeed [34], Bagri et al. [4], Guo et al. [18, 19], Jayswal et al. [20], Saeed and Treanţă [24], Shi et al. [28], Treanţă [32, 33], Sun et al. [30].

Over time, various researchers have been interested in obtaining certain solution techniques by considering symmetric-type duality for multiobjective programming problems. Among the first approaches to this concept of symmetric duality for quadratic programming were proposed by Dorn [13] and Dantzig et al. [11]. Then, Mond and Hanson [23] initially established a pair of symmetric duals and demonstrated the duality results by considering convexity and concavity hypotheses. Thereafter, Smart and Mond [29], by omitting the non-negativity constraints presented in Mond and Hanson [23], studied symmetric-type duality under invexity assumptions. Moreover, they also stated the static case, by relaxing the time dependency. Schaible [25–27] provided a unified approach of fractional programming. In [17], Gulati et al., following Mond and Hanson [23], extended the results for a family of minimax programming problems. Chandra et al. [7] introduced for the first time a symmetric dual in fractional nonlinear programming. Furthermore, Gulati et al. [16] have given a generalization of these results and considered the static and continuous fractional nonlinear models. For multiobjective fractional nonlinear extremization problems, a symmetric duality was established by Weir [38], where various duality theorems have been derived by using convexity hypotheses. These results were extended by Yang et al. [39] to the nondifferentiable situation by considering support-type functions. Also, Kim and Lee [21] presented a pair of symmetric duals for multiobjective variational problems by invexity assumptions. In [15], Gulati et al. presented other symmetric duals for multiobjective variational models under pseudoconvexity and pseudoconcavity hypotheses. A contribution was also made by Chen [9], who established results of symmetric duality for multiobjective mixed integer programs. Then, Chen [8] and Kim et al. [22] obtained duality results in symmetric multiple objective fractional models governed by convex functions. Recently, Abdulaleem and Treanţă [1] established optimality and duality results for E-differentiable multiobjective problems. Also, Antczak et al. [3] investigated efficiency and duality associated with nonconvex nondifferentiable optimization problems. Further, Das et al. [12] studied set-valued fractional extremization models by using contingent epi-derivative. In [31], Treanţă and Mititelu, studied duality governed by (ρ, b) – quasiinvexity in multi-dimensional multiple objective fractional control problems. Very recently, Upadhyay et al. [35–37] analyzed multiobjective semi-infinite programming problems on Hadamard manifolds.

The exploration of symmetric duality within the realm of multiobjective fractional problems has expanded into the broader landscape of continuous-time variational problems. Through this extension, a nuanced understanding of weak and strong dualities has been cultivated under generalized convexity assumptions. In this context, the integration of a control variable into the optimization process plays a pivotal role. This intricate interplay between optimization objectives and control variables introduces novel perspectives and avenues for analysis. Moreover, a profound connection between these variational problems and their counterparts in the realm of multiobjective symmetric dual control problems will be elucidated in this study. This network of relationships sheds light on the underlying principles governing optimization strategies in complex and multiobjective scenarios. More precisely, in this paper, we use weaker generalized convexity hypotheses than those formulated in Chen [8], Kim et al. [22], following Ahmad [2] (for classical multiobjective variational problems) to establish and provide characterizations of symmetric dual models associated with new multiple cost variational control problems. Concretely, we extend the framework to controlled variational models, and, therefore, the derived results are significantly stronger and more generous than those presented so far in the specialized literature.

The paper continues as follows. Section 2 presents notations, definitions, and preliminary tools for introducing the problem under study. Section 3 formulates the primal and dual models and establishes various duality results for this pair of problems. In Section 4, we state the conclusions associated with this study.

2. Preliminary tools

Let $K = [t_1, t_2]$ be a real interval, and $f^y(t, a(t), \dot{a}(t), \pi(t), b(t), \dot{b}(t), \zeta(t))$ and $g^y(t, a(t), \dot{a}(t), \pi(t), b(t), \dot{b}(t), \zeta(t))$, where $a\colon K \to \mathbb{R}^n$, $\pi\colon K \to \mathbb{R}^s$, $b\colon K \to \mathbb{R}^m$, $\zeta\colon K \to \mathbb{R}^l$ (see $\dot{a}$ and $\dot{b}$ as derivatives for a and b with respect to $t \in K$), are C^2-class functionals, for $y \in \{1, 2, \ldots, j\}$. In the paper, f^y_a, $f^y_{\dot{a}}$, f^y_π, f^y_b, $f^y_{\dot{b}}$ and f^y_ζ represent the gradients associated with the functional

$$f^y(t, a(t), \dot{a}(t), \pi(t), b(t), \dot{b}(t), \zeta(t))$$

with respect to $a, \dot{a}, \pi, b, \dot{b}$ and ζ,

$$f^y_a = \left(\frac{\partial f^y}{\partial a^1}, \ldots, \frac{\partial f^y}{\partial a^n}\right)^T, \qquad f^y_{\dot{a}} = \left(\frac{\partial f^y}{\partial \dot{a}^1}, \ldots, \frac{\partial f^y}{\partial \dot{a}^n}\right)^T,$$

$$f^y_b = \left(\frac{\partial f^y}{\partial b^1}, \ldots, \frac{\partial f^y}{\partial b^m}\right)^T, \qquad f^y_{\dot{b}} = \left(\frac{\partial f^y}{\partial \dot{b}^1}, \ldots, \frac{\partial f^y}{\partial \dot{b}^m}\right)^T,$$

$$f^y_\pi = \left(\frac{\partial f^y}{\partial \pi^1}, \ldots, \frac{\partial f^y}{\partial \pi^s}\right)^T, \qquad f^y_\zeta = \left(\frac{\partial f^y}{\partial \zeta^1}, \ldots, \frac{\partial f^y}{\partial \zeta^l}\right)^T,$$

for $y \in \{1, 2, \ldots, j\}$. Similarly, g^y_a, $g^y_{\dot{a}}$, g^y_π, g^y_b, $g^y_{\dot{b}}$ and g^y_ζ denote the gradient vectors of $g^y(t, a(t), \dot{a}(t), \pi(t), b(t), \dot{b}(t), \zeta(t))$ with respect to $a, \dot{a}, \pi, b, \dot{b}$ and ζ.

The following computations will be used to prove the strong duality theorem (see Theorem 2 in the next section):

$$\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b} = f^y_{\dot b t} + f^y_{\dot b b}\dot b + f^y_{\dot b \dot b}\ddot b + f^y_{\dot b a}\dot a + f^y_{\dot b \dot a}\ddot a + f^y_{\dot b \pi}\dot\pi + f^y_{\dot b \zeta}\dot\zeta,$$

involving

$$\begin{aligned}
&\frac{\partial}{\partial \zeta}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b \zeta}, && \frac{\partial}{\partial \dot\zeta}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = f^y_{\dot b \zeta},\\
&\frac{\partial}{\partial \pi}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b \pi}, && \frac{\partial}{\partial \dot\pi}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = f^y_{\dot b \pi},\\
&\frac{\partial}{\partial b}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b b}, && \frac{\partial}{\partial \dot b}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b \dot b} + f^y_{\dot b b},\\
&\frac{\partial}{\partial \ddot b}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = f^y_{\dot b \dot b}, && \frac{\partial}{\partial a}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b a},\\
&\frac{\partial}{\partial \dot a}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b \dot a} + f^y_{\dot b a}, && \frac{\partial}{\partial \ddot a}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^y_{\dot b}\right) = f^y_{\dot b \dot a}, \quad \text{for } \ y \in \{1, 2, \ldots, j\}.
\end{aligned}$$

Similarly, we can write the above-mentioned computations for the integral functional g.

