DOI 10.24425/aee.2025.154994

Investigation of linear periodically time-varying circuits based on their frequency characteristics

YURIY SHAPOVALOV [1], DARIYA BACHYK [1], ANNA PIWOWAR [2]✉

[1] *Department of Radioelectronic Technologies of Information Systems*
Lviv Polytechnic National University
2 Profesorska St., Building 11, Lviv, 79013, Ukraine

[2] *Department of Electrical Engineering and Computer Science*
Silesian University of Technology
ul. Akademicka 2A, 44-100 Gliwice, Poland

e-mail: {Yurii.I.Shapovalov/Dariia.R.Bachyk}@lpnu.ua, ✉ Anna.Piwowar@polsl.pl

(Received: 24.09.2024, revised: 27.10.2025)

Abstract: Linear circuits with constant parameters are typically analysed based on their frequency characteristics, assuming the invariance of harmonic signals as they pass through the system. However, in linear periodically time-varying circuits, this invariance does not hold. The output signal in such circuits is a periodic signal composed of harmonic components at various frequencies determined by both the input signal and the periodic variation of the circuit parameters. This results in a frequency spectrum that includes harmonics beyond the frequency of the input signal, influenced by the interaction between the input and the parameter variations. This paper investigates the behaviour of parametric devices by examining specific frequency ratios between the input signal and the variation in circuit parameters. The results are demonstrated using a parametric amplifier model and a long transmission line, analysed in the frequency domain.

Key words: frequency symbolic method, linear periodically time-varying circuits, LPTV

1. Introduction

In the context of modelling linear time-varying systems their analysis in the time domain is most often considered [1–3]. This is due to the fact that the most well-known software packages contain tools for numerical analysis of such circuits in the time domain, which is based on solving systems of differential equations with time-varying coefficients. Numerical analysis methods are characterised by such basic qualities as simplicity and versatility. However, they also have a significant defect due to the fact that the calculations must be repeated from scratch for each new value of the circuit parameter or even for a new value of the independent variable time.

Symbolic methods tend to be used to simulate circuits in the frequency domain [1, 4]. The main difference between symbolic methods and numeric ones is that some circuit parameters and variables, such as time or frequency, can be in the form of symbols [1] almost until the end of the calculation. At the same time, certain numeric-symbolic expressions are generated, in which numerical values instead of symbols are substituted only at the final stage of calculations. Substitution of specific numerical values into such expressions at the final stages and their calculation is quite fast, which makes such calculations convenient for a large number of symbolic parameter and variable values. Symbolic expressions carry a larger informative weight compared to numerical expressions, as they allow them to be algebraically simplified (similarity), differentiated, or integrated by symbolic parameters or variables, providing a more efficient solution to multivariate analysis problems, such as statistical studies or circuit optimisation.

However, until relatively recently, there were no frequency analysis methods used in the practice of analysis and synthesis of parametric circuits [5]. This state of affairs changed and the mathematical foundations were developed, which allowed us to create the so-called frequency symbolic (FS) method [1] for the analysis of linear periodically time varying (LPTV) circuits and, based on it, the system of user-defined functions "MAOPCs" [2]. A number of reports on the FS method and the "MAOPCs" system were made at scientific conferences [1, 2] and several articles were published in scientific journals [1, 2, 8].The result of applying the mentioned FS method and the "MAOPCs" system is the formation of symbolic transfer functions of LPTV circuits in the frequency domain, in which some parameters or variables are set by symbols. In addition, the "MAOPCs" system has tools for analysing the asymptotic stability of such circuits based on the generated transfer functions. According to us, the formation of frequency characteristics based on such parametric transfer functions and their use requires additional consideration, which is the goal of this paper. Thus, the article considers:

a) the method of forming the frequency characteristic of an LPTV circuit based on the presence of individual harmonic components in the output signal;
b) their possible coincidences at individual frequencies;
c) the possibilities of using such coincidences in the design of LPTV circuits.

In this regard, we believe that the frequency symbolic transfer functions of LPTV circuits have previously been formed by the "MAOPCs" system [2] in the MATLAB environment [11] and the objective is to build frequency responses on their basis.

This paper revolves around the construction of LPTV circuit frequency responses in the steady-state mode. Familiarity with the literature [6, 10–12], on the subject allows us to draw the following conclusions:

1. The term "frequency response" is primarily used in the literature in the context of linear circuits with constant parameters.
2. In a considerable quantity of literature sources, the definition of frequency characteristics is not given, which is obvious and unambiguous [13, 14].
3. Some sources [6, 10] define the frequency response as "the dependence on the frequency of the circuit parameters, as well as the values determined by these parameters".
4. Other sources [6, 11, 12] describe frequency response as the response of a circuit in steady-state mode to its input harmonic signal.

Thus, if there is a slight difference in the understanding of the frequency response for circuits with constant parameters, and it is not significant, but in essence it is the same thing, then for LPTV circuits, in our opinion, the understanding of the frequency response as a response to an input harmonic signal is determinative.

In the following material of the article, we adhere to the latter conclusion by the number 4 presented above.

The frequency characteristics of a linear electric circuit are dependencies that characterise the steady-state response of the circuit to a harmonic input signal $x_{\text{in}}(t) = A_{\text{in}} \cos(\omega t + \phi_{\text{in}})$, where t is the time, ω is the angular frequency, and A_{in}, ϕ_{in} are the amplitude and initial phase of this signal.

In the steady state of a circuit with constant parameters, when the input signal $x_{\text{in}}(t)$ is applied, the output will also be a harmonic signal $x_{\text{out}}(t) = A_{\text{out}} \cos(\omega t + \phi)$. In such a situation, the ratio $A(\omega) = A_{\text{out}}/A_{\text{in}}$, which depends on the frequency ω is called the amplitude-frequency response (AFR), and the difference $\psi(\omega) = \phi - \phi_{\text{in}}$ is called the phase-frequency response (PFR) of the circuit. The output signal is often delayed relative to the input signal, resulting in a negative PFR value.

