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Finite-time observer design of a class of nonlinear systems

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We consider the problem of global finite-time stability for a class of nonlinear systems. The novelty in this paper is to consider a nonlinear finite time observer design, which introduces a finite-time observer for nonlinear systems that can be put into a nonlinear canonical form up to an output injection. The proof is based on the Lyapunov theory for Finite-Time Stability and the observer design method.

Key words: global stabilization, finite-time observer, Lyapunov theory

1. Introduction

The problem of finite-time stabilization has been studied by many researchers, see for instance [2,13] in which it was proved that finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties.

Recently, in the work [14], the synchronization has been regarded as a special case of observer design problem, i.e. the state reconstruction from measurements of an output variable under the assumption that the system structure and parameters are known.

In this paper, we consider a family of nonlinear systems of the form:

$$\begin{cases} \dot{x}_1 = x_2^{r_1} + f_1(x,u,t), \\ \dot{x}_2 = x_3^{r_2} + f_2(x,u,t), \\ \quad\vdots \\ \dot{x}_{n-1} = x_n^{r_{n-1}} + f_{n-1}(x,u,t), \\ \dot{x}_n = u + f_n(x,u,t), \\ y = x_1\,, \end{cases} \tag{1}$$

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where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state and the system input and output respectively; $f_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuously differentiable function, $(i = 1, 2, ..., n)$ with $f_i(0, 0, t) = 0$, $\forall t$. In the present work, we suppose that $0 < r_i < 1$, for any $i \in \{1, 2, ..., n\}$ and $r_i = \dfrac{m_i}{n_i}$ with m_i, n_i are odd integers.

The purpose of this paper is to introduce a new type of finite time observer. We consider the nonlinear system (1) and under additionally assumptions, we construct an observer that makes the error between the considered system and the observer convergent to zero in finite time. In the works [1, 7, 14], the authors consider a family on nonlinear systems in the form (1) and choose $r_1 = r_2 = ... = r_n = p > 1$ with p is odd integer and in the work ([15]), the author chooses $(r_1 = p_1, r_2 = p_2, ..., r_n = p_n)$ with $p_i \geqslant 0$, $(i = 1, ..., n)$ is odd integer and construct an observer for the nonlinear system. In [10], Ji Li and C. Qian choose $(r_1 = r_2 = ... = r_n = r)$ with $0 < r < 1$ and $r = \dfrac{p}{q}$, $(p < q)$ with p, q are odd integers and design a finite time observer of the considered system. In the present work, we consider the nonlinear system (1) and we choose $(r_1 \neq r_2 \neq ... \neq r_n)$ with $0 < r_i < 1$ for $(i = 1, ..., n)$ and $r_i = \dfrac{m_i}{n_i}$ with m_i, n_i are odd integers to construct a finite time observer.

The paper is organized as follows. First, we recall some definitions and preliminary results of finite-time stability, stabilization and observer. Then, we proceed for the design of finite time observer for the considered system. The idea of the proof is to use Lyapunov theory for finite-time stability, and a recursive design method that yields a state finite-time observer. Then, a numerical example is given to illustrate the use of the main result.

Finally, the conclusion follows.

2. Notion of stability and Key technical lemmas

We consider the non-autonomous system,

$$\dot{x}(t) = f(t, x), \qquad t \geqslant 0, \quad x \in \mathbb{R}^n, \tag{2}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous. We denote by $x(t, t_0, x_0)$ the solution of the system (2) starting from $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 1. [13] (Finite time stability)
The origin of the system (2) is finite-time stable if:

a) it is stable (2);

b) for all $t_0 \geqslant 0$, there exists $\eta(t_0) > 0$, such that if $\|x_0\| < \eta(t_0)$, then *i)* $x(t, t_0, x_0)$ *is defined for* $t \geqslant t_0$*,*

ii) *there exists* $0 \leqslant T(t_0, x_0) < +\infty$ *such that* $x(t, t_0, x_0) = 0$ *for all* $t \geqslant t_0 + T(t_0, x_0)$.