Let $C^1(K, \mathbb{R}^n)$ and $C^1(K, \mathbb{R}^m)$ denote the spaces of piecewise smooth *state* functions a and b, respectively, equipped with norm $\|a\| = \|a\|_\infty + \|\dot a\|_\infty$ and $\|b\| = \|b\|_\infty + \|\dot b\|_\infty$, respectively. Also, let $C^0(K, \mathbb{R}^s)$ and $C^0(K, \mathbb{R}^l)$ denote the spaces of piecewise continuous *control* functions π and ζ, respectively, equipped with uniform norm, as well.

Consider the following multi-cost variational control problem (see Bector and Husain [5] for the classical multiobjective variational model):

$$\text{(P)} \quad \min_{(a,\pi)} \left(\int_{t_1}^{t_2} \phi^1(t, a(t), \pi(t))\,\mathrm{d}t, \ldots, \int_{t_1}^{t_2} \phi^j(t, a(t), \pi(t))\,\mathrm{d}t \right)$$

subject to

$$a(t_1) = \alpha = \text{given}, \quad a(t_2) = \beta = \text{given},$$

$$h(t, a(t), \pi(t)) \leqq 0, \quad t \in K,$$

$$\psi(t, a(t), \pi(t)) = \dot a(t), \quad t \in K,$$

where $\phi^y : C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \to \mathbb{R}$, $y = \overline{1, j}$, $h^\iota : C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \to \mathbb{R}$, $\iota = \overline{1, r}$ and $\psi^\tau : C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \to \mathbb{R}$, $\tau = \overline{1, n}$, are continuously differentiable functionals.

Let S denote the set of all feasible solutions of (P), that is,

$$S = \Big\{(a,\pi)\in C^1(K,\mathbb{R}^n)\times C^0(K,\mathbb{R}^s) \,|\, a(t_1)=\alpha,\ a(t_2)=\beta,\ h(t,a(t),\pi(t)) \leqq 0,$$
$$\psi(t,a(t),\pi(t)) = \dot{a}(t),\ t\in K\Big\}.$$

Definition 1 (Geoffrion [14]). *A point* $(a^0,\pi^0)\in S$ *is named* efficient point of (P) *if, for all* $(a,\pi)\in S$, *we have*

$$\int_{t_1}^{t_2} \phi^y\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t \geqslant \int_{t_1}^{t_2} \phi^y(t,a(t),\pi(t))\mathrm{d}t,\ \ \textit{for all } y\in\{1,2,\ldots,j\},$$
$$\Rightarrow \int_{t_1}^{t_2} \phi^y\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t = \int_{t_1}^{t_2} \phi^y(t,a(t),\pi(t))\mathrm{d}t$$

for all $y\in\{1,2,\ldots,j\}$. *The efficient point* $(a^0,\pi^0)\in S$ *is named* properly efficient point of (P) *if* $(\exists)$ $M>0$ *so that, for* $y\in\{1,2,\ldots,j\}$, *we have*

$$\int_{t_1}^{t_2} \phi^y\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t - \int_{t_1}^{t_2} \phi^y(t,a(t),\pi(t))\mathrm{d}t$$
$$\leqslant M\left(\int_{t_1}^{t_2} \phi^i(t,a(t),\pi(t))\mathrm{d}t - \int_{t_1}^{t_2} \phi^i\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t\right)$$

for some i, *such that*

$$\int_{t_1}^{t_2} \phi^i(t,a(t),\pi(t))\mathrm{d}t > \int_{t_1}^{t_2} \phi^i\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t$$

whenever $(a,\pi)\in S$, *and*

$$\int_{t_1}^{t_2} \phi^y(t,a(t),\pi(t))\mathrm{d}t < \int_{t_1}^{t_2} \phi^y\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t.$$

An efficient point that is not properly efficient point is called improperly efficient point. *Thus, for* $(a^0,\pi^0)\in S$ *to be improperly efficient point means that to every sufficiently large* $M>0$, *there is an* $(a,\pi)\in S$ *and an* $y\in\{1,2,\ldots,j\}$ *such that*

$$\int_{t_1}^{t_2} \phi^y(t,a(t),\pi(t))\mathrm{d}t < \int_{t_1}^{t_2} \phi^y\left(t,a^0(t),\pi^0(t)\right)\mathrm{d}t$$

and

$$\int_{t_1}^{t_2} \phi^y \left(t, a^0(t), \pi^0(t)\right) \mathrm{d}t - \int_{t_1}^{t_2} \phi^y(t, a(t), \pi(t)) \mathrm{d}t$$

$$> M\left(\int_{t_1}^{t_2} \phi^i(t, a(t), \pi(t)) \mathrm{d}t - \int_{t_1}^{t_2} \phi^i \left(t, a^0(t), \pi^0(t)\right) \mathrm{d}t\right)$$

for all $i \in \{1, 2, \ldots, j\}$, *satisfying*

$$\int_{t_1}^{t_2} \phi^i(t, a(t), \pi(t)) \mathrm{d}t > \int_{t_1}^{t_2} \phi^i \left(t, a^0(t), \pi^0(t)\right) \mathrm{d}t.$$

Definition 2 (Borwein [6])**.** *A point* $(a^0, \pi^0) \in S$ *is named* weak efficient point for (P) *if there exists no other* $(a, \pi) \in S$ *such that*

$$\int_{t_1}^{t_2} \phi^y \left(t, a^0(t), \pi^0(t)\right) \mathrm{d}t > \int_{t_1}^{t_2} \phi^y(t, a(t), \pi(t)) \mathrm{d}t, \quad \textit{for all } y \in \{1, 2, \ldots, j\}.$$

From the above-mentioned definitions, it follows that if $(a^0, \pi^0) \in S$ is efficient point of (P), it is a weak efficient point of (P), as well.

Definition 3. *The integral functional*

$$F(\bar{a}, b) = \int_{t_1}^{t_2} f(t, \bar{a}(t), \dot{\bar{a}}(t), \bar{\pi}(t), b(t), \dot{b}(t), \zeta(t)) \mathrm{d}t$$

is said to be pseudoinvex at $a, \dot{a}$ and π *if there exist the vector-valued functionals* $z \in \mathbb{R}^n$, *with* $z(t, a, \dot{a}, \pi, a, \dot{a}, \pi) = z|_{t=t_1, t=t_2} = 0$, *and* $\mu \in \mathbb{R}^s$ *with* $\mu(t, a, \dot{a}, \pi, a, \dot{a}, \pi) = 0$, *such that, for each* $b, \dot{b}$ *and* ζ, *we have*

$$\int_{t_1}^{t_2} \Big[z^T f_{\bar{a}}(t, a, \dot{a}, \pi, b, \dot{b}, \zeta) + \mu^T f_{\bar{\pi}}(t, a, \dot{a}, \pi, b, \dot{b}, \zeta)$$

$$+ \frac{\mathrm{d}z^T}{\mathrm{d}t} f_{\dot{\bar{a}}}(t, a, \dot{a}, \pi, b, \dot{b}, \zeta) \Big] \mathrm{d}t \geqslant 0$$

$$\Rightarrow \int_{t_1}^{t_2} f(t, \bar{a}, \dot{\bar{a}}, \bar{\pi}, b, \dot{b}, \zeta) \mathrm{d}t \geqslant \int_{t_1}^{t_2} f(t, a, \dot{a}, \pi, b, \dot{b}, \zeta) \mathrm{d}t,$$

for all $\bar{a}, \dot{\bar{a}}$ *and* $\bar{\pi}$.