In the presence of a transfer function of a circuit with constant parameters from a complex variable $\mathrm{j}\omega$ in the form

$$W(\mathrm{j}\omega) = \frac{X_{\text{out}}(\mathrm{j}\omega)}{X_{\text{in}}(\mathrm{j}\omega)}. \tag{1}$$

A complex exponential output signal $X_{\text{out}}(\mathrm{j}\omega, t)$ with a single complex exponential input signal $X_{\text{in}}(\mathrm{j}\omega, t) = \mathrm{e}^{\mathrm{j}\omega t}$ is defined as

$$X_{\text{out}}(\mathrm{j}\omega, t) = W(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t} = M(\omega)\mathrm{e}^{\mathrm{j}\varphi(\omega)}\mathrm{e}^{\mathrm{j}\omega t} = M(\omega)\mathrm{e}^{\mathrm{j}(\omega t + \varphi(\omega))}, \tag{2}$$

where $M(\omega)$, $\phi(\omega)$ – magnitude and phase of the transfer function $W(\mathrm{j}\omega)$, or, which is the same, the magnitude and initial phase (at $t = 0$) of the signal $X_{\text{out}}(\mathrm{j}\omega, t)$, respectively.

A harmonic signal passing through a linear circuit with constant parameters does not change its shape and remains harmonic with the frequency of the input signal. Therefore, since all voltages and currents in the circuit are harmonic and of the same frequency ω, their calculation is simplified by excluding the component $\mathrm{e}^{\mathrm{j}\omega t}$, in expressions such as (3) and considering $t = 0$. Therefore, the frequency characteristic of such a circuit is constructed independent of time by the expression [11]:

$$X_{\text{out}}(s)|_{s=\mathrm{j}\omega} = X_{\text{out}}(\mathrm{j}\omega) = M(\omega)\mathrm{e}^{\mathrm{j}\varphi(\omega)}. \tag{3}$$

Expression (2) defines two graphical dependencies on the frequency ω: frequency dependence $M(\omega)$ (called the amplitude-frequency response, or AFR), and frequency dependence $\varphi(\omega)$ (called the phase-frequency response, or PFR).

Using a similar approach, consider the formation of AFR and PFR of the LPTV circuit. Assuming that at least one of the circuit parameters changes periodically with a frequency Ω, the form of the complex output signal $X_{\text{out}}(\mathrm{j}\omega, t)$,in the steady-state mode changes significantly, and in the presence of the complex exponential input signal $X_{\text{out}}(\mathrm{j}\omega, t) = \mathrm{e}^{\mathrm{j}\omega t}$, also changes significantly compared to (1), because harmonic components with frequencies $(\omega \pm h\,\Omega)$ appear in it, where $h = 0, 1, 2\ldots$ Given the transfer function of a parametric circuit in the form $W(\mathrm{j}\omega, t)$ [1,2,15] the complex output signal $X_{\text{out}}(\mathrm{j}\omega, t)$ will be, by analogy with (1), the following:

$$X_{\text{out}}(\mathrm{j}\omega, t) = W(\mathrm{j}\omega, t)\,\mathrm{e}^{\mathrm{j}\omega t}. \tag{4}$$

If the transfer function $W(\mathrm{j}\omega, t)$ is approximated by a truncated complex Fourier series with $(2k+1)$ harmonic components in Ω [1]:

$$\begin{aligned} W(\mathrm{j}\omega, t) &= W_0(\mathrm{j}\omega) + \sum\nolimits_{h=1}^{k} \left(W_{-h}(\mathrm{j}\omega)\,\mathrm{e}^{-\mathrm{j}h\Omega t} + W_{+h}(\mathrm{j}\omega)\,\mathrm{e}^{+\mathrm{j}h\Omega t} \right) \\ &= M_0(\omega)\mathrm{e}^{\mathrm{j}\varphi_0(\omega)} + \sum\nolimits_{h=1}^{k} \left(M_{-h}(\omega)\,\mathrm{e}^{\mathrm{j}(-h\Omega t + \varphi_{-h}(\omega))} + M_{+h}(\omega)\mathrm{e}^{\mathrm{j}(+h\Omega t + \varphi_{+h}(\omega))} \right), \end{aligned} \tag{5}$$

then the complex output signal $X_{\mathrm{out}}(\mathrm{j}\omega, t)$ of the parametric circuit is an orthogonal series:

$$X_{\mathrm{out}}(\mathrm{j}\omega, t) = W_0(\mathrm{j}\omega)\,\mathrm{e}^{\mathrm{j}\omega t} + \sum\nolimits_{h=1}^{k} \left(W_{-h}(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}(-h\Omega+\omega)t} + W_{+h}(\mathrm{j}\omega)\mathrm{e}^{+\mathrm{j}(h\Omega+\omega)t} \right), \tag{6}$$

or:

$$\begin{aligned} X_{\mathrm{out}}(\mathrm{j}\omega, t) &= M_0(\omega)\,\mathrm{e}^{\mathrm{j}[\omega t + \varphi_0(\omega)]} \\ &+ \sum_{h=1}^{k} \left(M_{-h}(\omega)\,\mathrm{e}^{\mathrm{j}[(-h\Omega+\omega)t + \varphi_{-h}(\omega)]} + M_{+h}(\omega)\,\mathrm{e}^{\mathrm{j}[(h\Omega+\omega)t + \varphi_{+h}(\omega)]} \right), \end{aligned} \tag{7}$$

where $M_0(\omega)$, $M_{-h}(\omega)$, $M_{+h}(\omega)$ – magnitudes and $\phi_0(\omega)$, $\phi_{-h}(\omega)$, $\phi_{+h}(\omega)$ – the phases of the corresponding harmonic components of the transfer function $W(\mathrm{j}\omega, t)$ from (5), or, which is the same, the magnitudes and initial phases (at $t = 0$) of the harmonic components of the output signal $X_{\mathrm{out}}(\mathrm{j}\omega, t)$ with frequencies $(\omega \pm h\Omega)$ from (7), respectively. We consider the value Ω to be a given one, since in the model of an LPTV circuit it is its parameter, the value k is a positive integer such as $1, 2, \ldots$ for an LPTV circuit is chosen based on the required accuracy of its approximation $W(\mathrm{j}\omega, t)$.

