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Stabilization of linear discrete-time control systems with the Caputo convolution type difference fractional variable-order operator

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Abstract. The paper focuses on the stabilization of fractional variable-order linear discrete-time systems with the Caputo difference operator of the convolution type. The Z-transform is a powerful and widely used tool for analyzing the stability of linear control systems. It provides insight into the behavior of the system solutions, particularly in the discrete-time domain. Given the fundamental role of stability in control theory and its applications in automation, a central objective is the stabilization of control systems. Stabilization refers to the process of determining a suitable state-feedback law that ensures the asymptotic stability of the system. In situations where the open-loop system is not asymptotically stable, stability can often be achieved by introducing a properly designed state-feedback controller. This controller is typically constructed based on the eigenvalue spectrum of the closed-loop system matrix. The conditions for asymptotic stability, derived via the Z-transform approach, provide practical criteria for eigenvalue placement and serve as a guide for ensuring the desired system behavior.

Keywords: fractional discrete-time control system; asymptotic stability; stabilization.

1. INTRODUCTION

In classical fractional calculus, the use of constant-order derivatives/differences is common. Although this approach can effectively describe systems with fixed memory or hereditary properties, it often proves insufficient to accurately model real-world systems with dynamic or time-varying behavior. Constant order models assume a static nature of memory effects, which limits their ability to represent evolving dynamics in systems such as biological processes, viscoelastic materials, or adaptive control systems, see for instance [1, 2]. The dynamical behaviour of discrete-time fractional variable-order systems strongly depends on the order function, which introduces additional degrees of freedom in system modelling [1].

To address this, fractional variable-order (FVO) operators have been introduced independently by various authors. These allow the order of the derivative or a discrete-time operator to change as a function of time, making them more adaptable to complex system behaviors. In particular, variable-order models have shown success in characterizing anomalous diffusion, aging processes, and viscoelastic behavior where the system memory properties evolve over time [3].

The benefits of FVO extend to discrete-time systems as well. In such cases, using variable-order difference operators allows for the modeling of systems whose dynamic characteristics change from one time step to another. For example, in the field of artificial intelligence, a discrete-time neural network model with fractional operators of variable-order has been proposed and studied with proven stability properties [4].

Similarly, in control engineering, the implementation of a variable-order fractional discrete-time PID controller offers enhanced tuning and adaptation capabilities. Studies show that the variable-order approach improves the system response and adaptability compared to classical PID controllers [5–7].

Recent research has addressed the stability of discrete-time systems governed by fractional difference equations of variable-order. The necessary and sufficient conditions for asymptotic stability have been derived on the basis of spectral criteria, particularly using Grünwald–Letnikov-type operators [8]. Further investigations have considered systems involving convolution-type variable-order difference operators; see [9]. The application of these theoretical developments has also been demonstrated in the context of control in [10]. In addition, the foundational properties of the Grünwald–Letnikov backward difference operator with variable-order have been formally established in [11].

An important class of variable-order operators is based on the Caputo definition, particularly useful in initial-value problems because of its intuitive treatment of initial conditions. The stability of linear discrete-time systems defined by the Caputo fractional variable-order h-difference operator of convolution type has been investigated, where recurrence relations and spectral criteria were used to characterize the asymptotic behavior in [12, 13].

In the context of biological modeling, the Caputo variable-order operator has been applied to generalize the Rulkov neuron model, with results showing how variable memory can impact the dynamics and stability of neuron-like systems [14]. Moreover, in collective behavior modeling, the variable-order Caputo

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operator has been integrated into fractional Cucker–Smale-type models to capture the alignment and consensus of agents evolving over time, with convergence results supported by theoretical analysis [15].

2. PRELIMINARIES

We begin by revisiting the notation associated with a fractional variable-order operator derived from both: the Grüunwald-Letnikov and the Caputo-type differences approaches. This operator makes use of a so-called oblivion function, which is defined as follows.