Denote

$$T_0(x_0) = \inf\{T(t_0, x_0) \geqslant 0 : \ x(t, t_0, x_0) = 0, \ \forall t \geqslant t_0 + T(t_0, x_0)\},$$

$T_0(x_0)$ *is called the settling time of the solution* $x(t, t_0, x_0)$.

Definition 2. [13] (Stabilization in finite time)
The system (2) is finite-time stabilizable if there exists a feedback function $u \in C^0(\mathbb{R}^n)$ *such that:*

a) $u(0) = 0$*,*

b) the origin of the closed loop system $\dot{x} = f(t, x, u(x))$ *is finite-time stable.*

Definition 3. [13] (Uniform finite time stability)
The origin of the system (2) is uniformly finite-time stable if:

a) it is uniformly asymptotically stable,

b) it is finite-time stable,

c) there exists a positive definite continuous function $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that the settling time of the system (2):* $T_0(t, x) \leqslant \alpha(\|x\|)$, $\forall x \in \mathbb{R}^n$.

Definition 4. [9] (Finite time observer)
We consider the system

$$\begin{cases} \dot{x} = f(x, u, t), \\ y = x_1 \,, \end{cases} \tag{3}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$,$u \in \mathbb{R}$ *and* $y \in \mathbb{R}$ *are the system state and the system input and output respectively.*

We call an observer of the system (3) any auxiliary system in the following form:

$$\begin{cases} \dot{\xi}(t) = \hat{f}(\xi(t), y(t)), \\ \hat{x}(t) = \xi(t) \end{cases} \tag{4}$$

such that

$$\lim_{t \longrightarrow +\infty} \|e(t)\| = \|x(t) - \hat{x}(t)\| = 0,$$

$e(t)$ *is called the error of observation.*

The system (4) is a finite time observer for the system (3) if there exists a time constant τ *such that for all* $t \geqslant t_0 + \tau$*,*

$$\hat{x}(t) = x(t)$$

In the next, we recall the Lyapunov theory for finite time stability.

Theorem 1 (**Lyapunov Theory for Finite-Time Stability**).
Consider the nonlinear dynamical system (2). Then the following statements hold:

i) *If there exist a continuously differentiable function* $V\colon [0,\infty)\times\mathcal{D}\to\mathbb{R}$, *a class* $\mathcal{K}$ *function* $\alpha(\cdot)$, *a function* $k\colon [0,\infty)\to\mathbb{R}_+$*such that* $k(t)>0$ *for almost all* $t\in[0,\infty)$, *a real number* $\lambda\in(0,1)$, *and an open neighborhood* $\mathcal{M}\subseteq\mathcal{D}$ *of the origin such that:*

$$\begin{aligned} &V(t,0)=0, && t\in[0,\infty),\\ &\alpha(\|x\|)\leqslant V(t,x), && t\in[0,\infty),\quad x\in\mathcal{M},\\ &\dot V(t,x)\leqslant -k(t)(V(t,x))^{\lambda}, && t\in[0,\infty),\quad x\in\mathcal{M}, \end{aligned}$$

then the zero solution $x(t)\equiv 0$ *to (1) is finite-time stable.*

ii) *If* $\mathcal{N}=\mathcal{D}=\mathbb{R}^n$ *and there exist a continuously differentiable function* $V\colon [0,\infty)\times\mathcal{D}\to\mathbb{R}$, *a class* $\mathcal{K}_\infty$ *function* $\alpha(\cdot)$, *a function* $k\colon[0,\infty)\to\mathbb{R}$ *such that* $k(t)>0$ *for almost all* $t\in[0,\infty)$, *and an open neighborhood* $\mathcal{M}\subseteq\mathcal{D}$ *of the origin such that* (20)−(22) *hold, then the zero solution* $x(t)\equiv 0$ *of the system (1) is globally finite-time stable.*