2. Main concept

It can be concluded from the above that each term of expression (7) for values $h = 0, 1, 2, \ldots, k$ is the corresponding harmonic component of the output signal of frequency $(\omega \pm h\,\Omega)$:

– $X_{\mathrm{out},0}(\mathrm{j}\omega, t)$ – represents the "zero" harmonic component of the output signal at frequency ω and is given by the formula:

$$X_{\mathrm{out},0}(\mathrm{j}\omega, t) = M_0(\omega)\,\mathrm{e}^{\mathrm{j}[\omega t + \varphi_0(\omega)]}, \tag{8}$$

– $X_{\mathrm{out},-h}$ is the "negative" harmonic component of the output signal with frequencies $\omega - h\Omega$, is expressed by the equation:

$$X_{\mathrm{out},-h}(\mathrm{j}\omega, t) = M_{-h}(\omega)\,\mathrm{e}^{\mathrm{j}[(-h\Omega+\omega)t + \varphi_{-h}(\omega)]}, \tag{9}$$

– $X_{\mathrm{out},+i}$ is the "positive" harmonic component of the output signal with frequencies $\omega + i\,\Omega$ given by the formula:

$$X_{\mathrm{out,+h}}(\mathrm{j}\omega, t) = M_{+h}(\omega)\,\mathrm{e}^{\mathrm{j}[(+h\Omega+\omega)t + \varphi_{+h}(\omega)]}. \tag{10}$$

The total number of harmonic components in the output signal $X_{\mathrm{out}}(\mathrm{j}\omega, t)$ is $(2k+1)$, and each of them has a form close to expression (2) with certain alterations. For instance, the "zero" harmonic component $h = 0$ replicates the form of expression (2). The "negative" h-th harmonic component of the input signal with frequency ω forms the output harmonic component with frequency $\omega - h\,\Omega$, whereas the "positive" positive h-th harmonic component of the input signal with frequency ω forms the output harmonic component with frequency $\omega + h\,\Omega$. This fact determines the main difference between expression (6) and expression (2) and the key feature of the frequency characteristic of LPTV circuits: an input signal of one frequency in the output signal causes oscillations of the same frequency ω and other frequencies in the range $\omega \pm h\,\Omega$.

Let us consider the $\pm h$ term of the series (7) in more detail:

$$M_{\pm h}(\omega)\mathrm{e}^{\mathrm{j}[(\pm h\Omega+\omega)t+\varphi_{\pm h}(\omega)]}, \quad h = 0, 1, 2, \ldots, k \tag{11}$$

we see that, by analogy with (2) and (3), at $t = 0$, each term in (11) has its own frequency characteristic for the corresponding frequencies from the series $\omega \pm h\Omega$. Therefore, the frequency characteristic of the parametric circuit is formed as follows.

By analogy with expressions (2) and (3):

a) in the frequency characteristic of the LPTV circuit, we introduce not only the harmonic component with frequency ω, but also all other harmonic components present in the output signal with frequencies $\omega - k\,\Omega$, $\omega - (k-1)\,\Omega, \ldots, \omega - \Omega$, $\omega + \Omega$, $\omega + (k-1)\,\Omega$, $\omega + k\,\Omega$;

b) for convenience (by analogy with constant circuits), we construct the frequency characteristic of the LPTV circuit independent of time t, for which in (11), by analogy with (3), we choose $t = 0$ to simplify the representation. Thus, the frequency characteristic of a parametric circuit is not a single pair of functions, as in (3), but $(2k+1)$ pairs determined from complex expressions:

$$M_{\pm h}(\omega)\mathrm{e}^{\mathrm{j}(\varphi_{\pm h}(\omega))}, \quad h = 0, 1, 2, \ldots, k. \tag{12}$$

It is important to emphasize that each pair in expression (12) must invariably be assigned the same subscript – (0), $(-h)$, or $(+h)$ – as that used for the corresponding harmonic component in the output signal.

Definition: the frequency response of a linear parametric circuit (AFR and PFR) within $(2k+1)$ harmonic components can be presented as a set of separate graphical dependencies of magnitudes $M_0(\omega)$, $M_{\pm h}(\omega)$ values and phases $\phi_0(\omega)$, $\phi_{\pm h}(\omega)$ of all $(2k+1)$ harmonic components of the output signal (along the y-axis) on the frequency ω (along the x-axis). The convenience of this definition follows from the following implications.

Implication 1: the magnitude and initial phase of each individual harmonic component of the output signal can be treated as independent of time t and changes only as the input signal ω frequency changes.

Implication 2: since the output harmonic components are members of an orthogonal series, the magnitude and initial phase (vector) of the output signal can be represented as the sum of vectors, each defined by the magnitude and initial phase of the corresponding harmonic component.

Thus, we have determined that the frequency characteristic of an LPTV circuit does not consist of a single pair of graphical dependencies (as in the case of a circuit with constant parameters), but of $(2k+1)$ pairs of graphical dependencies of the magnitudes M_0, $M_{\pm h}$ and initial phases ϕ_0, $\phi_{\pm h}$ of each harmonic component of the output signal in the corresponding frequency range. Obviously, each such pair defines a vector that rotates around the origin with its frequency $\omega \pm h\,\Omega$ in the complex plane. If, for convenience, the frequency vector ω is depicted as a fixed vector by analogy with constant system (Fig. 1(a)), then all other vectors of the harmonic components present will rotate around it with frequencies $\pm h\,\Omega$, respectively.

If necessary, these vectors can be added to form a time-moving, equivalent output signal vector for each value of frequency ω (Fig. 1(b)).

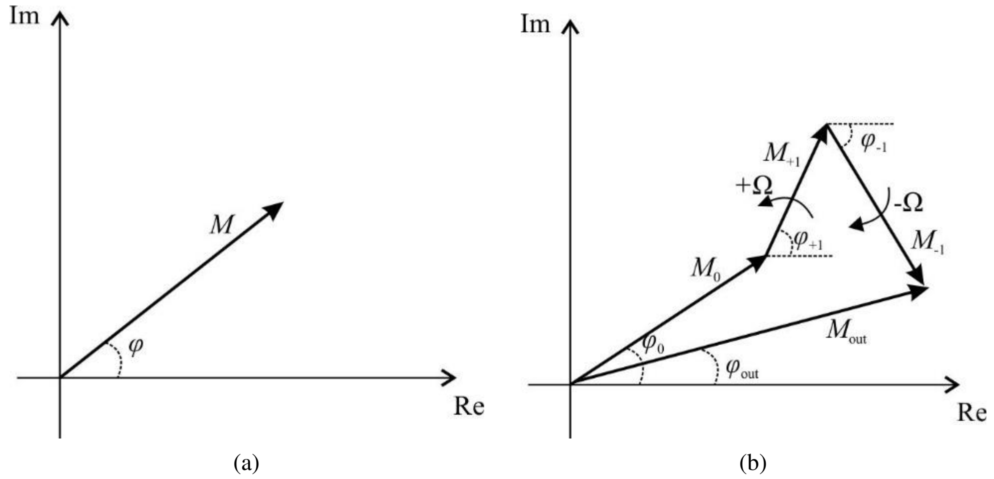


Fig. 1. The output signal vector: (a) for a circuit with constant parameters (stationary); (b) for an LPTV circuit for $k = 1$, formed by the sum of vectors (M_0, ϕ_0), (M_{+1}, ϕ_{+1}), (M_{-1}, ϕ_{-1}) of which the vector (M_0, ϕ_0) is stationary, the vector (M_{+1}, ϕ_{+1}) rotates with a frequency of $+\Omega$, the vector (M_{-1}, ϕ_{-1}) rotates in the opposite direction with the same frequency ω