Definition 1 [11, 16]**.** Given $k, i \in \mathbb{Z}$ and an order function $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$ represents the sampling interval, then the oblivion function is defined by

$$a^{\nu(k)}(i) = \begin{cases} 0, & i < 0, \\ 1, & i = 0, \\ (-1)^i \dfrac{\nu(k)\,(\nu(k)-1)\cdots(\nu(k)-i+1)}{i!}, & i > 0. \end{cases} \tag{1}$$

It is worth noting that expression (1) can be equivalently formulated for $i \geq 0$ as follows

$$\begin{aligned} a^{\nu(k)}(i) &= (-1)^i \binom{\nu(k)}{i} \\ &= (-1)^i \frac{\Gamma\big(\nu(k)+1\big)}{\Gamma(i+1)\,\Gamma\big(\nu(k)-i+1\big)}, \end{aligned} \tag{2}$$

where $\binom{\nu(k)}{i}$ denotes the generalized binomial coefficient and $\Gamma(\cdot)$ is the Gamma function. For $\nu(k)-i+1 \in \mathbb{Z}_- \cup \{0\}$ we assume $a^{\nu(k)}(i) = 0$. As in the paper we consider convolution operator with $(a^{\nu(i)}(i))_{i\in\mathbb{N}_0}$, we take $k = i$.

In this paper, we consider order functions $\nu(\cdot)$ that take values exclusively in the interval $(0,1]$. However, the definitions of fractional and variable-order h summations and h differences remain valid for any nonnegative order function $\nu(\cdot) : \mathbb{Z} \to \mathbb{R}_+$. As an order function we can use any function with values from interval $(0,1]$, for example we use $\nu(k) = 0.95 - \exp(-k)$, tending very fast to constant order 0.95.

Let $h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$. Given a function $x : h\mathbb{Z} \to \mathbb{R}$, the forward h difference operator is defined by (cf. [17]): $\Delta_h x(kh) = \dfrac{x(kh+h)-x(kh)}{h}$. For $q \in \mathbb{N}_0 := \{0,1,2,\ldots\}$, we denote by $\Delta_h^q := \Delta_h \circ \cdots \circ \Delta_h$ the q-fold composition of the forward difference operator. Then the q-th order h-difference can be expressed as:

$$\Delta_h^q x(kh) = \sum_{i=0}^{q} (-1)^{q-i} \binom{q}{i} x(kh+ih) h^{-q}.$$

Definition 2. Let $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$. The *fractional variable-order h-sum of convolution type (FVOS)* of a function $x : h\mathbb{Z} \to \mathbb{R}$ is defined as

$$\Delta_h^{-\nu(\cdot)} x(kh) := \sum_{i=0}^{k} h^{\nu(i)} a^{\nu(i)}(i) x(kh-ih),$$

where $k \in \mathbb{N}_0$.

Note that the FVOS operator represents a discrete convolution: $\Delta_h^{-\nu(\cdot)} x(kh) = (\mathfrak{a} * \overline{x})(k) = (\overline{x} * \mathfrak{a})(k)$, where "$*$" denotes the convolution operator, $\mathfrak{a}(i) := h^{\nu(i)} a^{\nu(i)}(i)$, and $\overline{x}(k) := x(kh)$. Using the convolution form, we can apply the $\mathscr{Z}$-transform and write:

$$\mathscr{Z}\left[\Delta_h^{-\nu(\cdot)} x\right](z) = X(z)\mathscr{A}(z), \tag{3}$$

where $X(z) := \mathscr{Z}[\overline{x}](z)$, and

$$\mathscr{A}(z) = \sum_{i=0}^{\infty} (-1)^i \binom{-\nu(i)}{i} z^{-i} h^{\nu(i)}.$$

- If $\nu(k) \equiv \alpha$ (i.e., constant order), then equation (3) simplifies to
$$\mathscr{Z}\left[\Delta_h^{-\alpha} x\right](z) = \left(\frac{hz}{z-1}\right)^{\alpha} X(z).$$

The definition of the Caputo fractional difference operator (for arbitrary $h > 0$) can be found, for instance, in [18]. Below, we introduce the Caputo fractional variable-order h-difference operator of convolution type.