Similarly, the integral functional $-\int_{t_1}^{t_2} f(t,a(t),\dot{a}(t),\pi(t),\bar{b}(t),\dot{\bar{b}}(t),\bar{\zeta}(t))\,\mathrm{d}t$ *is said to be* pseudoinvex at $b,\dot{b}$ and ζ *if there exist the vector-valued functionals* $\eta\in\mathbb{R}^m$, *with* $\eta(t,b,\dot{b},\zeta,b,\dot{b},\zeta)=\eta|_{t=t_1,t=t_2}=0$, *and* $\nu\in\mathbb{R}^l$ *with* $\nu(t,b,\dot{b},\zeta,b,\dot{b},\zeta)=0$, *such that, for each* $a,\dot{a}$ *and* π, *we have*

$$\int_{t_1}^{t_2}\Big[\eta^T f_{\bar{b}}(t,a,\dot{a},\pi,b,\dot{b},\zeta)+\nu^T f_{\bar{\zeta}}(t,a,\dot{a},\pi,b,\dot{b},\zeta)$$
$$+\frac{\mathrm{d}\eta^T}{\mathrm{d}t}f_{\dot{\bar{b}}}(t,a,\dot{a},\pi,b,\dot{b},\zeta)\Big]\mathrm{d}t\leqslant 0$$
$$\Rightarrow\int_{t_1}^{t_2} f(t,a,\dot{a},\pi,\bar{b},\dot{\bar{b}},\bar{\zeta})\,\mathrm{d}t\leqslant\int_{t_1}^{t_2} f(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t,$$

for all $\bar{b},\dot{\bar{b}}$ *and* $\bar{\zeta}$.

In the sequel, we will write $z(t,a,b)$ for $z(t,a,\dot{a},\pi,b,\dot{b},\zeta)$ and $\eta(t,a,b)$ for $\eta(t,a,\dot{a},\pi,b,\dot{b},\zeta)$. Also, we write (a,b) instead of (a,π,b,ζ).

3. Symmetric dual models

In this section, we introduce the controlled continuous analog associated with the static symmetric multiobjective dual programs (see Weir [38]):

$$\text{(P)}\quad \min_{(a,b)}\left(\frac{\int_{t_1}^{t_2} f^1(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t}{\int_{t_1}^{t_2} g^1(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t},\ \ldots,\ \frac{\int_{t_1}^{t_2} f^j(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t}{\int_{t_1}^{t_2} g^j(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t}\right)$$

subject to

$$a(t_1)=0=a(t_2),\qquad b(t_1)=0=b(t_2),$$
$$\dot{a}(t_1)=0=\dot{a}(t_2),\qquad \dot{b}(t_1)=0=\dot{b}(t_2),$$
$$\sum_{y=1}^{j}\Omega^y\left\{G^y(a,b)\left(f_b^y-\frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{b}}^y\right)-F^y(a,b)\left(g_b^y-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{b}}^y\right)\right\}\leqq 0,\quad t\in K,$$
$$\sum_{y=1}^{j}\Omega^y\left\{G^y(a,b)\left(f_\zeta^y-\frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{\zeta}}^y\right)-F^y(a,b)\left(g_\zeta^y-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{\zeta}}^y\right)\right\}\leqq 0,\quad t\in K,$$

$$b^T \sum_{y=1}^{j} \Omega^y \left\{ G^y(a,b) \left(f_b^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{b}}^y \right) - F^y(a,b) \left(g_b^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{b}}^y \right) \right\} \geqslant 0, \quad t \in K,$$

$$\zeta^T \sum_{y=1}^{j} \Omega^y \left\{ G^y(a,b) \left(f_\zeta^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\zeta}}^y \right) - F^y(a,b) \left(g_\zeta^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\zeta}}^y \right) \right\} \geqslant 0, \quad t \in K,$$

$$\Omega > 0,$$

and

$$\text{(D)} \qquad \max_{(u,v)} \left(\frac{\int_{t_1}^{t_2} f^1(t,u,\dot{u},\rho,v,\dot{v},\varrho)\,\mathrm{d}t}{\int_{t_1}^{t_2} g^1(t,u,\dot{u},\rho,v,\dot{v},\varrho)\,\mathrm{d}t}, \ldots, \frac{\int_{t_1}^{t_2} f^j(t,u,\dot{u},\rho,v,\dot{v},\varrho)\,\mathrm{d}t}{\int_{t_1}^{t_2} g^j(t,u,\dot{u},\rho,v,\dot{v},\varrho)\,\mathrm{d}t} \right)$$

subject to

$$u(t_1) = 0 = u(t_2), \quad v(t_1) = 0 = v(t_2),$$

$$\dot{u}(t_1) = 0 = \dot{u}(t_2), \quad \dot{v}(t_1) = 0 = \dot{v}(t_2),$$

$$\sum_{y=1}^{j} \Omega^y \left\{ G^y(u,v) \left(f_a^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{a}}^y \right) - F^y(u,v) \left(g_a^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{a}}^y \right) \right\} \geqq 0, \quad t \in K,$$

$$\sum_{y=1}^{j} \Omega^y \left\{ G^y(u,v) \left(f_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\pi}}^y \right) - F^y(u,v) \left(g_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\pi}}^y \right) \right\} \geqq 0, \quad t \in K,$$

$$u^T \sum_{y=1}^{j} \Omega^y \left\{ G^y(u,v) \left(f_a^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{a}}^y \right) - F^y(u,v) \left(g_a^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{a}}^y \right) \right\} \leqslant 0, \quad t \in K,$$

$$\rho^T \sum_{y=1}^{j} \Omega^y \left\{ G^y(u,v) \left(f_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\pi}}^y \right) - F^y(u,v) \left(g_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\pi}}^y \right) \right\} \leqslant 0, \quad t \in K,$$

$$\Omega > 0,$$

where, for $y = 1, 2, \ldots, j$, $f^y \colon K \times \mathbb{R}^{2n} \times \mathbb{R}^s \times \mathbb{R}^{2m} \times \mathbb{R}^l \to \mathbb{R}_+$, and $g^y \colon K \times \mathbb{R}^{2n} \times \mathbb{R}^s \times \mathbb{R}^{2m} \times \mathbb{R}^l \to \mathbb{R}_+ \setminus \{0\}$ are C^2-class functionals and

$$F^y(a,b) = \int_{t_1}^{t_2} f^y(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t, \quad G^y(a,b) = \int_{t_1}^{t_2} g^y(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t.$$

Remark 1. *If we consider $G^y(a,b) = 1$, $y = 1,2,\ldots,j$, and remove control functions, then the considered models (P) and (D) become the variational problems studied by Gulati et al. [15]. In addition, if we take $j = 1$ (and remove control functions), the above-mentioned models, (P) and (D), become the variational problems investigated in Gulati et al. [16].*

Let us notice that we do not consider (as in Kim et al. [22]) the restriction $\Omega^T e = 1, e = (1,1,\cdots,1) \in \mathbb{R}^j$, in the formulation of problems (P) and (D) since it does not interfere in establishing the following duality results. Moreover, its emergence creates difficulties in proving strong and converse duality theorems (see Remark 2).

