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Anisotropy component of electromagnetic force and torque

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Abstract. The paper deals with the problem of surface-integral representation of electromagnetic force/torque for magnetically anisotropic region. It is pointed out that in some anisotropic regions a component of electromagnetic force/torque appears - the so-called anisotropy component. The total electromagnetic field force/torque calculated with the help of Maxwell's and Lorentz's methods could lead to the different values for some anisotropic medium (homogeneous, without hysteresis). The coenergy method is used to evaluate total force/torque too. Analytical calculations of force/torque for isotropic and anisotropic media in electromagnetic field are presented. The condition for surface integral representation of Lorentz's either force or torque is formulated.

Key words: anisotropy component, electromagnetic force, electromagnetic torque.

1. Introduction

The electromechanical converters are one of the well-known electrical devices that are used for wide range of purposes [1]. The electromagnetic field distribution decides on force densities and hence enables calculation of total force and torque [2–4].

The problem of calculation of total forces and torques for different anisotropic structures of electromechanical converters is the centre of the carried out analysis. The different methods for total force and torque calculations are explored due to their efficiency. The virtual work principle is widely applied and discussed [5–7]. The Maxwell method (surface integral method) is successfully used and developed [8] for induction motor torque analysis and for force calculation [9]. Numerical approaches to forces and torques calculations are important tasks for electromechanical converter analysis. The local forces that constitute the total (so called integral) quantities (force, torque) are of great interest [10].

Parallel, the theoretical investigations have being carried out to constitute the background for numerical evaluation and to give some physical interpretations [11, 12]. The computations of local forces for magnetic regions are carried out by different methods in [13–17]. The force calculations are also provided by means of so-called sensitivity approach [18, 19].

New technologies offer still newer and newer materials for building electromechanical converters where either anisotropic materials [20–23] or laminated structure are used [24]. Magnetic anisotropy causes special physical effects on force and torques [25, 26]. Hence, the magnetic anisotropy should be considered in a theoretical way.

The paper shows that for anisotropic medium the Maxwellian surface representation could not be applied [27, 28]. In order to present this problem it is very convenient to develop the analytical analysis for some models of electromechanical converters which enable the sensitivity analysis thereof [29, 30] and constitute the benchmark tasks for numerical algorithms [31–33].

The paper approaches to the so-called anisotropy force/torque component that appears in electromechanical converters with magnetic anisotropy regions. The paper extends previous works [28, 32, 33]. There are presented both force and torque calculations, and the coenergy method is used for force/torque calculations. The force analysis is carried out for a linear converter model, and the torque analysis is shown basing on a cylindrical solid induction motor model. The both models are obviously simplified (in comparison with technical devices of them) in order to obtain analytical solution (to omit non numerical errors). The analytical solutions obtained confirm the theoretical statement, provided in the paper, and may be treated as benchmark task for electromechanical converters numerical analyses.

There is a considered electromagnetic field in anisotropic region for any anisotropy feature e.g. the magnetic reluctivity matrix can be either symmetrical or asymmetrical. However, the asymmetrical reluctivity matrix appears rarely, the well-known cases. Exemplary, for structure of samarium-cobalt, neodymium-iron-boron [4], gyromagnetic media [34] and ferrites [35] the reluctivity matrix is asymmetrical. For deformable bodies under pressure and so-called active bodies the reluctivity matrix could be asymmetrical too. Moreover, some equivalent structures defined for electromechanical converters with permanent magnets lead to the equivalent asymmetrical reluctivity matrix [36, 37].

The interesting problem is which of the methods is more suitable for calculating the electromagnetic forces when anisotropic media appear. The Maxwell's stress tensor and Lorentz's methods are considered. The force and torque are evaluated with the help of these methods and the results are compared. The basis constitutes the equation for electromagnetic field force volume density. That equation is proved basing on:

- Maxwell equations for electromagnetic field,
- constitutive relations (A.5) and (A.8),
- Lorentz force density formula,

as shown in Appendix.

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There will be pointed out that for certain cases these methods do not lead to the same result. Additionally, the coenergy method is applied for checking the results. The presented statement is proved mathematically (Appendix). There are developed two examples that confirm the statement.