Definition 3. Let $\nu : \mathbb{Z} \to (q-1, q]$ for some $q \in \mathbb{N}_1$. The *Caputo fractional variable-order h-difference operator of convolution type (CFVOD-h)* with order function $\nu(\cdot)$, acting on a function $x : h\mathbb{Z} \to \mathbb{R}$, is defined by:

$$\Delta_h^{\nu(\cdot)} x(kh) := \Delta_h^{-(q-\nu(\cdot))} \Delta_h^q x(kh). \tag{4}$$

Observe that the CFVOD-h given by (4) in Definition 3 simplifies in the special cases, for instance one gets:

- For $q = 1$, i.e., $\nu : \mathbb{Z} \to (0,1]$, the operator $\Delta_h^{\nu(\cdot)}$ simplifies to:

$$\begin{aligned} \Delta_h^{\nu(\cdot)} x(kh) &= \Delta_h^{-(1-\nu(\cdot))} \Delta_h x(kh) \\ &= \sum_{i=0}^{k} h^{-\nu(i)} a^{1-\nu(i)}(i) \big(x(kh+h-ih) - x(kh-ih)\big). \end{aligned} \tag{5}$$

- If $\nu(k) \equiv q$ for a fixed $q \in \mathbb{N}_1$, then:
$$\Delta_h^{\nu(\cdot)} x(kh) = \Delta_h^q x(kh).$$

- For $q = 1$, the $\mathscr{Z}$-transform of the operator is given by:
$$\mathscr{Z}\left[\Delta_h^{\nu(\cdot)} x\right](z) = \big((z-1)X(z) - zhx(0)\big)\mathscr{A}_2(z),$$
where $X(z) := \mathscr{Z}[\overline{x}](z)$, $\overline{x}(k) := x(kh)$, and
$$\mathscr{A}_2(z) := \sum_{i=0}^{\infty} (-1)^i \binom{\nu(i)-1}{i} z^{-i} h^{-\nu(i)}. \tag{6}$$

Observe that the Caputo fractional difference operator of variable-order introduced in Definition 3 can be understood as a natural extension of the constant-order case and is constructed analogously to the continuous Caputo fractional derivative. More precisely, for a given function $\nu : \mathbb{Z} \to \mathbb{R}_+$ and order $\nu(k) \in (q-1, q]$, the operator $\Delta_h^{\nu(\cdot)}$ is obtained by first applying the integer-order forward difference of order q, and then the fractional sum operator of order $q-\nu(k)$, see Definition 2. The operator $\Delta_h^{\nu(\cdot)}$ can be interpreted as a discrete convolution of the integer-order difference Δ^q with a kernel depending on the variable-order $\nu(\cdot)$. In this sense, it describes a nonlocal operator with memory, where past states of the system contribute to the current value with weights determined by $\nu(\cdot)$.

In this paper, we restrict our study to discrete-time systems. Therefore, the corresponding continuous-time operators for both fractional and variable-order integrals, as well as the Caputo fractional and variable-order differential operators, are not considered.

3. LINEAR DISCRETE-TIME SYSTEMS WITH THE CAPUTO FRACTIONAL VARIABLE-ORDER DIFFERENCE OPERATOR

Let us consider the following control system

$$\Delta_h^{\nu(\cdot)} x(kh) = Ax(kh) + Bu(kh), \quad k \geq 1, \tag{7}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $\nu\colon \mathbb{Z} \to (0,1]$ is an order function, $u\colon (h\mathbb{N})_0 \to \mathbb{R}^m$ is the input, $x : (h\mathbb{N})_0 \to \mathbb{R}^n$ is the state, and the matrices are $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Then by (5) system (7) can be rewritten in the following form

$$\sum_{i=0}^{k} h^{-\nu(i)} a^{1-\nu(i)}(i) \big(x(kh+h-ih) - x(kh-ih)\big) = Ax(kh) + Bu(kh), \quad k \geq 1,$$

or equivalently,

$$\begin{aligned} x(kh+h) - x(kh) &= h^{\nu(0)} Ax(kh) \\ &- \sum_{i=1}^{k} h^{h^{\nu(0)}-\nu(i)} a^{1-\nu(i)}(i) (x(kh+h-ih) - x(kh-ih)) \\ &+ h^{\nu(0)} Bu(kh), \quad k \geq 1. \end{aligned}$$

Consequently,

$$\begin{aligned} x(kh+h) &= (I + h^{\nu(0)} A) x(kh) \\ &- \sum_{i=1}^{k} h^{h^{\nu(0)}-\nu(i)} a^{1-\nu(i)}(i) \big(x(kh+h-ih) - x(kh-ih)\big) \\ &+ h^{\nu(0)} Bu(kh), \quad k \geq 1, \end{aligned} \tag{8}$$