Next, we give an equivalent variant of the above-mentioned considered models:

$$\text{(P}'\text{)} \quad \min_{(a,b)} \left(Y^1, Y^2, \ldots, Y^j\right)$$

subject to

$$a(t_1) = 0 = a(t_2), \qquad b(t_1) = 0 = b(t_2), \tag{1}$$

$$\dot{a}(t_1) = 0 = \dot{a}(t_2), \qquad \dot{b}(t_1) = 0 = \dot{b}(t_2), \tag{2}$$

$$\int_{t_1}^{t_2} f^y(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t - Y^y \int_{t_1}^{t_2} g^y(t,a,\dot{a},\pi,b,\dot{b},\zeta)\,\mathrm{d}t = 0, \quad y = 1,2,\ldots,j, \tag{3}$$

$$\sum_{y=1}^{j} \Omega^y \left\{\left(f_b^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{b}}^y\right) - Y^y \left(g_b^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{b}}^y\right)\right\} \leqq 0, \quad t \in K, \tag{4}$$

$$\sum_{y=1}^{j} \Omega^y \left\{\left(f_\zeta^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\zeta}}^y\right) - Y^y \left(g_\zeta^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\zeta}}^y\right)\right\} \leqq 0, \quad t \in K, \tag{4$'$}$$

$$b^T \sum_{y=1}^{j} \Omega^y \left\{\left(f_b^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{b}}^y\right) - Y^y \left(g_b^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{b}}^y\right)\right\} \geqslant 0, \quad t \in K, \tag{5}$$

$$\zeta^T \sum_{y=1}^{j} \Omega^y \left\{\left(f_\zeta^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\zeta}}^y\right) - Y^y \left(g_\zeta^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\zeta}}^y\right)\right\} \geqslant 0, \quad t \in K, \tag{5$'$}$$

$$\Omega > 0, \tag{6}$$

and

$$\text{(D}'\text{)} \qquad \max_{(u,v)} \left(X^1, X^2, \ldots, X^j \right)$$

subject to

$$u(t_1) = 0 = u(t_2), \qquad v(t_1) = 0 = v(t_2), \tag{7}$$

$$\dot{u}(t_1) = 0 = \dot{u}(t_2), \qquad \dot{v}(t_1) = 0 = \dot{v}(t_2), \tag{8}$$

$$\int_{t_1}^{t_2} f^y(t, u, \dot{u}, \rho, v, \dot{v}, \varrho)\,\mathrm{d}t - X^y \int_{t_1}^{t_2} g^y(t, u, \dot{u}, \rho, v, \dot{v}, \varrho)\,\mathrm{d}t = 0, \quad y = 1, 2, \ldots, j, \tag{9}$$

$$\sum_{y=1}^{j} \Omega^y \left\{ \left(f_a^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{a}}^y \right) - X^y \left(g_a^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{a}}^y \right) \right\} \geqq 0, \quad t \in K, \tag{10}$$

$$\sum_{y=1}^{j} \Omega^y \left\{ \left(f_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\pi}}^y \right) - X^y \left(g_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\pi}}^y \right) \right\} \geqq 0, \quad t \in K, \tag{10$'$}$$

$$u^T \sum_{y=1}^{j} \Omega^y \left\{ \left(f_a^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{a}}^y \right) - X^y \left(g_a^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{a}}^y \right) \right\} \leqslant 0, \quad t \in K, \tag{11}$$

$$\rho^T \sum_{y=1}^{j} \Omega^y \left\{ \left(f_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{\pi}}^y \right) - X^y \left(g_\pi^y - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{\pi}}^y \right) \right\} \leqslant 0, \quad t \in K, \tag{11$'$}$$

$$\Omega > 0. \tag{12}$$

Further, let us consider A and B as the feasible solution sets for (P) and (D), respectively.

The following two results (weak and strong type duality) are formulated in accordance with (P′) and (D′) but, of course, these results apply equally to (P) and (D).

Theorem 1. *Let* $(a(t), b(t), \Omega, Y) \in A$ *and* $(u(t), v(t), \Omega, X) \in B$ *be feasible solutions in (P′) and (D′), respectively. Assume that:*

(i) $\sum_{y=1}^{j} \Omega^y \int_{t_1}^{t_2} \left\{ f^y(t, (\cdot), (\cdot), (\cdot), v, \dot{v}, \varrho) - X^y g^y(t, (\cdot), (\cdot), (\cdot), v, \dot{v}, \varrho) \right\} \mathrm{d}t$ *is pseudoinvex at* $u, \dot{u}$ *and* ρ*, with* $z(t, a, u) + u(t) \geqq 0$*,* $\mu(t, a, u) + \rho(t) \geqq 0$*,* $t \in K$*;*

(ii) $-\sum_{y=1}^{j}\Omega^y\int_{t_1}^{t_2}\left\{f^y(t,u,\dot{u},\rho,(\cdot),(\cdot),(\cdot))-Y^y g^y(t,u,\dot{u},\rho,(\cdot),(\cdot),(\cdot))\right\}\mathrm{d}t$ *is pseudoinvex at* $v,\dot{v}$ *and* ϱ*, with* $\eta(t,v,b)+b(t)\geqq 0$*,* $\nu(t,v,b)+\zeta(t)\geqq 0$*,* $t\in K$*.*

Then, the relation $Y\nleq X$ *is valid.*

Proof. Relations given in (10) and (10′), joint with $z(t,a,u)+u(t)\geqq 0$, $\mu(t,a,u)+\rho(t)\geqq 0$, $t\in K$, involves

$$[z(t,au)+u(t)]^T\left[\Omega\left\{\left(f_a-\frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{a}}\right)-X\left(g_a-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{a}}\right)\right\}\right]\geqslant 0,$$

$$[\mu(t,a,u)+\rho(t)]^T\left[\Omega\left\{\left(f_\pi-\frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{\pi}}\right)-X\left(g_\pi-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{\pi}}\right)\right\}\right]\geqslant 0,$$

and, by using (11) and (11′), we get

$$(z(t,a,u))^T\left[\Omega\left\{\left(f_a-\frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{a}}\right)-X\left(g_a-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{a}}\right)\right\}\right]\geqslant 0,\qquad t\in K,$$

$$(\mu(t,a,u))^T\left[\Omega\left\{\left(f_\pi-\frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{\pi}}\right)-X\left(g_\pi-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{\pi}}\right)\right\}\right]\geqslant 0,\qquad t\in K,$$

which imply

$$\begin{aligned}0&\leqslant\int_{t_1}^{t_2}z(t,a,u)^T\left[\Omega\left\{(f_a-Xg_a)-\frac{\mathrm{d}}{\mathrm{d}t}(f_{\dot{a}}-Xg_{\dot{a}})\right\}\right]\mathrm{d}t\\&=\int_{t_1}^{t_2}\left[z(t,a,u)^T\Omega(f_a-Xg_a)+\frac{\mathrm{d}z(t,a,u)^T}{\mathrm{d}t}\Omega\,(f_{\dot{a}}-Xg_{\dot{a}})\right]\mathrm{d}t\\&\quad-z(t,a,u)^T\Omega\,(f_{\dot{a}}-Xg_{\dot{a}})\Big|_{t=t_1}^{t=t_2},\end{aligned}$$

and

$$\int_{t_1}^{t_2}\left[\mu(t,a,u)^T\Omega(f_\pi-Xg_\pi)+\frac{\mathrm{d}\mu(t,a,u)^T}{\mathrm{d}t}\Omega\,(f_{\dot{\pi}}-Xg_{\dot{\pi}})\right]\mathrm{d}t\geqslant 0.$$

Since $z(t,a,u)=0$, at $t=t_1$ and $t=t_2$, it follows

$$\int_{t_1}^{t_2}\left[z(t,a,u)^T\Omega(f_a-Xg_a)+\frac{\mathrm{d}z(t,a,u)^T}{\mathrm{d}t}\Omega\,(f_{\dot{a}}-Xg_{\dot{a}})\right]\mathrm{d}t\geqslant 0$$

and

$$\int_{t_1}^{t_2} \left[\mu(t,a,u)^T \Omega(f_\pi - Xg_\pi) + \frac{\mathrm{d}\mu(t,a,u)^T}{\mathrm{d}t} \Omega\left(f_{\dot\pi} - Xg_{\dot\pi}\right)\right] \mathrm{d}t \geqslant 0,$$

involving

$$\int_{t_1}^{t_2} \left[z(t,a,u)^T \Omega(f_a - Xg_a) + \mu(t,a,u)^T \Omega(f_\pi - Xg_\pi) \right.$$
$$\left. + \frac{\mathrm{d}z(t,a,u)^T}{\mathrm{d}t} \Omega\left(f_{\dot a} - Xg_{\dot a}\right)\right] \mathrm{d}t \geqslant 0.$$