2. Electromagnetic field forces

Basing on Maxwell's equations and Lorentz's force density formula

$$\vec{f}_L = \rho\vec{E} + \vec{j} \times \vec{B}, \tag{1}$$

where $\vec{j}$ is forced current density which satisfies (A.1), ρ is charge density which satisfies Gauss law, $\vec{E}$ denotes electric field strength, $\vec{B}$ means magnetic flux density.

The total force density in electromagnetic field can be presented (see Appendix) in the form of

$$\vec{f} = \vec{f}_L + \vec{f}_P + \vec{N} + \vec{Q} + \vec{M}, \tag{2}$$

where the Poynting component (of electromagnetic field momentum) equals to

$$\vec{f}_P = \frac{\partial(\vec{D} \times \vec{B})}{\partial t}, \tag{3}$$

the non-homogeneous force component (it appears in non-homogeneous region) equals to

$$\vec{N} = -\frac{1}{2}E_u E_w \mathrm{grad}(\varepsilon_{uw}) + \frac{1}{2}B_u B_w \mathrm{grad}(\nu_{uw}), \tag{4}$$

where u, v mean co-ordinates.

Subsequently, the hysteresis component (it is caused by polarization/magnetization of region) is equal to

$$\begin{aligned}\vec{Q} &= \frac{1}{2}\mathrm{grad}(\Delta P_u E_u) - E_u \mathrm{grad}\left(\Delta P_u\right) \\ &+ \frac{1}{2}\mathrm{grad}(B_u \Delta I_u) - B_u \mathrm{grad}\left(\Delta I_u\right),\end{aligned} \tag{5}$$

and the anisotropy component is given by following relation

$$\vec{M} = \frac{1}{2}(\nu_{vu} - \nu_{uv})B_v \mathrm{grad}(B_u) - \frac{1}{2}(\varepsilon_{uv} - \varepsilon_{vu})E_u \mathrm{grad}(E_v), \tag{6}$$

where all vectors and material parameters are introduced and defined in Appendix (it is used vector notation [3, 4]).

The force density $\vec{f}$ given by Maxwell stress tensor (A.12) leads the total force/torque value. For checking the total force/torque value the coenergy method is applied.

Equation (2) is valid for orthogonal curvilinear co-ordinate system. It should be emphasized that the operator $\mathrm{div}_u(\,)$ in Eq. (2) differs from the operator $\mathrm{div}(\,)$ and is useful for surface representation of total electromagnetic torque as shown below.

Main theoretical problem presented in this paper is to consider the surface representation of total force/torque by surface integral of electromagnetic field vector components.

For electromechanical converters for which it can be neglected:

a) electric field in comparison with magnetic field (considering energy field density),
b) Poynting force (low field frequency) – $\vec{f}_P = 0$,
c) non-homogeneous force – no reluctance force/torques ("smooth" construction, no saturation) $\vec{N} = 0$, and
d) hysteresis component (no hysteresis phenomenon, no permanent magnets) $\vec{Q} = 0$,

the method both Lorentz's and Maxwell's do not lead to the same result for magnetically anisotropic region with asymmetrical reluctivity matrix ($u \neq v$)

$$\nu_{vu} \neq \nu_{uv}. \tag{7}$$

The difference between them describes the anisotropy component of force or torque Eq. (6).

The mathematical proof of this statement bases on Eq. (2).

Namely, basing on the assumptions specified above the Eq. (2) takes the important for anisotropy regions form as follows

$$\vec{f} = \vec{f}_L + \vec{M}. \tag{8}$$

According to Eq. (6), the force density is not equal to Lorentz force density for anisotropic asymmetrical region Eq. (7). This conclusion for forces densities leads to the same conclusion for total electromagnetic field force and torque.

However, the statement concerns a rare group of magnetically anisotropic materials but it is important from theoretical point of view.

3. Electromagnetic field equations

Linear and cylindrical electromechanical converters are considered for the analysis of electromagnetic force and for torque analysis, respectively. The electric field displacement is neglected due to the fact that for electromechanical converters magnetic field, taken into account, is predominant. The lack of hysteresis phenomena is also assumed.