Lemma 1. [7, 11] *For* $x,\ y\in\mathbb{R}$, *and* $0<b\leqslant 1$, *we have the inequality*

$$\big(\,|\,x\,|+|\,y\,|\,\big)^b\leqslant|\,x\,|^b+|\,y\,|^b, \tag{5}$$

Therefore, for all real x_i, $i=1,2,\ldots,n$

$$\big(\,|\,x_1\,|+|\,x_2\,|+\ldots+|\,x_n\,|\,\big)^b\leqslant|\,x_1\,|^b+|\,x_2\,|^b+\ldots+|\,x_n\,|^b, \tag{6}$$

where $b=\dfrac{p}{q}\leqslant 1$, *with* $p,q>0$ *are odd integers,*

$$|\,x^b+y^b\,|\leqslant 2^{1-b}\,|\,x+y\,|^b\,. \tag{7}$$

Lemma 2. [7, 11] *Let* n, m *be positive real numbers and* a, b, d *continuous and positive real functions, then* $\forall c>0$ *we have:*

$$|\,a\,|^n|\,b\,|^m\,d\leqslant c\,|\,a\,|^{n+m}+\frac{m}{n+m}\left[\frac{n}{c(n+m)}\right]^{\frac{n}{m}}|\,b\,|^{n+m}\,d^{\frac{n+m}{m}}. \tag{8}$$

3. Design of a finite-time observer

Before the design of finite time observer, we introduce the following assumption.

Assumption 1. *For $i = 1, 2, ..., n$, there exist positive constants $\bar{a}_i$ such that $\forall (x, u, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$ and $y \in \mathbb{R}^n$, we have:*

$$|f_i(x, u, t) - f_i(y, u, t)| \leqslant \bar{a}_i\left(|x_1 - y_1|^{r_i} + |x_2 - y_2|^{r_i} + \ldots + |x_i - y_i|^{r_i}\right).$$

Assume that the system (1) is observable, then an observer for this system is designed as:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2^{r_1} + f_1\left(\hat{x}_1\right) + L_1\chi_1\left(x_1 - \hat{x}_1\right), \\ \dot{\hat{x}}_2 = \hat{x}_3^{r_2} + f_2\left(\hat{x}_1, \hat{x}_2\right) + L_2\chi_2\left(x_2 - \hat{x}_2\right), \\ \quad\vdots \\ \dot{\hat{x}}_n = u + f_n\left(\hat{x}_1, \ldots, \hat{x}_n\right) + L_n\chi_n\left(x_n - \hat{x}_n\right), \end{cases} \tag{9}$$

where, for $i = 1, ..., n$, $L_i > 0$, and $\chi_i(.)$ is a continuous function which will be chosen later.

For $(i = 1, ..., n)$, denote $e_i = x_i - \hat{x}_i$, then the observation error dynamics are given by

$$\begin{cases} \dot{e}_1 = x_2^{r_1} - \hat{x}_2^{r_1} + f_1\left(x_1\right) - f_1\left(\hat{x}_1\right) - L_1\chi_1\left(e_1\right), \\ \dot{e}_2 = x_3^{r_2} - \hat{x}_3^{r_2} + f_2\left(x_1, x_2\right) - f_2\left(\hat{x}_1, \hat{x}_2\right) - L_2\chi_2\left(e_2\right), \\ \quad\vdots \\ \dot{e}_n = f_n(x_1, ..., x_n) - f(\hat{x}_1, ..., \hat{x}_n) - L_n\chi_n(e_n). \end{cases} \tag{10}$$

Theorem 2. *If the assumption (1) is satisfied, the system (10) is globally finite-time stable.*

For the proof of the main result, we need the following steps.

First step: In the first step, we prove the following result:

Proposition 1. *Let $V_1(e_1) = \dfrac{e_1^{3-r_1}}{3-r_1}$, and $\chi(e_1) = e_1^{r_1} + \Phi_1(e_1)$, where ϕ_1 is a continuous function which will be chosen later. Then V_1 is non negative and the derivative of V_1 along the trajectories of the system (1) satisfies:*

$$\dot{V}_1 \leqslant -\left(L_1 - \left(\bar{a}_1 + 1\right)\right)e_1^2 + 4^{\frac{1}{r_1}}e_2^2 - L_1 e_1^{2-r_1}\Phi_1\left(e_1\right). \tag{11}$$

Proof. Let $V_1(e_1) = \dfrac{e_1^{3-r_1}}{3-r_1}$.