3. Single-circuit parametric amplifier model analysis

A model of a parametric amplifier with a single variable coefficient was considered, and its structure is shown in Fig. 2.

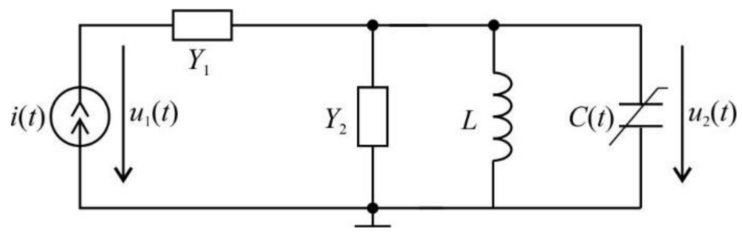


Fig. 2. Model of single-circuit parametric amplifier

The time-varying parameter of the system is the capacitance, which is described by a parametric function:

$$C(t) = c_0 \left(1 + m_c \cos(\Omega t)\right), \quad c_0 = 10 \text{ pF}. \tag{13}$$

Figure 2 shows the model of the analysed LPTV system and its parameters are as follows: the angular frequency of the parametric function is denoted as $\Omega = 2\pi F$, where the frequency of the parametric function is $F = 200$ MHz and the coefficient $m_c = 0.01$, inductance $L = 0.2533$ pH, conductance $Y_1 = 0.25$ S and $Y_2 = 0.4$ mS. The system is forced by a current source described by:

$$i(t) = I_m \cos\left(\omega_{\text{in}} t - \varphi_{\text{in}}\right), \tag{14}$$

where: $I_m = 0.1$ mA, $\varphi_{\text{in}} = 45°$, $\omega_{\text{in}} = 2\pi f_{\text{in}}$ and $f_{\text{in}} = 100$ MHz.

In the both experiments, the number of harmonic components k in the transfer functions was chosen from the condition that the values of the output signal calculated by the system "MAOPCs" and the Micro-Cap program [16, 17] coincide within 99.5%. The plots in Fig. 3 and Fig. 4 show the AFR individual harmonic components of the single-circuit parametric amplifier model from Fig. 2 for $k = 3$, calculated by the system "MAOPCs" [2] for the input signal frequency range from 75 to 525 dMHz.

Each graph in Figs. 3 and 4 presents the dependence of the magnitude $Mh(h = 0, \pm 1, \pm 2)$ of the h-th harmonic component on frequency F: $M_{-3}(f)$, $M_{-2}(f)$, $M_{-1}(f)$, $M_0(f)$, $M_{-1}(f)$, $M_{+2}(f)$, $M_{+3}(f)$, respectively. The frequency ranges shown in the graphs are different, reflecting the variation in the frequencies of the generated harmonic components according to the expression $f \pm hF$. In the graphs in Figs. 3(a), 3(b), and 3(c), the frequency axis includes negative values. This indicates that for these cases, the condition $f - hF < 0$ holds. Naturally, the actual frequency of such a harmonic component is $|f - hF|$, which corresponds to the absolute value of the negative frequency indicated on the frequency axis.

The program for plotting graphs displays a slider below the graphs (in this case, under Fig. 4(d)), where the cursor position ⌂ indicates the frequency of the input signal that is currently being applied to the circuit. Let's denote the frequency pointed to by the cursor as f_r. The graphing program shows the frequencies of the seven harmonic components generated by the input signal with frequency f_r with a red point on all seven graphs. By changing f_r on the slider, the user sees how the location of the red points changes on the graphs. In addition, to the right of each graph, the numerical values of the magnitude and phase of the corresponding harmonic component are displayed in black text, which corresponds to the position of the red point. On the slider, we see that the cursor indicates the frequency $f_r = 100$ MHz, so according to the expression $f \pm hF$, the red points in Figs. 3(c), 4(a), 3(b), 4(b), 3(a), 4(c), 4(d) indicate frequencies of 100 MHz, 300 MHz, 500 MHz, 700 MHz, respectively

In the model of the amplifier from Fig. 2, the frequency pumping F of the capacitance is 200 MHz and does not change during our computer experiment. Accordingly, for $f_r = 200$ MHz, the graphs from Fig. 3(c) and Fig. 4(a) show that, based on the expression $f + hF$, the red points are located at frequencies $(f - 1 \cdot F) = (100 - 1 \cdot 200) = |-100| = 100$ MHz and $(f + 0 \cdot F) = (100 + 0 \cdot 200) = 100$ MHz, respectively. As we can see, these frequencies are identical. This indicates that, at the input signal frequency $f_r = 100$ MHz, components M_0 and M_1 have the same frequency, and, in the output signal of the circuit, these components form a combined harmonic component at the same frequency of 100 MHz. Fixing this fact, the graph output program on both graphs from Fig. 3(c) and Fig. 4(a) displays a message in red text on the right that the frequency of the component M_{-1} from Fig. 3(c) is equal to the frequency of the component M_0 from Fig. 4(a), and conversely, the frequency of the component M_0 from Fig. 4(a) is equal to the frequency of the component M_{-1} from Fig. 3(c), respectively. This text is in red and can be seen in Fig. 3(c) and Fig. 4(a). As follows from the graphs in Fig. 3 and Fig. 4, similar coincidences of harmonic component frequencies at $f_r = 100$ MHz are observed at 300 MHz for components M_{-2} and M_{+1} corresponding to $(f - 2 \cdot F = (100 - 2 \cdot 200) = |-300| = 300$ MHz and $(f + 1 \cdot F) = (100 + 1 \cdot 200) = 300$ MHz, respectively. The same applies to the frequency of 500 MHz for components M_{-3} and M_{+2}, respectively. The same applies to the frequency of 500 MHz for components M_{-3} and M_{-2}. They are also added and the output of the circuit they will