The Maxwell equation

$$\mathrm{curl}\vec{E} = -\dot{\vec{B}} \tag{9}$$

and

$$\mathrm{curl}\vec{H} = \vec{j}, \tag{10}$$

are the base for electromagnetic field analysis for considers electromechanical converters. Constitutive relations for electromagnetic field vectors for non-hysteresis medium are

$$H_u = \nu_{uv} B_v, \tag{11}$$

where ν_{uv} are magnetic reluctivity, u, v indicate co-ordinates of curvilinear system (summation due to twice appearing indices is accepted). For the two considered cases of electromechanical converters only one component of magnetic vector potential does not vanish, which was denoted as the z-component i.e.

- for linear converter (Fig. 1, Cartesian co-ordinate: 1-x, 2-y, 3-z, $\vec{i}_x \times \vec{i}_y = \vec{i}_z$, L_x=L_y=L_z=1)

$$\vec{A} = \vec{A}_z = A_z\vec{i}_z = A\vec{i}_z, \tag{12a}$$

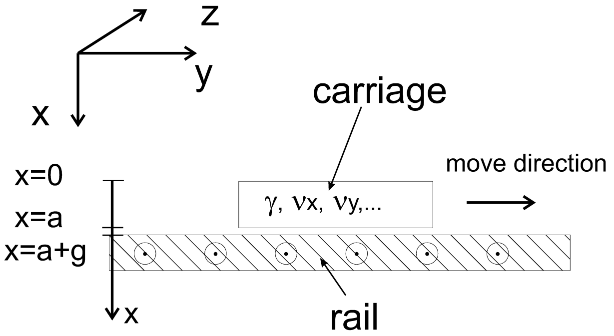

Fig. 1. Linear motor

- for cylindrical converter (Fig.2, cylindrical co-ordinate: 1-r, 2-α, 3-z, $\vec{i}_r \times \vec{i}_\alpha = \vec{i}_z$, L_r=1, L_α=r, L_z=1)

$$\vec{A} = \vec{A}_z = A_z \vec{i}_z = A \vec{i}_z. \tag{12b}$$

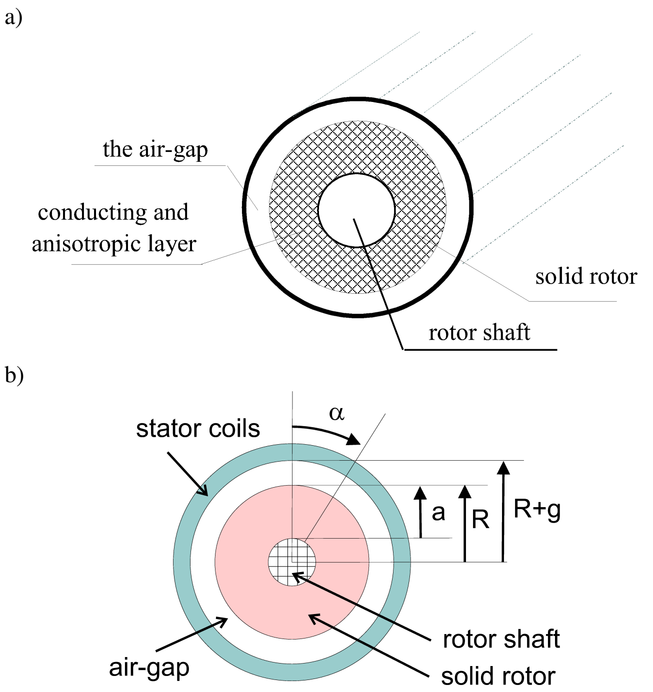

Fig. 2. Cylindrically shaped electromechanical converter – a) its cross section – b)

The magnetic vector potential placement (along z-axis) results from the shape of magnetomotive force pattern and adequate co-ordinate system orientation. The accuracy of such assumptions for magnetic vector potential symmetry results from simplified construction of mechanical converter linear and cylindrical – Figs. 1, 2.