We have $r_1 = \dfrac{m_1}{n_1}$. m_1, n_1 are odd integers, we obtain $3 - r_1 = \dfrac{3n_1 - m_1}{n_1}$, then V_1 is a positive function of class C^1, where $V(e_1) \leqslant e_1^{3-r_1}$. Then the derivative of

V_1 along the trajectories of the system (10) is:

$$\begin{aligned}
\dot{V}_1 &= e_1^{2-r_1}\dot{e}_1 \\
&= e_1^{2-r_1}\left(x_2^{r_1} - \hat{x}_2^{r_1}\right) + e_1^{2-r_1}\left(f_1(x_1) - f_1(\hat{x}_1)\right) - L_1 e_1^{2-r_1}\chi_1\left(e_1\right). \\
&\leqslant 2^{1-r_1}|e_1|^{2-r_1}\,|e_2|^{r_1} + \bar{a}_1|e_1|^{2-r_1}\left(|x_1 - \hat{x}_1|^{r_1}\right) - L_1\chi_1\left(e_1\right)e_1^{2-r_1} \\
&\leqslant (\bar{a}_1 + 1)e_1^2 + 2^{\frac{2}{r_1}}e_2^2 - L_1 e_1^{2-r_1}\chi_1\left(e_1\right).
\end{aligned}$$

Let $\chi_1\left(e_1\right) = e_1^{r_1} + \Phi_1\left(e_1\right)$, we obtain:

$$\dot{V}_1 \leqslant -\left(L_1 - (\bar{a}_1 + 1)\right)e_1^2 + 4^{\frac{1}{r_1}}e_2^2 - L_1 e_1^{2-r_1}\Phi_1\left(e_1\right). \qquad \square$$

Second step: Let $1 \leqslant k \leqslant n-1$, we assume at step $k-1$, there exists a positive definite and proper function of class C^1, $V_{k-1}(e_1, \ldots, e_{k-1})$, which satisfies:

$$\begin{aligned}
&V_{k-1}(e_1, \ldots, e_{k-1}) > 0, \quad \text{for } (e_1, \ldots, e_{k-1}) \neq 0, \\
&V_{k-1}(e_1, \ldots, e_{k-1}) \leqslant e_1^{3-r_1} + \ldots + e_{k-1}^{3-r_{k-1}}
\end{aligned}$$

and

$$\dot{V}_{k-1} \leqslant -\sum_{i=1}^{k-1}\left(L_i - (\bar{a}_i + 1)\right)e_i^2 - \sum_{i=1}^{k-1} L_i e_i^{2-r_i}\Phi_i\left(e_i\right) + 4^{\frac{1}{r_{k-1}}}e_k^2$$

with

$$\begin{cases}
\chi_1\left(e_1\right) = e_1^{r_1} + \Phi_1\left(e_1\right), \\
\chi_2\left(e_2\right) = e_2^{r_2} + \Phi_2\left(e_2\right), \\
\qquad\vdots \\
\chi_{k-1}\left(e_{k-1}\right) = e_{k-1}^{r_{k-1}} + \Phi_{k-1}\left(e_{k-1}\right)
\end{cases}$$

with $\Phi_i(e_i)$,$i = (1, \ldots, k-1)$ is a continuous function which will be chosen later.

Proposition 2. *We consider*

$$V_k(e_1, \ldots, e_k) = V_{k-1}(e_1, \ldots, e_{k-1}) + \frac{e_k^{3-r_k}}{3 - r_k}.$$

Then

$$\dot{V}_k(e_1, \ldots, e_k) \leqslant -\sum_{i=1}^{k}(L_i - C_i)e_i^2 - \sum_{i=1}^{k} L_i e_i^{2-r_i}\Phi_i(e_i) + 4^{\frac{1}{r_k}}e_{k+1}^2$$

with C_i is a positive constant, for $i \in \{1, \ldots, k\}$.