Hence taking $k \geq 2$ and replacing k by $k-1$ in (8) systems (7), can be rewritten in a recursive form as follows:

$$\begin{aligned} x(h) &= \Big(I + h^{\nu(0)} A\Big) x(0) + h^{\nu(0)} Bu(0), \\ x(kh) &= \Big(I + h^{\nu(0)} A\Big) x(kh-h) \\ &- \sum_{i=1}^{k-1} h^{\nu(0)-\nu(k-i)} a^{\nu(k-i)-1}(k-i) \, (x(ih) - x(ih-h)) \\ &+ h^{\nu(0)} Bu(kh-h), \qquad k \geq 2 \end{aligned} \tag{9}$$

and $x(0) = x_0 \in \mathbb{R}^n$ is given.

By applying the $\mathscr{Z}$-transform to system (7), we obtain

$$X(z) = \Big((z-1)\mathscr{A}_2(z) I - A\Big)^{-1} \Big[zh\mathscr{A}_2(z) x(0) + BU(z)\Big], \tag{10}$$

where $U(z) = \mathscr{Z}[\overline{u}](z)$ with $\overline{u}(k) = u(kh)$, and $X(z)$ together with $\mathscr{A}_2(z)$ are defined in Section 2. To solve system (7), one must compute the inverse $\mathscr{Z}$–transform of (10).

3.1. Stability of fractional variable-order linear discrete-time system

The problem of stability for discrete-time linear systems of fractional variable-order requires conditions that guarantee stability. Let us formulate conditions for the asymptotic stability of linear systems of fractional variable-order $\nu\colon \mathbb{Z} \to (0,1]$. Observe that the autonomous system of the following form:

$$\Delta_h^{\nu(\cdot)} x(kh) = Ax(kh) \tag{11}$$

is a particular type of (7) where B is the zero matrix.

Let us recall that system (11) is *stable* if it is stable in the Lyapunov sense, i.e., for all $\epsilon > 0$ there exists $\delta > 0$, such that if $\|x_0\| < \delta$, then the solution $x\colon h\mathbb{N}_0 \to \mathbb{R}^n$ of system (11) with the initial condition $x(0) = x_0$ satisfies $\|x(kh)\| < \epsilon$ for $k \in \mathbb{N}_0$. The system (11) is called *asymptotically stable*, if it is stable (Lyapunov) and $\lim_{k\to\infty} x(kh) = 0$.

Additionally, system (7) is (asymptotically) stable if and only if the corresponding autonomous system (11) is (asymptotically) stable.

Observe that directly from the above definition one gets that if a system is asymptotically stable, then it is stable.

3.2. Conditions of stability and asymptotic stability

Now, let us give the sufficient condition for the asymptotic stability of the systems (7) and (11).

Theorem 1. Let $\mathrm{spec}(A) := \{\lambda_i\colon \det(A - \lambda_i I) = 0, i = 1,\ldots,k\}$, $k \leq n$. If for all $i = 1,\ldots,k$ we have

$$\lambda_i \in \{(z-1)\mathscr{A}_2(z)\colon |z| < 1\},$$

then the systems (7) and (11) are asymptotically stable.

Proof. Observe that if all roots of the equation $\det((z-1)\mathscr{A}_2(z)I - A) = 0$ are within the unit circle, then the system (11) (or equivalently, (7)) is asymptotically stable; see, for instance, [19]. Note that $A = PJP^{-1}$, where P is invertible and $J = \mathrm{diag}(J_1,\ldots,J_s)$ and J_l are Jordan blocks of order $r_l, l = 1,\ldots,s$. Since $\det((z-1)\mathscr{A}_2(z)I - A) = \det\big((z-1)\mathscr{A}_2(z)I - PJP^{-1}\big) = \det((z-1)\mathscr{A}_2(z)I - J)$, all the roots of the equation $\det((z-1)\mathscr{A}_2(z)I - A) = 0$ are within the unit circle if and only if all the eigenvalues of A belong to $\{(z-1)\mathscr{A}_2(z)\colon |z| < 1\}$. □