The magnetic flux density by means of vector magnetic potential can be presented as follows

$$\vec{B} = \vec{i}_x \frac{\partial A}{\partial y} - \vec{i}_y \frac{\partial A}{\partial x}, \tag{13a}$$

$$\vec{B} = \frac{\vec{i}_r}{r} \frac{\partial A}{\partial \alpha} - \vec{i}_\alpha \frac{\partial A}{\partial r}. \tag{13b}$$

Equations (11) and (12a, b) can be rewritten in unified form with the help of numbered Lame coefficients (Table 1) as follows

$$\vec{B} = \frac{\vec{i}_1}{L_2 L_3} \frac{\partial (A L_3)}{\partial x_2} - \frac{\vec{i}_2}{L_1 L_3} \frac{\partial (A L_3)}{\partial x_1}. \tag{14}$$

Table 1
Lame coefficients for Cartesian and cylindrical co-ordinate systems

Co-ordinate system	L_1	L_2	L_3
Cartesian ($x_1 = x$, $x_2 = y$, $x_3 = z$)	1	1	1
Cylindrical ($x_1 = r$, $x_2 = \alpha$, $x_3 = z$)	1	r	1

The presented above notation simplifies the analysis for cases considered. The magnetic field strength components due to Eq. (A.5) for nonhysteresis and anisotropic region can be shown in the form of

$$H_1 = \nu_{11} B_1 + \nu_{12} B_2, \tag{15a}$$

$$H_2 = \nu_{21} B_1 + \nu_{22} B_2, \tag{15b}$$

because the third component of magnetic field strength disappears $B_3 = 0$ due to Eqs. 13a,b.

The Maxwell's equation for conductive region if electric displacement current vanishes (low field frequency) takes the form of

$$\mathrm{curl}(\vec{H}) = \vec{j} = \gamma \vec{E} = -\gamma \dot{\vec{A}}, \tag{16a}$$

and for third component (z-component) it leads to the following relationship

$$\frac{1}{L_1 L_2} \left(\frac{\partial (L_2 H_2)}{\partial x_1} - \frac{\partial (L_1 H_1)}{\partial x_2} \right) = -\gamma \dot{A}_3 = -\gamma \dot{A}. \tag{16b}$$

Combining Eqs. (14) and (16b) Eq. for vector magnetic component it is obtained

$$\begin{aligned} &\frac{1}{L_1 L_2} \frac{\partial}{\partial x_1} (L_2 \nu_{21} B_1 + L_2 \nu_{22} B_2) \\ &- \frac{1}{L_1 L_2} \frac{\partial}{\partial x_2} (L_1 \nu_{11} B_1 + L_1 \nu_{12} B_2) = -\gamma \dot{A}. \end{aligned} \tag{16c}$$

For homogeneous region magnetic reluctivities are spatially constant, thus

$$\begin{aligned} &\frac{\partial}{\partial x_1} \left(\nu_{21} \frac{\partial A}{\partial x_2} - \frac{L_2 \nu_{22}}{L_3} \frac{\partial (A L_3)}{\partial x_1} \right) \\ &- \frac{\partial}{\partial x_2} \left(\frac{\nu_{11}}{L_2} \frac{\partial A}{\partial x_2} - \frac{\nu_{12}}{L_3} \frac{\partial (A L_3)}{\partial x_1} \right) = -L_2 \gamma \dot{A}, \end{aligned} \tag{17}$$

where it was taken into account (see Table 1)

$$\frac{\partial L_3}{\partial x_2} = 0 \quad \text{and} \quad L_1 = 1, \tag{18}$$

and finally

$$\begin{aligned} &\nu_{21} \frac{\partial^2 A}{\partial x_1 \partial x_2} - \nu_{22} \frac{\partial}{\partial x_1} \frac{L_2}{L_3} \frac{\partial (A L_3)}{\partial x_1} \\ &- \nu_{11} \frac{\partial}{\partial x_2} \frac{\partial A}{L_2 \partial x_2} + \nu_{12} \frac{\partial}{\partial x_2} \frac{\partial (A L_3)}{L_3 \partial x_1} = -L_2 \gamma \dot{A}. \end{aligned} \tag{19}$$

The Eq. (19) leads to relation for the linear converter (Cartesian co-ordinate system) as follows

$$\nu_{yy}\frac{\partial^2 A}{\partial x^2}-(\nu_{xy}+\nu_{yx})\frac{\partial^2 A}{\partial x\partial y}+\nu_{xx}\frac{\partial^2 A}{\partial y^2}=\gamma\dot{A}, \tag{20}$$

and for the cylindrical converter (cylindrical co-ordinate system) in the form of

$$\frac{\nu_{\alpha\alpha}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)-\frac{\nu_{r\alpha}+\nu_{\alpha r}}{r}\frac{\partial^2 A}{\partial r\partial\alpha}+\frac{\nu_{rr}}{r^2}\frac{\partial^2 A}{\partial\alpha^2}=\gamma\dot{A}. \tag{21}$$

For complex analysis the time-partial derivative of A is presented as multiplication of the operand $i\omega$ (i means imaginary unit) and the complex magnetic potential A at the steady state for time-sinusoidal varying fields as follows

$$\dot{\overrightarrow{A}}\rightarrow i\omega\overrightarrow{A}, \tag{22}$$

where ω means field pulsation (this means that the magnetomotive force for considered converter is monoharmonic). As a result the magnetic vector potential henceforth has got the complex form.