Proof.

Proposition 3. *Let* $V_k = V_{k-1} + \dfrac{e_k^{3-r_k}}{3-r_k}$, *is a positive function of* C^1, *then the derivative of* V_k *along the trajectories of the system (10) gives*

$$\begin{aligned}\dot V_k &= \dot V_{k-1} + e_k^{2-r_k}\dot e_k\\ &= \dot V_{k-1} + \underbrace{e_k^{2-r_k}(x_{k+1}^{r_k} - \hat x_{k+1}^{r_k})}_{(\mathbf{a})} + \underbrace{e_k^{2-r_k}(f_k(x_1,...,x_k) - f_k(\hat x_1,...,\hat x_k))}_{(\mathbf{b})}\\ &\quad - \underbrace{L_k e_k^{2-r_k}\chi_k(e_k)}_{(\mathbf{c})}.\end{aligned}$$

*Now we estimate the expression (**a**):*

$$\begin{aligned}|e_k^{2-r_k}(x_{k+1}^{r_k} - \hat x_{k+1}^{r_k})| &\leqslant 2^{1-r_k}|e_k|^{2-r_k}|x_{k+1} - \hat x_{k+1}|^{r_k}\\ &\leqslant 2^{1-r_k}|e_k|^{2-r_k}|e_{k+1}|^{r_k}\\ &\leqslant e_k^2 + 4^{\frac{1}{r_k}} e_{k+1}^2 .\end{aligned}$$

*Then, according to Lemma (2) we estimate the expression (**b**): we have, for* $x, \hat x \in \mathbb{R}^k$

$$\begin{aligned}e_k^{2-r_k}(f_k(x) - f_k(\hat x)) &\leqslant \bar a_k|e_k|^{2-r_k}[(|x_1 - \hat x_1|^{r_k}) + ... + (|x_k - \hat x_k|^{r_k})]\\ &\leqslant \bar a_k|e_k|^{2-r_k}[|e_1|^{r_k} + ... + |e_k|^{r_k}]\\ &\leqslant \left(\bar a_k^{\frac{2}{r_k}}\right)\sum_{i=1}^{k-1} e_i^2 + 2(\bar a_k + 1)e_k^2 .\end{aligned}$$

Now we let $\chi_k(e_k) = e_k^{r_k} + \Phi_k(e_k)$; *We obtain*

$$\begin{aligned}\dot V_k \leqslant &-\sum_{i=1}^{k-1}(L_i - (\bar a_i + 1))\, e_i^2 - \sum_{i=1}^{k-1} L_i e_i^{2-r_i}\Phi_i(e_i) + 4^{\frac{1}{r_{k-1}}} e_k^2\\ &+ e_k^2 + 4^{\frac{1}{r_k}} e_{k+1}^2 + \left(\bar a_k^{\frac{2}{r_k}}\right)\sum_{i=1}^{k-1} e_i^2 + 2(\bar a_k + 1)e_k^2 - L_k e_k^2 - L_k e_k^{2-r_k}\Phi_k(e_k)\\ \leqslant &-\sum_{i=1}^{k}(L_i - C_i)e_i^2 - \sum_{i=1}^{k} L_i e_i^{2-r_i}\Phi_i(e_i) + 4^{\frac{1}{r_k}} e_{k+1}^2 .\end{aligned}$$

With $C_i = 2\bar a_i + 3 + 4^{\frac{1}{r_{k-1}}} + \bar a_k^{\frac{2}{r_k}}$, *for* $1 = 1, ..., k$

Last step: Using the inductive argument above, at the nth step, we obtain the following result.