Note that $\{z \in \mathbb{C}\colon |z| = 1\} = \{z \in \mathbb{C}\colon z = e^{j\varphi}, \varphi \in [0,2\pi]\} = \{e^{j\varphi}\colon \varphi \in [0,2\pi]\}$, where j denotes the imaginary unit. Hence it follows that the set

$$\{(e^{j\varphi} - 1)\mathscr{A}_2(e^{j\varphi})\colon \varphi \in [0,2\pi)\} \tag{12}$$

is the boundary of the set $\{(z-1)\mathscr{A}_2(z): |z|<1\}$. If the eigenvalues of the matrix A are within the set (12), then the system (11) is asymptotically stable.

Proposition 1. Let $\operatorname{spec}(A)=\{\lambda_i: i=1,\dots,k\}$, $k\le n$ and

$$w_i := 2\left|\sin\frac{\varphi_i}{2}\right|\left(\left(\sum_{k=0}^{\infty} h^{-\nu(k)}a^{\nu(k)-1}(k)\cos(k\varphi_i)\right)^2 + \left(\sum_{k=0}^{\infty} h^{-\nu(k)}a^{\nu(k)-1}(k)\sin(k\varphi_i)\right)^2\right)^{0.5}, \tag{13}$$

where $\varphi_i=\arg(\lambda_i)$. Then, if there is $\lambda_i\in\operatorname{spec}(A)$ such that

$$|\lambda_i|>w_i\,, \tag{14}$$

then systems (7) and (11) are unstable.

Proof. Observe that

$$\begin{aligned} w_i &= 2\left|\sin\frac{\varphi_i}{2}\right|\left(\left(\sum_{k=0}^{\infty} h^{-\nu(i)}a^{\nu(k)-1}(k)\cos(k\varphi_i)\right)^2 + \left(\sum_{k=0}^{\infty} h^{-\nu(i)}a^{\nu(k)-1}(k)\sin(k\varphi_i)\right)^2\right)^{0.5} \\ &= \left|(e^{j\varphi}-1)\mathscr{A}_2(e^{j\varphi})\right|. \end{aligned}$$

where $\varphi\in\mathbb{R}$. Then by condition (14) we get $\lambda_i\notin\{(z-1)\mathscr{A}_2(z): |z|\le 1\}$. Hence, the eigenvalue λ_i of the matrix A lies outside the image of the unit circle in the mapping $z\mapsto(z-1)\mathscr{A}_2(z)$ where $\mathscr{A}_2:\mathbb{C}\to\mathbb{C}$ is given by (6).

Consequently, the systems (7) and (11) are unstable. □

Condition (14) seems to be difficult to check. However, we can do that in some approximations in series as the values of coefficients $a^{\nu(\cdot)}(\cdot)$ tend very fast to zero. Hence, it is enough to calculate the right-hand side for some steps.

4. STABILIZABILITY OF FRACTIONAL ORDER LINEAR DISCRETE-TIME CONTROL SYSTEMS

The theory of classical linear control systems provides the following well-known result.

Theorem 2 [20]**.** Let $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$. The following conditions are equivalent:

(a) $\operatorname{rank}\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix}=n$;

(b) $\operatorname{rank}\begin{bmatrix}\lambda I-A & B\end{bmatrix}=n$ for all $\lambda\in\mathbb{C}$;

(c) For any real numbers $\alpha_0,\alpha_1,\dots,\alpha_{n-1}$, there exists a matrix $K\in\mathbb{R}^{m\times n}$ such that the characteristic polynomial of $A+BK$ is given by

$$p_{A+BK}(\lambda)=\lambda^n+\alpha_{n-1}\lambda^{n-1}+\dots+\alpha_1\lambda+\alpha_0\,.$$

Therefore, if the pair (A,B) satisfies any of the conditions in Theorem 2, and the desired eigenvalues of the matrix $A+BK$ are known, it is possible to construct a feedback matrix K such that $A+BK$ has these eigenvalues.