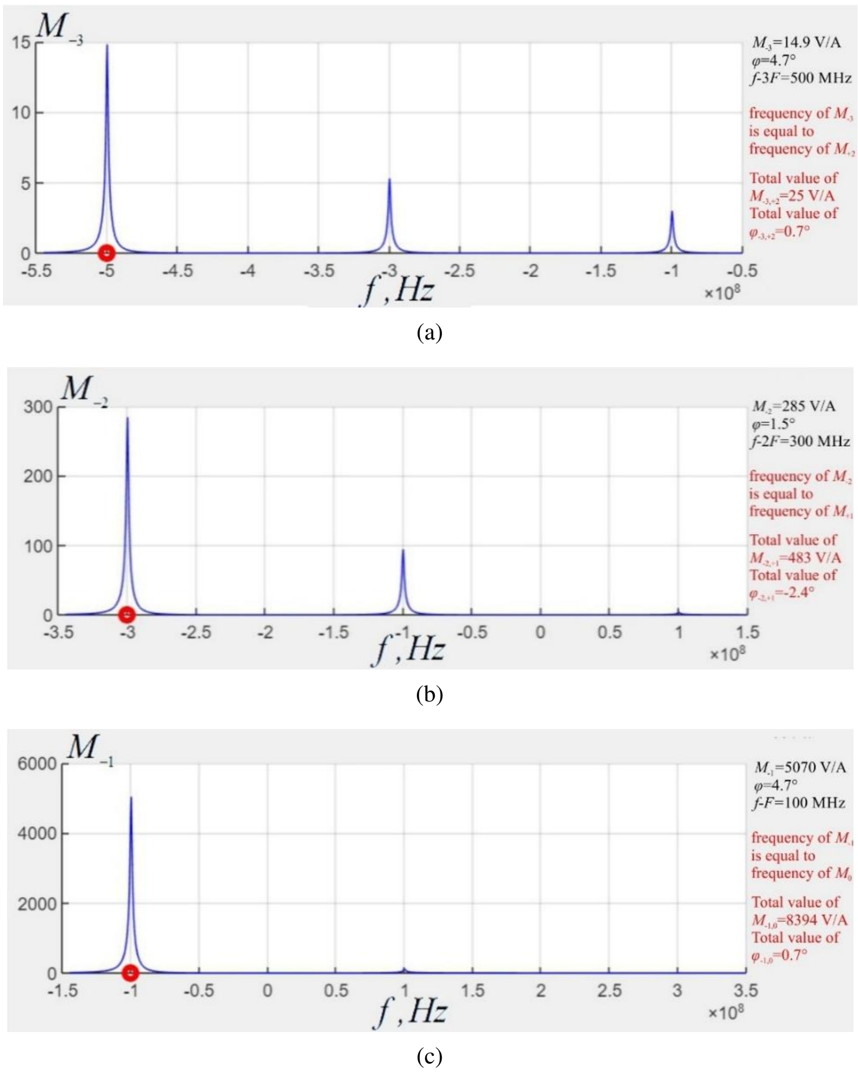
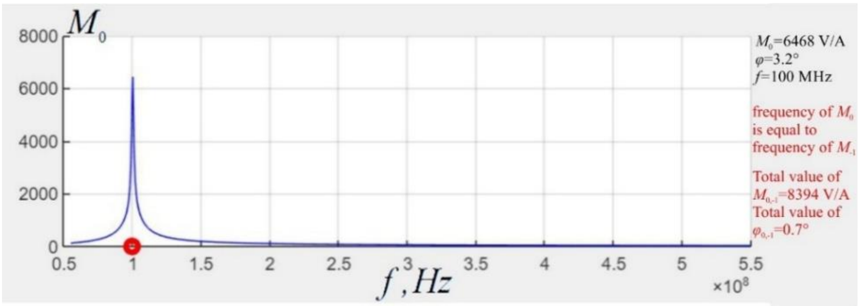


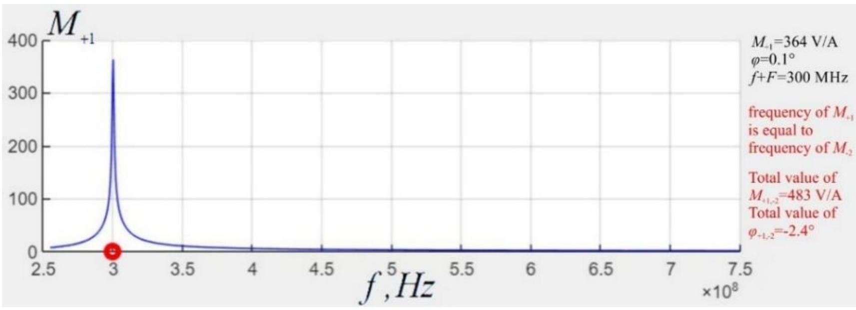
Fig. 3. The amplitude-frequency response of the single-circuit parametric amplifier for $k = 3$. The amplitude-frequency response in the frequency range containing $(2k + 1) = 7$ harmonic components, respectively: M_{-3} (a); M_{-2} (b); M_{-1} (c)

be represented by the corresponding sums. From now on, the frequencies at which coincidences occur will be called characteristic frequencies. The harmonic component M_{+3} has no pair, so it will be represented in the output signal only by itself.