Proposition 4. *Let* $V_n(e_1, ..., e_n) = V_{n-1}(e_1, ..., e_{n-1}) + \frac{e_n^{3-r_n}}{3-r_n}$ *and* $\chi_n(e_n) = e_n^{r_n} + \Phi(e_n)_n$. *Then*

$$\dot{V}_n(e_1, ..., e_n) \leqslant -\sum_{i=1}^{n}(L_i - C_i)e_i^2 - \sum_{i=1}^{n} L_i e_i^{2-r_i}\Phi_i(e_i)$$

and the closed-loop system (10) is globally finite-time stable.

Proof. Let

$$V_n = V_{n-1} + \frac{e_n^{3-r_n}}{3-r_n},$$

V_n is a positive function of C^1. Then

$$\begin{aligned}
\dot{V}_n &= \dot{V}_{n-1} + e_n^{2-r_n}\dot{e}_n \\
&= \dot{V}_{n-1} + e_n^{2-r_n}(f_n(x) - f_n(\hat{x})) - L_n e_n^{2-r_n}\chi_n(e_n) \\
&\leqslant \dot{V}_{n-1} + \bar{a}_n|e_n|^{2-r_n}[(|x_1 - \hat{x}_1|^{r_n}) + ... + (|x_n - \hat{x}_n|^{r_n})] - L_n e_n^{2-r_n}\chi_n(e_n) \\
&\leqslant \dot{V}_{n-1} + 2^{1-r_n}\bar{a}_n|e_n|^{2-r_n}(|e_1|^{r_n} + ... + |e_n|^{r_n}) - L_n e_n^{2-r_n}\chi_n(e_n) \\
&\leqslant \dot{V}_{n-1} + \bar{a}_n^{\frac{2}{r_n}}\sum_{i=1}^{n-1} e_i^2 + 2(\bar{a}_n + 1)e_n^2 - L_n e_n^{2-r_n}\chi_n(e_n).
\end{aligned}$$

Let $\chi_n(e_n) = e_n^{r_n} + \Phi_n(e_n)$. We obtain

$$\dot{V}_n \leqslant -\sum_{i=1}^{n}(L_i - C_i)e_i^2 - \sum_{i=1}^{n} L_i e_i^{2-r_i}\Phi_i(e_i).$$

With $C_i = 2\bar{a}_i + 3 + 4^{\frac{1}{r_{n-1}}} + \bar{a}_n^{\frac{2}{r_n}}$, for $1 = 1, ..., n$.

We have

$$\begin{aligned}
V_n &= \sum_{i=1}^{n}\frac{e_i^{3-r_i}}{3-r_i} \\
&\leqslant \sum_{i=1}^{n} e_i^{3-r_i}.
\end{aligned}$$

Let $\Phi_i(e_i) = e_i^{\frac{r_i}{3}}$, for $(i = 1, ..., n)$ and $\alpha = \frac{2}{3} \in]0, 1[$. We obtain:

$$\dot{V}_n + V_n^{\alpha} \leqslant -\sum_{i=1}^{n}(L_i - C_i)e_i^2 - \sum_{i=1}^{n} L_i e_i^{2-\frac{2r_i}{3}} + \sum_{i=1}^{n} e_i^{2-\frac{2r_i}{3}}.$$

If we choose $L_1 > C_1;L_2 > C_2;...;L_n > C_n$, we obtain

$$\dot{V}_n + V_n^{\alpha} \leqslant 0.$$

Then the system (10) is a globally finite time stable.

4. Example

Consider the following nonlinear system of the for

$$\begin{cases} \dot{x}_1 = x_2^{5/7} + \dfrac{1}{2(1+x_1^2)}, \\ \dot{x}_2 = x_3^{3/5}, \\ \dot{x}_3 = u, \\ y = x_1 \end{cases} \tag{12}$$

with $f(x_1) = \dfrac{1}{2(1+x_1^2)}$ is a continuously differentiable function and for $x,\ y \in \mathbb{R}$ we have $|f_1(x) - f_1(y)| \leqslant 2|x-y|^{5/7}$.