Let $h>0$ and let $\nu:\mathbb{N}_0\to(0,1]$. Consider the following linear discrete-time control system:

$$\Delta_h^{\nu(\cdot)}x(t)=Ax(t)+Bu(t), \tag{15}$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ with $m<n$, $u: h\mathbb{N}_0\to\mathbb{R}^m$ is the control input, and $x: h\mathbb{N}_0\to\mathbb{R}^n$ is the state.

Definition 4. The control system (15) is said to be (asymptotically) stable if the associated autonomous system of the form

$$\Delta_h^{\nu(\cdot)}x(t)=Ax(t) \tag{16}$$

is (asymptotically) stable.

Since system (15) includes a control input u, it is possible to enforce asymptotic stability through a suitable choice of state feedback. The task of designing such feedback to ensure system stability is referred to in the literature as the *stabilization problem*. Following the classical approach for continuous- and discrete-time dynamical systems (see [20, 21]), we now introduce the definition of stabilizability for system (15).

Definition 5. System (15) is said to be *stabilizable* if there exists a state feedback control law of the form

$$u(t)=Kx(t),\qquad t\in h\mathbb{N}_0\,, \tag{17}$$

with $K\in\mathbb{R}^{m\times n}$, such that the resulting closed-loop system

$$\Delta_h^{\nu(\cdot)}x(t)=(A+BK)x(t) \tag{18}$$

is asymptotically stable.

Therefore, the problem of stabilizability for system (15) reduces to finding a matrix $K\in\mathbb{R}^{m\times n}$ such that the closed-loop system (18) becomes asymptotically stable. If the original system is already stable, designing a feedback matrix K is not strictly necessary. Nevertheless, even in such cases, introducing state feedback of the form $u=Kx$ may enhance the qualitative behavior of the system trajectories. On the other hand, if the system is unstable, the objective is to determine the entries of the matrix K such that the closed-loop system (18) achieves asymptotic stability.

4.1. Stabilizability conditions and stabilization

By Definition 5 and Theorem 1, we obtain the following result:

Proposition 2. If there exists a matrix K such that all roots of the equation

$$\det\left((z-1)\mathscr{A}_2(z)I-(A+BK)\right)=0$$

lie strictly inside the unit circle, then the system (15) is stabilizable.

From the equivalence of conditions (a), (b), and (c) in Theorem 2, it follows that for any monic polynomial $w(\lambda)$ there exists a matrix K such that $w(\lambda) = p_{A+BK}(\lambda)$ if and only if

$$\operatorname{rank}\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = n.$$

This leads to the following result for the case $m = 1$:

Proposition 3. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and let $\mathscr{C} = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$. If $\operatorname{rank}\mathscr{C} = n$, then the system (15) is stabilizable via state-feedback control $u = Kx$, where

$$K = -\begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \cdot \mathscr{C}^{-1} \cdot \left(A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_0 I\right), \quad (19)$$

and $\alpha_i \in \mathbb{R}$, for $i = 0, 1, \ldots, n-1$, are such that the roots of the polynomial

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$$

lie within the set $\{(z-1)\mathscr{A}_2(z) : |z| < 1\}$.

Proof. Assume that $\operatorname{rank}\mathscr{C} = n$. Then the matrix $\mathscr{C} \in \mathbb{R}^{n\times n}$ is invertible. According to Theorem 2, for any given coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, there exists a gain matrix $K \in \mathbb{R}^{1\times n}$ such that the characteristic polynomial of $A + BK$ satisfies

$$p_{A+BK}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0.$$

Let us choose $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that the spectrum of $A + BK$ satisfies

$$\operatorname{spec}(A + BK) = \{\lambda_1, \ldots, \lambda_n\} \subset \{(z-1)\mathscr{A}_2(z) : |z| < 1\}.$$

Since every square matrix satisfies its own characteristic equation, we have:

$$(A + BK)^n + \alpha_{n-1}(A + BK)^{n-1} + \ldots + \alpha_1(A + BK) + \alpha_0 I = 0.$$

Expanding this expression and rearranging terms gives:

$$A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I = -\mathscr{C} \cdot \begin{bmatrix} \vdots \\ K \end{bmatrix}.$$

Since $\mathscr{C}$ is invertible, we can solve for the vector formed by the entries of K as:

$$\begin{bmatrix} \vdots \\ K \end{bmatrix} = -\mathscr{C}^{-1} \cdot \left(A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_1 A + \alpha_0 I\right).$$