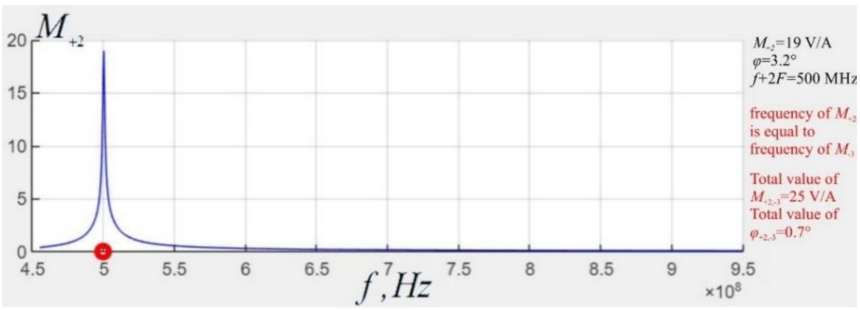
We can see that not all the running frequencies f_r of the input signal form pairs of harmonic signals whose frequencies coincide. Table 1 shows all possible values of the frequency f_r at which the above-described frequency matches when $k = 3$. These are the considered coincidences at $f_r = 100$ MHz, as well as coincidences at $f_r = 200$, 300, 400, and 500 MHz. It is important to note that the coincidences described can determine the purpose of the LPTV circuit. Thus, the above-described coincidence of components M_0 and M_{-1} at the frequency $f_r = 100$ MHz with $F = 2 \cdot f_r = 200$ Hz explains the formation of a combined harmonic component at $f_r = 100$ MHz in the output signal. This is the reason for the amplification of the output signal at $f_r = 100$ MHz in a single-circuit parametric amplifier.



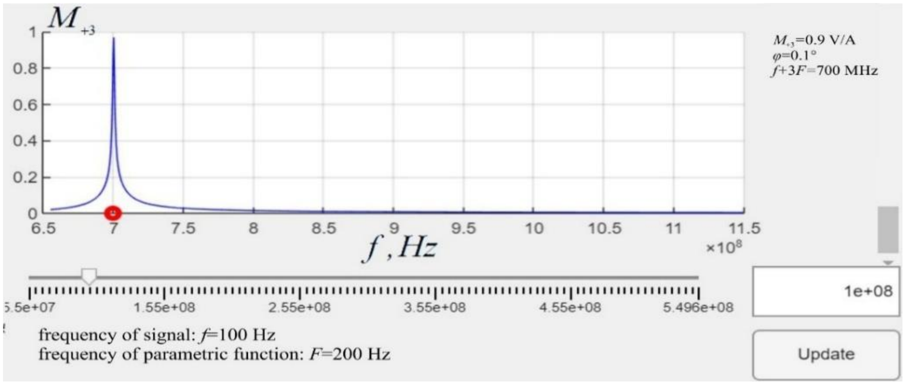
(a)



(b)



(c)



(d)

Fig. 4. The amplitude-frequency response of the single-circuit parametric amplifier for $k = 3$. The amplitude-frequency response in the frequency range containing $(2k + 1) = 7$ harmonic components, respectively: M_0 (a); M_{+1} (b); M_{+2} (c); M_{+3} (d)

Table 1. Frequency values of the harmonic components

Values of the running frequency f_r, MHz	**Frequencies of harmonic components of output signal ($\omega - h\,\Omega$) for $\Omega = 2\pi F$, $\omega = 2\pi f$, $F = 200$ MHz, $0 \le h \le k$, $k = 3$**						
	$f_r - 3F$	$f_r - 2F$	$f_r - F$	f_r	$f_r + F$	$f_r + 2F$	$f_r + 3F$
1	2	3	4	5	6	7	8
100	500	300	100	100	300	500	700
200	400	200	0	200	400	600	800
300	300	100	100	300	500	700	900
400	200	0	200	400	600	800	1000
500	100	100	300	500	700	900	1100

From Fig. 3, Fig. 4 and Table 1 the following conclusions can be drawn:

– The harmonic components marked in Table 1 with the same colour, at the corresponding values of the running frequency f_r in the output signal are added (as vectors), since they have the same frequency. This can be decisive in terms of the studied parametric circuit application.
– With a harmonic input signal, the output signal of the circuit is formed by both its individual harmonic components and their pairwise sums, which have the same frequency (nine colour-coded pairs in Table 1).
– The sum of the individual harmonic components of the output signal can determine the purpose of the circuit – to amplify the signal, multiply or divide the frequency of the input signal.
– The output signal can be investigated at a frequency that differs from the input signal frequency.

– By feeding the appropriate sums of harmonic signals to the input of the circuit, due to linearity, at its output, we can form total pairs of different harmonic components, obtaining various useful effects.

Table 2 presents pairs of magnitude M_0, $M_{\pm h}$ values of the corresponding harmonic components in the output signal that share the same frequency, along with the resulting magnitudes $M_{\pm h,\pm h}$ of such pairs. Such resulting magnitude is determined according to the vector summation rule applied to the vectors representing these magnitudes.

Table 2. Pairs of magnitude values of the corresponding harmonic components in the output signal that share the same frequency, along with the resulting magnitudes of such pairs

f_r, MHz	**Pairs of magnitude values of the corresponding harmonic components and the resulting magnitudes of such pairs at specific frequencies for single values of the running frequency f_r and $\Omega = 2\pi F$, $F = 200$ MHz**														
	100 MHz			200 MHz			300 MHz			400 MHz			500 MHz		
100	M_0	M_{-1}	$M_{0,-1}$				M_{+1}	M_{-2}	$M_{+1,-2}$				M_2	M_{-3}	$M_{+2,-3}$
	6468	5070	8394				364	285	483				19	14	25
200				M_0	M_{-2}	$M_{0,-2}$				M_{-2}	M_1	$M_{-2,1}$			
				106	0	106				0	5.7	5.7			
300	M_{-1}	M_{-2}	$M_{-1,-2}$				M_0	M_{-3}	$M_{0,-3}$						
	121	95	157				59	5.4	54.5						
400				M_{-1}	M_{-3}	$M_{-1,-3}$									
				2.8	0	2.8									
500	M_{-2}	M_{-3}	$M_{-2,-3}$												
	3.8	2.9	4.8												

For example, at $f_r = 100$ MHz MHz and $F = 200$ MHz (Table 2, row no. 1), it follows from the first row of Table 1 that the circuit output contains pairs of harmonic components: $(f_r, f_r - F)$ are shown in orange, $(f_r - F, f_r + F)$ in blue, and $(f_r - 3F, f_r + 2F)$ in green, corresponding to frequencies of 100, 300, and 500 MHz, respectively. Their magnitudes are denoted as (M_{-1}, M_0), (M_{-2}, M_{+1}), (M_{-3}, M_{+2}). To the right of these in Table 2, the corresponding resulting magnitudes are given, denoted as $M_{-1,0}$, $M_{-2,+1}$, $M_{-3,+2}$. The remaining rows have the analogous content.