Now, by using the pseudoinvexity assumption given in (i), we obtain

$$\Omega \int_{t_1}^{t_2} \{f(t,a,\dot a,\pi,v,\dot v,\varrho) - Xg(t,a,\dot a,\pi,v,\dot v,\varrho)\} \,\mathrm{d}t$$
$$\geqslant \Omega \int_{t_1}^{t_2} \{f(t,u,\dot u,\rho,v,\dot v,\varrho) - Xg(t,u,\dot u,\rho,v,\dot v,\varrho)\} \,\mathrm{d}t.$$

Taking into account the relation in (9), the above inequality gives

$$\Omega \int_{t_1}^{t_2} \{f(t,a,\dot a,\pi,v,\dot v,\varrho) - Xg(t,a,\dot a,\pi,v,\dot v,\varrho)\} \,\mathrm{d}t \geqslant 0. \tag{13}$$

On the other hand, relations given in (4) and (4′), together with $\eta(t,v,b) + b(t) \geqq 0$, $\nu(t,v,b) + \zeta(t) \geqq 0$, $t \in K$, implies

$$[\eta(t,v,b) + b(t)]^T \left[\Omega\left\{\left(f_b - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot b}\right) - Y\left(g_b - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot b}\right)\right\}\right] \leqslant 0,$$

$$[\nu(t,v,b) + \zeta(t)]^T \left[\Omega\left\{\left(f_\zeta - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot\zeta}\right) - Y\left(g_\zeta - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot\zeta}\right)\right\}\right] \leqslant 0,$$

and, by using (5) and (5′), we get

$$(\eta(t,v,b))^T \left[\Omega\left\{\left(f_b - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot b}\right) - Y\left(g_b - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot b}\right)\right\}\right] \leqslant 0, \quad t \in K,$$

$$(\nu(t,v,b))^T \left[\Omega\left\{\left(f_\zeta - \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot\zeta}\right) - Y\left(g_\zeta - \frac{\mathrm{d}}{\mathrm{d}t} g_{\dot\zeta}\right)\right\}\right] \leqslant 0, \quad t \in K,$$

which imply

$$0 \geqslant \int_{t_1}^{t_2} \eta(t,v,b)^T \left[\Omega\left\{(f_b - Yg_b) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}} - Yg_{\dot{b}}\right)\right\}\right] \mathrm{d}t$$

$$= \int_{t_1}^{t_2} \left[\eta(t,v,b)^T \Omega(f_b - Yg_b) + \frac{\mathrm{d}\eta(t,v,b)^T}{\mathrm{d}t}\Omega\left(f_{\dot{b}} - Yg_{\dot{b}}\right)\right] \mathrm{d}t$$

$$- \ \eta(t,v,b)^T \Omega\left(f_{\dot{b}} - Yg_{\dot{b}}\right)\Big|_{t=t_1}^{t=t_2},$$

and

$$\int_{t_1}^{t_2} \left[\nu(t,v,b)^T \Omega(f_\zeta - Yg_\zeta) + \frac{\mathrm{d}\nu(t,v,b)^T}{\mathrm{d}t}\Omega\left(f_{\dot{\zeta}} - Yg_{\dot{\zeta}}\right)\right] \mathrm{d}t \leqslant 0.$$

Since $\eta(t,v,b) = 0$, at $t = t_1$ and $t = t_2$, it follows

$$\int_{t_1}^{t_2} \left[\eta(t,v,b)^T \Omega(f_b - Yg_b) + \frac{\mathrm{d}\eta(t,v,b)^T}{\mathrm{d}t}\Omega\left(f_{\dot{b}} - Yg_{\dot{b}}\right)\right] \mathrm{d}t \leqslant 0$$

and

$$\int_{t_1}^{t_2} \left[\nu(t,v,b)^T \Omega(f_\zeta - Yg_\zeta) + \frac{\mathrm{d}\nu(t,v,b)^T}{\mathrm{d}t}\Omega\left(f_{\dot{\zeta}} - Yg_{\dot{\zeta}}\right)\right] \mathrm{d}t \leqslant 0,$$

involving

$$\int_{t_1}^{t_2} \left[\eta(t,v,b)^T \Omega(f_b - Yg_b) + \nu(t,v,b)^T \Omega(f_\zeta - Yg_\zeta)\right.$$

$$\left. + \frac{\mathrm{d}\eta(t,v,b)^T}{\mathrm{d}t}\Omega\left(f_{\dot{b}} - Yg_{\dot{b}}\right)\right] \mathrm{d}t \leqslant 0.$$

Now, by using the pseudoinvexity assumption given in (ii), we obtain

$$\Omega \int_{t_1}^{t_2} \left\{f(t,a,\dot{a},\pi,v,\dot{v},\varrho) - Yg(t,a,\dot{a},\pi,v,\dot{v},\varrho)\right\} \mathrm{d}t$$

$$\leqslant \Omega \int_{t_1}^{t_2} \left\{f(t,a,\dot{a},\pi,b,\dot{b},\zeta) - Yg(t,a,\dot{a},\pi,b,\dot{b},\zeta)\right\} \mathrm{d}t.$$

Taking into account the relation in (3), the above inequality gives

$$\Omega \int_{t_1}^{t_2} \{f(t,a,\dot{a},\pi,v,\dot{v},\varrho) - Y g(t,a,\dot{a},\pi,v,\dot{v},\varrho)\}\,\mathrm{d}t \leqslant 0.$$

The above inequality along with (13) yields

$$\sum_{y=1}^{j} \Omega^y \left(Y^y - X^y\right) \int_{t_1}^{t_2} g^y(t,a,\dot{a},\pi,v,\dot{v},\varrho)\mathrm{d}t \geqslant 0. \tag{14}$$

If, for some $y \in \{1,2,\dots,j\}$, we have $Y^y < X^y$, and, for $i \in \{1,2,\dots,j\}$, with $i \neq y$, we have $Y^i \leqslant X^i$, then since $\int_{t_1}^{t_2} g^y(t,a,\dot{a},\pi,v,\dot{v},\varrho)\mathrm{d}t > 0$ and $\Omega > 0$, it results a contradiction with relation given in (14). □

Theorem 2. *Consider the hypotheses stated in Theorem 1 are satisfied and assume that:*

(i) $(\bar{a},\bar{b},\bar{\Omega},\bar{Y})$ *is a proper efficient point in* (P')*;*

(ii) consider the following relations

$$\Psi(t)^T \left\{ \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f^y_{bb} - \bar{Y}^y g^y_{bb}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}b} - \bar{Y}^y g^y_{\dot{b}b}\right) \right\} \right] \right.$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}\left[\Psi(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}\dot{b}} - \bar{Y}^y g^y_{\dot{b}\dot{b}}\right) \right\} \right]$$
$$+ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{ -\Psi(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left(f^y_{\dot{b}\dot{b}} - \bar{Y}^y g^y_{\dot{b}\dot{b}}\right) \right\} + \bar{\Psi}(t)^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left(f^y_{\zeta b} - \bar{Y}^y g^y_{\zeta b}\right) \right]$$
$$\left. - \frac{\mathrm{d}}{\mathrm{d}t}\left[\bar{\Psi}(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left(f^y_{\zeta\dot{b}} - \bar{Y}^y g^y_{\zeta\dot{b}}\right) \right] \right\} \Psi(t) = 0,$$

$$\left\{ \left(f^y_b - \bar{Y}^y g^y_b\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}} - \bar{Y}^y g^y_{\dot{b}}\right) \right\} \Psi(t)^T$$
$$+ \left\{ \left(f^y_\zeta - \bar{Y}^y g^y_\zeta\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{\zeta}} - \bar{Y}^y g^y_{\dot{\zeta}}\right) \right\} \bar{\Psi}(t)^T = 0, \quad y \in \{1,2,\dots,j\}, \quad t \in K,$$

implies $\Psi(t) = \bar{\Psi}(t) = 0$, $t \in K$*;*

(iii) the following elements

$$\left\{\left(\left(f_b^1 - \bar{Y}^1 g_b^1\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^1 - \bar{Y}^1 g_{\dot{b}}^1\right)\right), \ldots, \left(\left(f_b^j - \bar{Y}^j g_b^j\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^j - \bar{Y}^j g_{\dot{b}}^j\right)\right)\right\}$$

are linear independent.