Finally, extracting K from this expression leads to formula (19), which defines a feedback matrix such that the closed-loop system (18) is asymptotically stable, since the eigenvalues of $A + BK$ are within the stability region $\{(z-1)\mathscr{A}_2(z) : |z| < 1\}$. □

Analogously to the classical difference and differential systems, the following statement occurs:

Proposition 4. System (15) is stabilizable by (17) if and only if the following implication holds

$$\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} < n \;\Rightarrow\; \lambda \in \{(z-1)\mathscr{A}_2(z) : \; |z| < 1\}. \quad (20)$$

Proof. "⇒" Let us prove necessity by contradiction. Assume that system (15) is stabilizable by (17) and (20) does not hold. Then for some $\lambda \in \mathbb{C}$ we get $\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} < n$ and $\lambda \notin \{(z-1)\mathscr{A}_2(z) : \; |z| < 1\}$. Since $\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \lambda I - A - BK & B \end{bmatrix}$ for arbitrary matrix $K \in \mathbb{R}^{m\times n}$, from inequality

$$\operatorname{rank}\begin{bmatrix} \lambda I - (A + BK) & B \end{bmatrix} < n$$

we get that $\lambda \in \operatorname{spec}(A + BK)$ and $\lambda \notin \{(z-1)\mathscr{A}_2(z) : \; |z| < 1\}$. Hence, system (18) is not asymptotically stable, so system (15) is not stabilizable by (17). We get a contradiction with the assumption, that is, implication (20) occurs.

"⇐" Assume that the system (15) is not stabilizable and (20) holds. Then for arbitrary matrix $K \in \mathbb{R}^{m\times n}$ we have $\operatorname{spec}(A + BK) \not\subseteq \{(z-1)\mathscr{A}_2(z) : \; |z| < 1\}$. If $\operatorname{rank}\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} < n$, then there exists $\lambda \in \mathbb{R}$, such that $\lambda \notin \{(z-1)\mathscr{A}_2(z) : \; |z| < 1\}$. □

Corollary 1. System (15) is not stabilizable by state feedback control (17) if and only if $\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} < n$ and $\lambda \in \{(z-1)\mathscr{A}_2(z) : \; |z| \geq 1\}$.

Observe that the stabilizability of an unstable system consists in determining the values of the coefficients of the matrix $K \in \mathbb{R}^{m\times n}$, so that the eigenvalues of the matrix $A + BK$ are in the set $\{(z-1)\mathscr{A}_2(z) : \; |z| < 1\}$. Thus, the stability problem is that given an unstable system (15), we want to find the feedback that makes the system (18) asymptotically stable. Since $A + BK \in \mathbb{R}^{n\times n}$, the eigenvalues of the matrix $A + BK$ are roots of polynomials with real coefficients. Note that $\operatorname{spec}(A + BK)$ is the set of complex numbers that self-conjugate, which follows from the fact that the set of zeros of a polynomial is self-conjugate if and only if the polynomial has the real coefficient.

Lemma 1. The set of zeros of a polynomial p is self-conjugate if and only if the polynomial p has the real coefficient.

Proof. "⇐" Let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, $a_i \in \mathbb{R}$. Assume that $\alpha \in \mathbb{C}$ is the root of the polynomial p, then

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0. \quad (21)$$

Taking the conjugate of both sides of (21) we get $\bar{\alpha}^n + a_{n-1}\bar{\alpha}^{n-1} + \cdots + a_1\bar{\alpha} + a_0 = 0$. Hence $p(\bar{\alpha}) = 0$ and we get $\bar{\alpha}$ is the root of polynomial p.