As follows from Table 2, the resulting magnitude is not equal to the sum of the individual magnitudes. This indicates that the vectors of the harmonic components of the output signal being added have different initial phases.

Since the parameters of the circuit change over time, the output signal is usually affected by the phase of the input signal (in the time domain, which corresponds to the moment when the input signal is applied to the circuit). A change in this phase influences the amplification of the signal magnitude. Figure 5 shows the well-known fact [18] that the output signal of the single-circuit parametric amplifier depends on the phase φ_{in} of the input signal but also on the time related to the change of the amplifier [18].

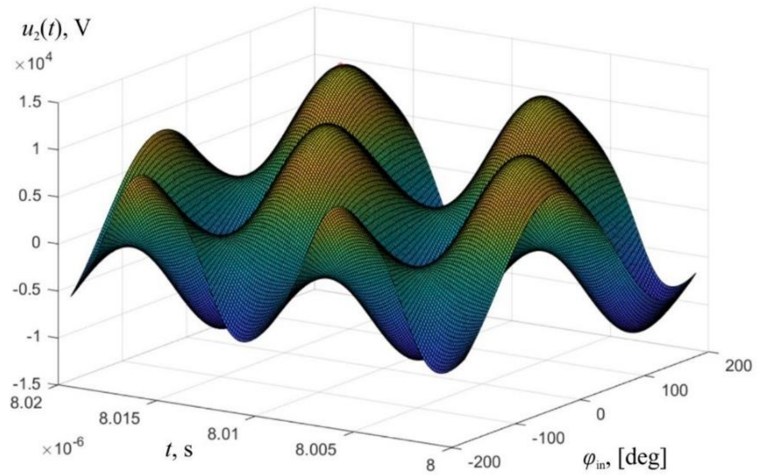


Fig. 5. Time waveform of the output signal of the analyzed amplifier as a function of the input signal phase φ_{in}

4. Parametric transmission line

The considered model of a parametric transmission line is presented in Fig. 6. The structure is typical of circuit theory, and the authors' proposal to introduce the time-varying inductance into this structure is analysed. The system consists of $l = 32$ elementary sections (Fig. 6(a)), each consisting of a stationary capacitor (C = const) and a non-stationary coil with time-varying parameter (Fig. 7(b)).

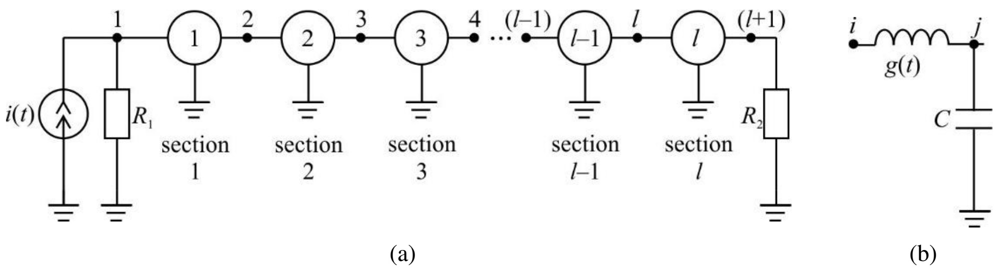


Fig. 6. Transmission line model: structure composed of l sections (a), elementary parametric section (b)

The parametric function is described by the formula:

$$g(t) = L(t)^{-1} = g_0\,(1 + m\cos(\Omega t))\,. \tag{15}$$

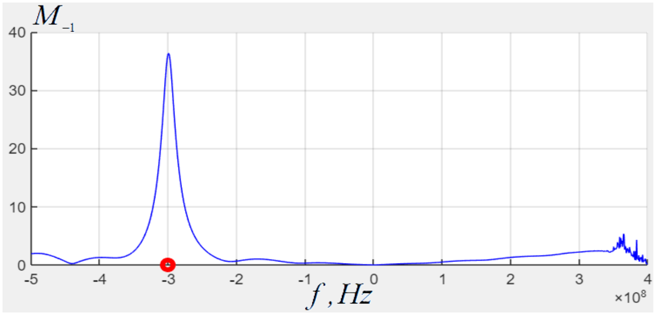
where: $m = 0.5$, $\Omega = 2\pi F$, $F = 600$ MHz. The values of the other parameters and coefficients of the system are $R_1 = R_2 = 50\ \Omega$, $C = 3.125$ nF, $g_0 = 1/(7.91210^{-9})$ 1/H, and forcing signal:

$$i(t) = I_m \cos(\omega_{\text{in}} t - \varphi_{\text{in}})\,, \tag{16}$$

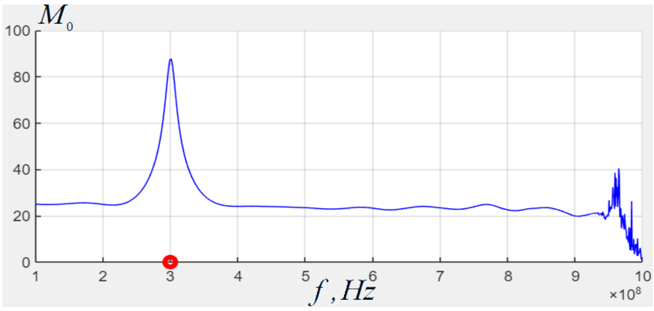
where: $I_m = 0.1$ mA, $\varphi_{\text{in}} = 45^\circ$, $\omega_{\text{in}} = 2 \cdot \pi \cdot f_{\text{in}}$, $f_{\text{in}} = 300$ MHz.

The frequency response of the transmission line model for $k = 1$ and $l = 32$ is shown in Fig. 7.