In this case, the point $(\bar{a}, \bar{b}, \Omega = \bar{\Omega}, \bar{Y})$ *is a proper efficient point in (D′).*

Proof. Since $(\bar{a}, \bar{b}, \bar{\Omega}, \bar{Y})$ is a properly efficient point of (P′), we get $(\bar{a}, \bar{b}, \bar{\Omega}, \bar{Y})$ is a weak efficient point of (P′). Thus, by Fritz John criteria (see Craven [10]), there exist the multipliers $\alpha, \beta, \Upsilon \in \mathbb{R}^j$, and the piecewise smooth functions $\Gamma\colon K \to \mathbb{R}^m$, $\omega\colon K \to \mathbb{R}^l$, δ, $\varsigma\colon K \to \mathbb{R}$, such that

$$\begin{aligned} L = &\sum_{y=1}^{j} \alpha^y \bar{Y}^y + \sum_{y=1}^{j} \beta^y \left(f^y - \bar{Y}^y g^y\right) \\ &+ [\Gamma(t) - \delta(t)\bar{b}(t)]^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{\left(f_b^y - \bar{Y}^y g_b^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y - \bar{Y}^y g_{\dot{b}}^y\right)\right\}\right] \\ &+ [\omega(t) - \varsigma(t)\bar{\zeta}(t)]^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{\left(f_\zeta^y - \bar{Y}^y g_\zeta^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}}^y - \bar{Y}^y g_{\dot{\zeta}}^y\right)\right\}\right] - \Upsilon^T \bar{\Omega} \end{aligned}$$

fulfils the next conditions at $(\bar{a}, \bar{b}, \bar{\Omega}, \bar{Y})$:

$$L_a - \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{a}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} L_{\ddot{a}} = 0, \quad t \in K, \tag{15}$$

$$L_b - \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{b}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} L_{\ddot{b}} = 0, \quad t \in K, \tag{16}$$

$$L_\Omega = 0, \quad t \in K, \tag{17}$$

$$L_Y = 0, \quad t \in K, \tag{18}$$

$$\int_{t_1}^{t_2} \beta^y \left(f^y - \bar{Y}^y g^y\right) \mathrm{d}t = 0, \quad y \in \{1, 2, \ldots, j\}, \quad t \in K, \tag{19}$$

$$\Gamma(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left\{\left(f_b^y - \bar{Y}^y g_b^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y - \bar{Y}^y g_{\dot{b}}^y\right)\right\} = 0, \quad t \in K, \tag{20}$$

$$\delta(t)\bar{b}(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left\{\left(f_b^y - \bar{Y}^y g_b^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y - \bar{Y}^y g_{\dot{b}}^y\right)\right\} = 0, \quad t \in K, \tag{21}$$

$$\omega(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_\zeta^y - \bar{Y}^y g_\zeta^y \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(f_{\dot{\zeta}}^y - \bar{Y}^y g_{\dot{\zeta}}^y \right) \right\} = 0, \quad t \in K, \tag{22}$$

$$\varsigma(t) \bar{\zeta}(t)^T \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_\zeta^y - \bar{Y}^y g_\zeta^y \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(f_{\dot{\zeta}}^y - \bar{Y}^y g_{\dot{\zeta}}^y \right) \right\} = 0, \quad t \in K, \tag{23}$$

$$\Upsilon^T \bar{\Omega} = 0, \tag{24}$$

$$((\alpha, \Gamma(t), \delta(t), \omega(t), \Upsilon, \varsigma(t)) \geqq 0, \quad t \in K, \tag{25}$$

$$(\alpha, \beta, \Gamma(t), \delta(t), \omega(t), \Upsilon, \varsigma(t)) \neq 0, \quad t \in K, \tag{26}$$

$$L_\pi = 0, \quad t \in K, \tag{27}$$

$$L_\zeta = 0, \quad t \in K. \tag{28}$$

The relations given above are valid in K, excepting the peaks associated with $(\bar{a}(t), \bar{b}(t), \bar{\Omega}, \bar{Y})$, where the relations (15), (16) are satisfied for unique left and right-hand limits.

Next, by using the computations on $\frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{b}}^y$ and $\frac{\mathrm{d}}{\mathrm{d}t} g_{\dot{b}}^y$, $y \in \{1, 2, \ldots, j\}$, from Section 2, the relations given in (15)–(18) become:

$$\begin{aligned}
&\sum_{y=1}^{j} \beta^y \left\{ (f_a^y - \bar{Y}^y g_a^y) - \frac{\mathrm{d}}{\mathrm{d}t} (f_{\dot{a}}^y - \bar{Y}^y g_{\dot{a}}^y) \right\} + (\Gamma(t) - \delta(t) \bar{b}(t))^T \\
&\quad \times \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{ba}^y - \bar{Y}^y g_{ba}^y \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(f_{\dot{b}a}^y - \bar{Y}^y g_{\dot{b}a}^y \right) \right\} - \frac{\mathrm{d}}{\mathrm{d}t} (\Gamma(t) - \delta(t) \bar{b}(t))^T \\
&\quad \times \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{b\dot{a}}^y - \bar{Y}^y g_{b\dot{a}}^y \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(f_{\dot{b}\dot{a}}^y - \bar{Y}^y g_{\dot{b}\dot{a}}^y \right) - \left(f_{\dot{b}a}^y - \bar{Y}^y g_{\dot{b}a}^y \right) \right\} \\
&\quad + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left\{ -(\Gamma(t) - \delta(t) \bar{b}(t))^T \sum_{y=1}^{j} \bar{\Omega}^y \left(f_{\dot{b}\dot{a}}^y - \bar{Y}^y g_{\dot{b}\dot{a}}^y \right) \right\} + (\omega(t) - \varsigma(t) \bar{\zeta}(t))^T \\
&\quad \times \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{\zeta a}^y - \bar{Y}^y g_{\zeta a}^y \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(f_{\dot{\zeta}a}^y - \bar{Y}^y g_{\dot{\zeta}a}^y \right) \right\} - \frac{\mathrm{d}}{\mathrm{d}t} (\omega(t) - \varsigma(t) \bar{\zeta}(t))^T \\
&\quad \times \sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{\zeta\dot{a}}^y - \bar{Y}^y g_{\zeta\dot{a}}^y \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(f_{\dot{\zeta}\dot{a}}^y - \bar{Y}^y g_{\dot{\zeta}\dot{a}}^y \right) - \left(f_{\dot{\zeta}a}^y - \bar{Y}^y g_{\dot{\zeta}a}^y \right) \right\} \\
&\quad + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left\{ -(\omega(t) - \varsigma(t) \bar{\zeta}(t))^T \sum_{y=1}^{j} \bar{\Omega}^y \left(f_{\dot{\zeta}\dot{a}}^y - \bar{Y}^y g_{\dot{\zeta}\dot{a}}^y \right) \right\} = 0, \quad t \in K,
\end{aligned} \tag{29}$$