"⇒" Let $\alpha = a + bj$, $a, b \in \mathbb{R}$ Then $\bar{\alpha} = a - bj$ and

$$(\lambda - \alpha)(\lambda - \bar{\alpha}) = \lambda^2 - 2a\lambda + a^2 + b^2,$$

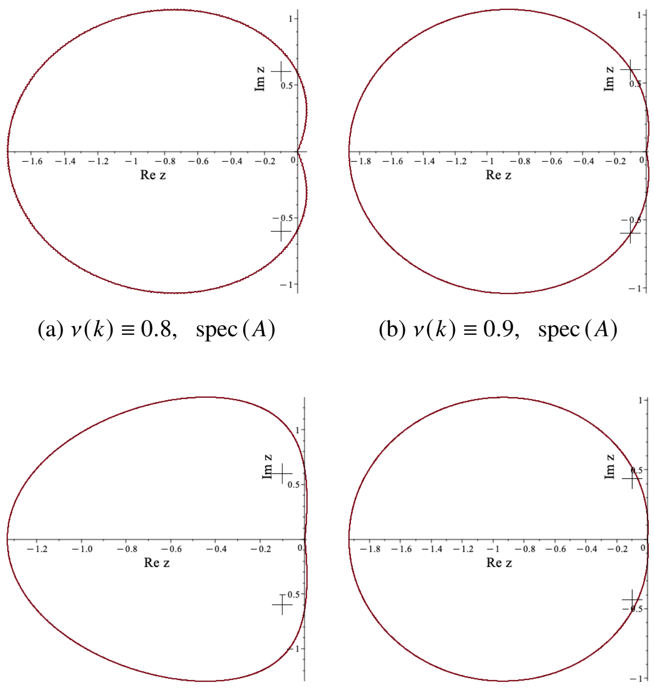
where $-2a \in \mathbb{R} \wedge a^2 + b^2 \in \mathbb{R}$. Using the fact that for all $\alpha \in \mathbb{C}$ we have the following implication

$$p(\alpha) = 0 \;\Rightarrow\; p(\bar{\alpha}) = 0,$$

we get that p is the polynomial with real coefficient. □

5. EXAMPLE

We consider a discrete-time linear system (so $h = 1$) with $A = \begin{bmatrix} 0.8 & -1.17 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For the matrix A we have $\operatorname{spec}(A) = \{-0.1 \pm 0.6j\}$. We take into account the following order functions: (a) $\nu(k) \equiv 0.8$, (b) $\nu(k) \equiv 0.9$, (c) $\nu(k) = 0.95 - \exp(-k)$. And for (a) we have a stable system (see Fig. 1a), for (b) the situation is shown in Fig. 1b showing eigenvalues on the border of the area of stability but still stable, and for (c) as we see in Fig. 1c the system is stable. For constant order 0.95 we have instability, but with variable-order $\nu(k) = 0.95 - \exp(-k)$, when the values of an order function at the beginning are small enough, it stability occurs. In all cases, both sequences of coefficients $a^{\nu(k)}(k)$ and $a^{\nu(k)-1}(k)$ are decreasing. Then, all series used in the domain of an image are convergent. Note that $\operatorname{rank}\begin{bmatrix} B & AB \end{bmatrix} = 2$, so in case when the order is greater than 0.9, for example 0.95 we choose the eigenvalues of the matrix $A + BK$, such that $\operatorname{spec}(A + BK)$ is inside the region of stability. Let $K = \begin{bmatrix} 0 & 0.17 \end{bmatrix}$ what slightly change eigenvalues positions of original from matrix A. The position of eigenvalues of $A + BK$ is illustrated by Fig. 1d. Therefore, our system is stabilized also for higher order. Although the example, is presented in a general mathematical form, it corresponds to a discrete-time realization of a second-order physical system, such as an RLC circuit. Hence, it illustrates the applicability of the proposed results to practical dynamical systems encountered in engineering.



(a) $\nu(k) \equiv 0.8$, $\operatorname{spec}(A)$ (b) $\nu(k) \equiv 0.9$, $\operatorname{spec}(A)$

(c) $\nu(k) = 0.95 - \exp(-k)$, $\operatorname{spec}(A)$ (d) $\nu(k) = 0.95$, $\operatorname{spec}(A + BK)$

Fig. 1. The boundary of the stability regions and and the position of eigenvalues of matrix A and $\operatorname{spec}(A + BK) = \{-0.0(9) \pm 0.435889j\}$

6. CONCLUSIONS

The paper addresses the problem of the stabilization of fractional order linear discrete-time systems with the Caputo type operators in planar systems. In the future, we intend to investigate the stabilization of fractional linear discrete-time systems of variable-order in $\mathbb{R}^n$ and study the problem of output feedback controls that are easier to implement compared to state feedback controls.

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