The observer is given by:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2^{5/7} + \dfrac{1}{2(1+\hat{x}_1^2)} + L_1\chi(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_2 = \hat{x}_3^{3/5} + L_2\chi(x_2 - \hat{x}_2), \\ \dot{\hat{x}}_3 = u + L_3\chi(x_3 - \hat{x}_3). \end{cases}$$

Now, let $e_i = x_i - \hat{x}_i$. We obtain,

$$\begin{cases} \dot{e}_1 = x_2^{5/7} - \hat{x}_2^{5/7} + \dfrac{1}{2(1+x_1^2)} - \dfrac{1}{2(1+\hat{x}_1^2)} - L_1\chi(e_1), \\ \dot{e}_2 = x_3^{3/5} - \hat{x}_3^{3/5} - L_2\chi(e_2), \\ \dot{e}_3 = -L_3\chi(e_3) \end{cases} \tag{13}$$

with

$$\begin{cases} \chi(e_1) = e_1^{5/7} + \Phi_1(e_1), \\ \chi(e_2) = e_2^{3/5} + \Phi_2(e_2), \\ \chi(e_3) = e_3^{7/11} + \Phi_3(e_3). \end{cases}$$

In the first step
let $V_1(e_1) = \frac{7}{16} e_1^{16/7}$ and we estimate $\dot{V}_1$

$$\dot{V}_1(e_1) \leqslant |e_1|^{9/7} \left| x_2^{5/7} - \hat{x}_2^{5/7} \right| + |e_1|^{9/7} \left| \frac{1}{2(1+x_1^2)} - \frac{1}{2(1+\hat{x}_1^2)} \right|$$
$$- L_1 e_1^{9/7} \left(e_1^{5/7} + \Phi_1(e_1) \right)$$
$$\leqslant -(L_1 - 3)e_1^2 + \frac{1}{4} e_2^2 - L_1 e_1^{9/7} \Phi_1(e_1).$$

In the second step,
We let,

$$V_2(e_1, e_2) = V_1(e_1) + \frac{5}{12} e_2^{12/5}$$

then

$$\dot{V}_2(e_1, e_2) \leqslant -(L_1 - 3)e_1^2 - (L_2 - \frac{5}{4})e_2^2 - L_1 e_1^{9/7} \Phi_1(e_1) - L_2 e_2^{7/5} \Phi_2(e_2) + e_3^2$$

and

$$V_2(e_1, e_2) \leqslant e_1^{16/7} + e_2^{12/5}.$$

In the last step,
Let, $V_3(e_1, e_2, e_3) = V_2(e_1, e_2) + \frac{11}{26} e_3^{26/11}$. Then:

$$\dot{V}_3 \leqslant \dot{V}_2(e_1, e_2) - L_3 e_3^{15/11} (e_3^{7/11} + \Phi_3(e_3))$$
$$\leqslant -(L_1 - 3)e_1^2 - \left(L_2 - \frac{5}{4} \right) e_2^2 - (L_3 - 1)e_3^2 - L_1 e_1^{9/7} \Phi_1(e_1)$$
$$- L_2 e_2^{7/5} \Phi_2(e_2) - L_3 e_3^{15/11} \Phi_3(e_3)$$

and

$$V_3(e_1, e_2, e_3) \leqslant e_1^{16/7} + e_2^{12/5} + e_3^{26/11}.$$

Now, we let

$$\begin{cases} \Phi_1(e_1) = e_1^{5/21}, \\ \Phi_2(e_2) = e_2^{1/5}, \\ \Phi_3(e_3) = e_3^{7/33}. \end{cases}$$

If we choose $\alpha = 2/3$ and $L_1 = 4$, $L_2 = 3$, $L_3 = 2$, we obtain,

$$\dot{V}_3(e_1, e_2, e_3) + V_3^{2/3}(e_1, e_2, e_3) < 0.$$

Then, the system (13) is globally finite-time stable.

5. Conclusion

In this paper, we give a simpler design finite-time observer method to achieve globally finite-time stabilization of a family of nonlinear system (1), under the assumption (1). Finally, a numerical example has also been given to demonstrate the use of our main result.

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