$$\sum_{y=1}^{j}(\beta^y-\delta(t)\bar{\Omega}^y)\left\{\left(f_b^y-\bar{Y}^y g_b^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y-\bar{Y}^y g_{\dot{b}}^y\right)\right\}$$
$$+(\Gamma(t)-\delta(t)\bar{b}(t))^T\left[\sum_{y=1}^{j}\bar{\Omega}^y\left\{\left(f_{bb}^y-\bar{Y}^y g_{bb}^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}b}^y-\bar{Y}^y g_{\dot{b}b}^y\right)\right\}\right]$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\left[(\Gamma(t)-\delta(t)\bar{b}(t))^T\sum_{y=1}^{j}\bar{\Omega}^y\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}\dot{b}}^y-\bar{Y}^y g_{\dot{b}\dot{b}}^y\right)\right\}\right]$$
$$+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{-(\Gamma(t)-\delta(t)\bar{b}(t))^T\sum_{y=1}^{j}\bar{\Omega}^y\left(f_{\dot{b}\dot{b}}^y-\bar{Y}^y g_{\dot{b}\dot{b}}^y\right)\right\}$$
$$+(\omega(t)-\varsigma(t)\bar{\zeta}(t))^T\left[\sum_{y=1}^{j}\bar{\Omega}^y\left\{\left(f_{\zeta b}^y-\bar{Y}^y g_{\zeta b}^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}b}^y-\bar{Y}^y g_{\dot{\zeta}b}^y\right)\right\}\right]$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\left[(\omega(t)-\varsigma(t)\bar{\zeta}(t))^T\sum_{y=1}^{j}\bar{\Omega}^y\left(f_{\zeta\dot{b}}^y-\bar{Y}^y g_{\zeta\dot{b}}^y\right)\right]=0,\quad t\in K, \tag{30}$$

$$\left\{\left(f_b^y-\bar{Y}^y g_b^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y-\bar{Y}^y g_{\dot{b}}^y\right)\right\}(\Gamma(t)-\delta(t)\bar{b}(t))^T$$
$$+\left\{\left(f_\zeta^y-\bar{Y}^y g_\zeta^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}}^y-\bar{Y}^y g_{\dot{\zeta}}^y\right)\right\}(\omega(t)-\varsigma(t)\bar{\zeta}(t))^T-\Upsilon^T=0,$$
$$y\in\{1,2,\ldots,j\},\quad t\in K, \tag{31}$$

and

$$\alpha^y-\beta^y g^y-(\Gamma(t)-\delta(t)\bar{b}(t))^T\bar{\Omega}^y\left(g_b^y-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{b}}^y\right)$$
$$-(\omega(t)-\varsigma(t)\bar{\zeta}(t))^T\bar{\Omega}^y\left(g_\zeta^y-\frac{\mathrm{d}}{\mathrm{d}t}g_{\dot{\zeta}}^y\right)=0,\quad y\in\{1,2,\ldots,j\},\ t\in K. \tag{32}$$

By considering $\Upsilon \geqq 0$, from $\bar{\Omega}>0$ and relation (24), we get $\Upsilon=0$. Consequently, relations given in (31) become

$$\left\{\left(f_b^y-\bar{Y}^y g_b^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y-\bar{Y}^y g_{\dot{b}}^y\right)\right\}(\Gamma(t)-\delta(t)\bar{b}(t))^T$$
$$+\left\{\left(f_\zeta^y-\bar{Y}^y g_\zeta^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}}^y-\bar{Y}^y g_{\dot{\zeta}}^y\right)\right\}(\omega(t)-\varsigma(t)\bar{\zeta}(t))^T=0,$$
$$y\in\{1,2,\ldots,j\},\ t\in K, \tag{33}$$

involving, as a special case, the following

$$\left\{\left(f^y_b - \bar{Y}^y g^y_b\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}} - \bar{Y}^y g^y_{\dot{b}}\right)\right\}(\Gamma(t) - \delta(t)\bar{b}(t))^T = 0,$$
$$y \in \{1, 2, \ldots, j\}, \quad t \in K, \tag{34}$$

and

$$\left\{\left(f^y_\zeta - \bar{Y}^y g^y_\zeta\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{\zeta}} - \bar{Y}^y g^y_{\dot{\zeta}}\right)\right\}(\omega(t) - \varsigma(t)\bar{\zeta}(t))^T = 0,$$
$$y \in \{1, 2, \ldots, j\}, \quad t \in K.$$

Multiplying by $(\Gamma(t) - \delta(t)\bar{b}(t))$ relation (30) and by considering (34), it results

$$\begin{aligned}
&(\Gamma(t) - \delta(t)\bar{b}(t))^T \left\{\left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{\left(f^y_{bb} - \bar{Y}^y g^y_{bb}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}b} - \bar{Y}^y g^y_{\dot{b}b}\right)\right\}\right]\right. \\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\left[(\Gamma(t) - \delta(t)\bar{b}(t))^T \sum_{y=1}^{j} \bar{\Omega}^y \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}\dot{b}} - \bar{Y}^y g^y_{\dot{b}\dot{b}}\right)\right\}\right] \\
&+ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{-(\Gamma(t) - \delta(t)\bar{b}(t))^T \sum_{y=1}^{j} \bar{\Omega}^y \left(f^y_{\dot{b}\dot{b}} - \bar{Y}^y g^y_{\dot{b}\dot{b}}\right)\right\} \\
&+ (\omega(t) - \varsigma(t)\bar{\zeta}(t))^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left(f^y_{\zeta b} - \bar{Y}^y g^y_{\zeta b}\right)\right] \\
&\left.- \frac{\mathrm{d}}{\mathrm{d}t}\left[(\omega(t) - \varsigma(t)\bar{\zeta}(t))^T \sum_{y=1}^{j} \bar{\Omega}^y \left(f^y_{\zeta\dot{b}} - \bar{Y}^y g^y_{\zeta\dot{b}}\right)\right]\right\} \\
&\times (\Gamma(t) - \delta(t)\bar{b}(t)) = 0.
\end{aligned} \tag{35}$$

This, following hypothesis (ii), yields

$$\Gamma(t) - \delta(t)\bar{b}(t) = 0, \quad t \in K, \tag{36}$$
$$\omega(t) - \varsigma(t)\bar{\zeta}(t) = 0, \quad t \in K. \tag{37}$$

From (30), we have

$$\sum_{y=1}^{j} (\beta^y - \delta(t)\bar{\Omega}^y)\left\{\left(f^y_b - \bar{Y}^y g^y_b\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f^y_{\dot{b}} - \bar{Y}^y g^y_{\dot{b}}\right)\right\} = 0$$

which, by considering hypothesis (iii), implies

$$\beta^y - \delta(t)\bar{\Omega}^y = 0, \quad y \in \{1, 2, \ldots, j\}$$

or

$$\beta = \delta(t)\bar{\Omega}. \tag{38}$$

Now, by (29), (36) and (38) we obtain

$$\sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_a^y - \bar{Y}^y g_a^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{a}}^y - \bar{Y}^y g_{\dot{a}}^y\right) \right\} = 0, \quad t \in K. \tag{39}$$

From (36), we get

$$\bar{b}(t) = \frac{\Gamma(t)}{\delta(t)} \geqq 0, \quad t \in K. \tag{40}$$

By considering the relation formulated in (28), we obtain

$$\begin{aligned}
&\sum_{y=1}^{j} \left(\beta^y - \varsigma(t)\bar{\Omega}^y\right) \left\{ \left(f_\zeta^y - \bar{Y}^y g_\zeta^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}}^y - \bar{Y}^y g_{\dot{\zeta}}^y\right) \right\} \\
&+ (\Gamma(t) - \delta(t)\bar{b}(t))^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{b\zeta}^y - \bar{Y}^y g_{b\zeta}^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}\zeta}^y - \bar{Y}^y g_{\dot{b}\zeta}^y\right) \right\} \right] \\
&+ (\omega(t) - \varsigma(t)\bar{\zeta}(t))^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{\zeta\zeta}^y - \bar{Y}^y g_{\zeta\zeta}^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}\zeta}^y - \bar{Y}^y g_{\dot{\zeta}\zeta}^y\right) \right\} \right] = 0, \ t \in K.
\end{aligned} \tag{41}$$