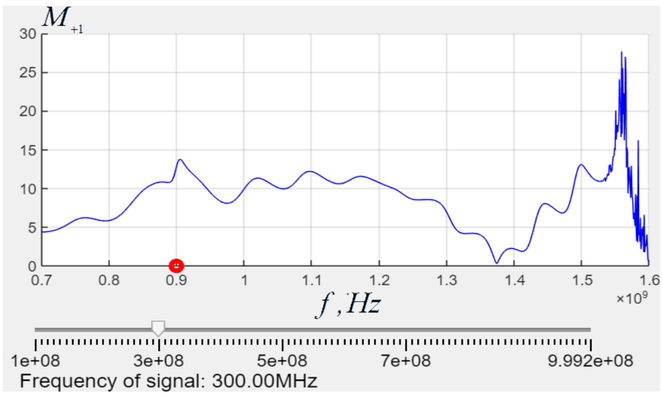
Table 3 presents pairs of magnitude values M_0, $M_{\pm h}$ of the corresponding harmonic components in the output signal that share the same frequency, along with the resulting magnitudes $M_{\pm h,\pm h}$ of such pairs. For example, at $f_r = 300$ MHz and $F = 600$ MHz (Table 3, row no. 1), the circuit output contains pairs of harmonic components: $(f_r, f_r - F)$, $(f_r - 2F, f_r + F)$ and $(f_r - 3F, f_r + F)$,



(a)



(b)



(c)

Fig. 7. The amplitude-frequency response of the parametric transmission line for $k = 1$. The frequency response in the frequency range contains $(2k + 1) = 3$ harmonic components, respectively: M_{-1} (a); M_0 (b); M_{+1} (c)

corresponding to frequencies of 300, 900, and 1500 MHz, respectively. Their magnitudes are denoted as (M_{-1}, M_0), (M_{-2}, M_{+1}), (M_{-3}, M_{+2}). To the right of these in Table 3, the corresponding resulting magnitudes are given, denoted as $M_{-1,0}$, $M_{-2,+1}$, $M_{3,+2}$.

Table 3. Pairs of magnitude values of the corresponding harmonic components in the output signal that share the same frequency, along with the resulting magnitudes of such pairs

f_r, MHz	**Pairs of magnitude values of the corresponding harmonic components and the resulting magnitudes of such pairs at specific frequencies for single values of the running frequency f_r and $\Omega = 2\pi F$, $F = 600$ MHz**														
	300 MHz			600 MHz			900 MHz			1200 MHz			1500 MHz		
300	M_0	M_{-1}	$M_{-1,0}$				M_{+1}	M_{-2}	$M_{-2,+1}$				M_{+2}	M_{-3}	$M_{-3,+2}$
	84	34	119				9.6	3.2	6.7				2.1	4.3	6.2
600				M_0	M_{-2}	$M_{-2,0}$				M_{-3}	M_{+1}	$M_{-3,+1}$			
				29	5.4	25				2.64	14	13			
900	M_{-1}	M_{-2}	$M_{-2,-1}$				M_0	M_{-3}	$M_{-3,0}$						
	6.3	5.0	11.4				22	1.2	24						
1200				M_{-1}	M_{-3}	$M_{-3,-1}$									
				5.9	1.8	5.2									
1500	M_{-2}	M_{-3}	$M_{-3,-2}$												
	0.02	0.22	0.23												

Figure 8 shows the dependence of the output signal $u_2(t, \phi_{\text{in}})$ of a parametric transmission line on the phase of the input signal ϕ_{in} and time t. This dependence not only on the phase of the input signal, but also on time. Varying this parameter has an effect on the dynamic properties of the system. The results of such an analysis can be used in solving further optimization problems.

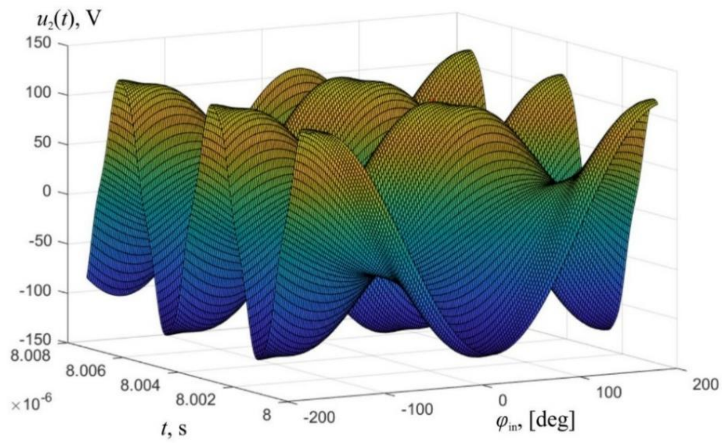


Fig. 8. Time waveform of the output signal of the analyzed transmission line as a function of the input signal phase φ_{in}

5. Conclusions

The following conclusions follow from the material presented in the article.

The paper considers the method of forming the frequency characteristics of an LPTV circuit (AFR and PFR) within $(2k + 1)$ harmonic components of its output signal, since the input signal is assumed to be a single complex exponential. By analogy with linear circuits with constant parameters in the method:

a) the magnitude and initial phase of each individual harmonic component of the output signal does not depend on time t and changes only when the frequency of the input signal changes;

b) as is typical for the theory of circuits with the proposed definition of frequency characteristics, the output signal of the circuit can be represented by a vector that is the sum of the vector representation of its individual harmonic components.

A distinction is demonstrated between the frequency characteristic (AFR and PFR) of a circuit with constant parameters, which is represented by a single pair of frequency dependencies, and the frequency characteristic definition proposed in this work for an LPTV circuit, which is represented by a pair of frequency dependencies for each of the $(2k + 1)$ harmonic components of the output signal and is adequate within this number $(2k + 1)$ of components.

The paper considers the case of possible coincidences of individual harmonic components of the output signal at some frequencies, which may make it possible to use such coincidences in further studies when designing LPTV circuits for various functional purposes.

The proposed method for forming the frequency response of an LPTV circuit is implemented in the system of user-defined functions "MAOPCs", which preliminarily generates the specified symbolic parametric transfer functions with respect to the voltages and currents in the circuit. These symbolic parametric transfer functions enable the user, via a dialog window, to analyze the desired frequency responses and their dependence on the values of the parametric element parameters or changes in the frequency range. A detailed description of the process of forming the frequency response of the LPTV circuit in the "MAOPCs" system is rather cumbersome, so it will be presented in future works.

By varying the phase of the input harmonic signal in the system of user-defined functions "MAOPCs", it is possible to analyze the changes in the circuit's output signal or in individual harmonic components of the signal.

Summarising the above, we argue that the calculation of the frequency characteristics of LPTV circuits in the environment of the the system of user-defined functions "MAOPCs" and their analysis generally improves the quality of modelling and study of parametric circuit models in the practice of designing devices modelled by such models. In particular, we will use this method to simulate complex LPTV circuits containing thousands of nodes and elements with fixed and time-varying parameters. The method is currently being used to model a low-noise amplifier on a long line.

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