Equations (20) and (21) will be solved with the help of separation method [2–4]. The separated functions for all problems are collected in Table 2.

Table 2
Separation method chosen for Eqs. (20) and (21)

Co-ordinate system	$A=A(x_1,x_2,x_3)$	the equation for function
Cartesian ($x_1=x$, $x_2=y$, $x_3=z$)	$A=X(x)Y(y)$	$Y(y)=\exp(-iky)$
Cylindrical ($x_1=r$, $x_2=\alpha$, $x_3=z$)	$A=R(r)S(\alpha)$	$S(\alpha)=\exp(-ip\alpha)$

These equations take the forms given below.

- For the linear converter (Cartesian co-ordinate system)

$$\frac{d^2X}{dx^2}+ik\frac{\nu_{xy}+\nu_{yx}}{\nu_{yy}}\frac{dX}{dx}-a_0X=0. \tag{23}$$

where

$$a_0=k^2\frac{\nu_{xx}}{\nu_{yy}}+\frac{i\omega\gamma}{\nu_{yy}}, \tag{24}$$

$$a_1=\frac{\nu_{xy}+\nu_{yx}}{2\nu_{yy}}ki, \tag{25}$$

$$\lambda_{1,2}=-a_1\pm\sqrt{a_1^2+a_0}, \tag{26}$$

with the solutions in the form of (Tables 3, 4)

$$\underline{X(x)=a_a\exp(\lambda_1 x)+b_a\exp(\lambda_2 x).} \tag{27}$$

- For the cylindrical converter (cylindrical co-ordinate system)

$$\frac{d^2R}{dr^2}+\frac{(1-2c)}{r}\frac{dR}{dr}=\left[\frac{\nu_{rr}p^2}{\nu_{\alpha\alpha}r^2}+\beta^2\right]R, \tag{28}$$

where

$$c=-pi(\nu_{r\alpha}+\nu_{\alpha r})/2\nu_{\alpha\alpha}, \tag{29}$$

$$\beta=\sqrt{i\omega\gamma/\nu_{\alpha\alpha}}, \tag{30}$$

$$p_B=\sqrt{c^2+p^2\nu_{rr}/\nu_{\alpha\alpha}}, \tag{31}$$

the solution is in the form [38, p.362] (Tables 5, 6) given by means of McDonalds functions

$$\underline{R(\beta r)=a_a(\beta r)^c I_{pB}(\beta r)+b_a(\beta r)^c K_{pB}(\beta r).} \tag{32}$$

- For the non-conductive region ($\gamma=0$) and isotropic region i.e. air-gap region it is satisfied for the linear converter (Cartesian co-ordinate system)

$$\frac{d^2X}{dx^2}-k^2X=0. \tag{33}$$

with the solutions in the form of

$$\underline{X(x)=a_\delta\exp(kx)+b_\delta\exp(-kx).} \tag{34}$$

- For the cylindrical co-ordinate system

$$\frac{d^2R}{dr^2}+\frac{dR}{rdr}=\frac{p^2}{r^2}R, \tag{35}$$

with the solution in the form

$$\underline{R(r)=a_\delta r^p+b_\delta r^{-p}.} \tag{36}$$

The solutions presented should be combined with the boundary conditions.

4. Boundary conditions for electromagnetic field problems

Four boundary conditions for electromagnetic field vectors are defined, that enable to calculate the four unknown constants a$_a$, b$_a$, a$_\delta$, b$_\delta$ (Tables 4, 6). The boundary conditions are physically motivated.

The magnetic field strength disappears at the inner conductive layer surface ($x=0$, $r=R-a$ – see Fig. 1 and Figs. 2a, b)

$$H_2=\nu_{21}B_1+\nu_{22}B_2=0. \tag{37}$$

For linear and cylindrical converters that conditions result from the fact that it is assumed that magnetic reluctivity of rail or rotor core is infinite.