By considering the relation formulated in (27), we obtain

$$\begin{aligned}
&\sum_{y=1}^{j} \beta^y \left(f_\pi^y - \bar{Y}^y g_\pi^y\right) \\
&+ (\Gamma(t) - \delta(t)\bar{b}(t))^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{b\pi}^y - \bar{Y}^y g_{b\pi}^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}\pi}^y - \bar{Y}^y g_{\dot{b}\pi}^y\right) \right\} \right] \\
&+ (\omega(t) - \varsigma(t)\bar{\zeta}(t))^T \left[\sum_{y=1}^{j} \bar{\Omega}^y \left\{ \left(f_{\zeta\pi}^y - \bar{Y}^y g_{\zeta\pi}^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}\pi}^y - \bar{Y}^y g_{\dot{\zeta}\pi}^y\right) \right\} \right] = 0, \ t \in K.
\end{aligned} \tag{42}$$

From relations (36) and (38), for $t \in K, \delta(t) = 0$, we get $\beta = 0$ and $\Gamma(t) = 0$, respectively. Moreover, relation (32) involves $\alpha = 0$. A similar analysis can be established for $\varsigma(t) = 0$, $t \in K$. Hence $(\alpha, \beta, \Gamma(t), \delta(t), \omega(t), \Upsilon, \varsigma(t)) = 0$, contradicting the Fritz John condition (26). Thus, it results $\delta(t),\ \varsigma(t) > 0, t \in K$.

By (39) and (40)–(42), we obtain $(\bar{a}, \bar{b}, \bar{\Omega}, \bar{Y}) \in B$ i.e., $(\bar{a}, \bar{b}, \bar{Y})$ is feasible point in (D′) with $\Omega = \bar{\Omega}$, and, moreover, the cost functionals become equal.

If $(\bar{a}, \bar{b}, \bar{Y})$ is not an efficient point for (D′) with $\Omega = \bar{\Omega}$, then there exists a point $\left(u^0, v^0, X^0\right)$ feasible for (D′) such that

$$X^0 \geqslant \bar{Y}$$

which contradicts Theorem 1. Next, let us as show that $(\bar{a}, \bar{b}, \bar{\Omega}, \bar{Y})$ is a properly efficient point for (D′). By contrary, if it is not so, therefore, for some $\left(u^0, v^0, \bar{\Omega}, X^0\right) \in B$ and some y, $X^{0y} - \bar{Y}^y > M$ for any $M > 0$, since

$$\int_{t_1}^{t_2} g^y \left(t, \bar{a}, \dot{\bar{a}}, \bar{\pi}, v^0, \dot{v}^0, \varrho^0\right) \mathrm{d}t > 0$$

and $\bar{\Omega}^y > 0$, $y \in \{1, 2, \ldots, j\}$, it follows

$$\sum_{y=1}^{j} \bar{\Omega}^y \left(\bar{Y}^y - X^{0y}\right) \int_{t_1}^{t_2} g^y \left(t, \bar{a}, \dot{\bar{a}}, \bar{\pi}, v^0, \dot{v}^0, \varrho^0\right) \mathrm{d}t < 0,$$

which is in contradiction with Theorem 1. Consequently, $(\bar{a}, \bar{b}, \bar{Y})$ is properly efficient point of (D′) with $\Omega = \bar{\Omega}$. □

Remark 2. *As we established previously, unlike Kim et al. [22], we can not consider $\Omega^T e = 1$ in the primal model since we need a new multiplier $\epsilon \in R$ associated with it and, thus, (31) is formulated as follows*

$$\begin{aligned}
&\left\{\left(f_b^y - \bar{Y}^y g_b^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{b}}^y - \bar{Y}^y g_{\dot{b}}^y\right)\right\} (\Gamma(t) - \delta(t)\bar{b}(t))^T \\
&+ \left\{\left(f_\zeta^y - \bar{Y}^y g_\zeta^y\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\zeta}}^y - \bar{Y}^y g_{\dot{\zeta}}^y\right)\right\} (\omega(t) - \varsigma(t)\bar{\zeta}(t))^T - \Upsilon^T + \epsilon = 0, \\
&y \in \{1, 2, \ldots, j\}, \quad t \in K.
\end{aligned}$$

By the above computation, we obtain $\Upsilon = 0$ but we can not get $\epsilon = 0$. Thus, we do not have (33), playing a pivotal role in establishing the strong-type duality.

Next, we state a converse-type duality theorem. Its proof is in the same manner as in Theorem 2 given above.

Theorem 3. *Consider the assumptions given in Theorem 1 are fulfilled and suppose that:*

(i) $(\bar{u}, \bar{v}, \bar{\Omega}, \bar{X})$ *is a proper efficient point in (D′);*

(ii) consider the following relations

$$\Psi(t)^T\Bigg\{\left[\sum_{y=1}^{j}\bar{\Omega}^y\left\{\left(f_{aa}^y-\bar{Y}^y g_{aa}^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{a}a}^y-\bar{Y}^y g_{\dot{a}a}^y\right)\right\}\right]$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\left[\Psi(t)^T\sum_{y=1}^{j}\bar{\Omega}^y\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{a}\dot{a}}^y-\bar{Y}^y g_{\dot{a}\dot{a}}^y\right)\right\}\right]$$
$$+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{-\Psi(t)^T\sum_{y=1}^{j}\bar{\Omega}^y\left(f_{\dot{a}\dot{a}}^y-\bar{Y}^y g_{\dot{a}\dot{a}}^y\right)\right\}$$
$$+\bar{\Psi}(t)^T\left[\sum_{y=1}^{j}\bar{\Omega}^y\left(f_{\pi a}^y-\bar{Y}^y g_{\pi a}^y\right)\right]$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\left[\bar{\Psi}(t)^T\sum_{y=1}^{j}\bar{\Omega}^y\left(f_{\pi\dot{a}}^y-\bar{Y}^y g_{\pi\dot{a}}^y\right)\right]\Bigg\}\Psi(t)=0,$$

$$\left\{\left(f_a^y-\bar{Y}^y g_a^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{a}}^y-\bar{Y}^y g_{\dot{a}}^y\right)\right\}\Psi(t)^T$$
$$+\left\{\left(f_\pi^y-\bar{Y}^y g_\pi^y\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{\pi}}^y-\bar{Y}^y g_{\dot{\pi}}^y\right)\right\}\bar{\Psi}(t)^T=0,$$
$$y\in\{1,2,\ldots,j\},\qquad t\in K,$$

implies $\Psi(t)=\bar{\Psi}(t)=0,\ t\in K$*;*

(iii) the following elements

$$\left\{\left(\left(f_a^1-\bar{Y}^1 g_a^1\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{a}}^1-\bar{Y}^1 g_{\dot{a}}^1\right)\right),\ldots,\left(\left(f_a^j-\bar{Y}^j g_a^j\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{\dot{a}}^j-\bar{Y}^j g_{\dot{a}}^j\right)\right)\right\}$$

are linear independent.

In this case, the point $(\bar{u}, \bar{v}, \Omega=\bar{\Omega}, \bar{X})$ *is a proper efficient point in (P′).*

4. Conclusions

In this study, we have employed weaker generalized convexity hypotheses than those formulated in Chen [8], Kim et al. [22], following Ahmad [2] (for classical multiobjective variational problems) in order to establish and provide characterizations of symmetric dual models associated with new multiple cost variational control problems. More precisely, we have extended the framework to controlled variational models, and, therefore, the derived results are significantly stronger and more generous than those presented so far in the specialized literature.

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