The continuity for normal magnetic flux density components ($x=a$, $r=R$)

$$B_{\delta 1}=B_{a1} \tag{38}$$

and for tangential components of magnetic field strength

$$\nu_o B_{\delta 2}=\nu_{22}B_{a2}+\nu_{21}B_{a1}. \tag{39}$$

The magnetomotive force constituted by converter currents leads to the following condition for tangential component of magnetic field strength at the rail/stator surface ($x=a+g$, $r=R+g$)

$$\nu_o B_{\delta 2}=-\frac{\partial\Theta_s}{L_2\partial x_2}, \tag{40}$$

which is derived under the assumption that the magnetic field strength vanishes in the outer side of winding surface (it is assumed infinitely magnetic reluctivity for stator frame).

5. Solutions to electromagnetic field problems

The analysis of electromagnetic field due to the relations presented above and boundary conditions complete the analytical solution to field problems. For the both chosen converter models the unknown constants are calculated. The solutions are grouped in Tables 3, 5.

5.1. Linear electromechanical converter. Solutions of the Eqs. (23) and (33) are given in Table 3. The four unknown constants a_a, b_a, a_δ, b_δ can be evaluated by formulating the boundary conditions and are grouped in Table 4.

The solutions presented in Tables 3 and 4 enable to continue electromagnetic field and electromagnetic force/torque analyses, subsequently.

5.2. Cylindrically shaped cylindrical electromechanical converter. The solutions of Eqs. (28) and (35) are given in Table 5. The four unknown constants a_a, b_a, a_δ, b_δ can be evaluated by formulating the boundary conditions and they are grouped in Table 6. The analytical solution to the cylindrical motor can be presented in terms of separated function $R(r)$ and $S(\alpha)$ obtained with the help of variable separation method.

The accuracy of the obtained solutions of two partial differential equations are checked (for conductive and nonconductive region) in the following way. Firstly, the boundary conditions fulfilments are checked for four defined boundary conditions (37)–(40). Secondly, the solutions for ordinary differential equation for separated functions Z( ) and D( ), that denote solution for vector magnetic potential separated function R( ) for conductive region ($\gamma \neq 0$) and in air-gap ($\gamma = 0$), respectively. LZ (RZ) means value of left-hand (right-hand) side of ordinary differential Eq. (28) for conductive region. LD (RD) means value of left-hand (right-hand) sides of ordinary differential Eq. (35) for the air-gap region, respectively. The accuracy for both ordinary differential equations is presented in Fig. 3 (for exact solutions LZ/RZ = 1 and LD/RD = 1 should be satisfied).

Table 3
olution of the differential equations

Region	anisotropic carriage (index a)	air-gap – (index δ)
Solution	$X(x) = a_a \exp(\lambda_1 x) + b_a \exp(\lambda_2 x)$	$X(x) = a_\delta \exp(kx) + b_\delta \exp(-kx)$
constants	a_a, b_a	a_δ, b_δ

Table 4
The boundary conditions for magnetic field

Boundary condition (Fig. 1)	Field excited by stator currents	Constants for solutions
Rail mmf $x = a + g$	$\nu_o B_{\delta y} = -\dfrac{\partial \Theta_s}{\partial y}$	$\underline{a_a = \Theta_s \nu_o^{-1}\{U e^{\lambda_1(a+g)} - W e^{\lambda_2(a+g)}\}^{-1}}$ $\underline{b_a = -a_a S}$, $\underline{a_\delta = a_a U}$, $\underline{b_\delta = a_a W}$ $S = \dfrac{\nu_{yy}\lambda_1 + \nu_{yx} ki}{\nu_{yy}\lambda_2 + \nu_{yx} ki}$, $Q = e^{\lambda_1 a} - S e^{\lambda_2 a}$ $P = \dfrac{\nu_{yy}}{k\nu_o}(\lambda_1 e^{\lambda_1 a} - S\lambda_2 e^{\lambda_2 a}) + \dfrac{i\nu_{yx}}{\nu_o}(e^{\lambda_1 a} - S e^{\lambda_2 a})$, $U = \dfrac{1}{2}(P+Q)e^{-ka}$, $W = \dfrac{1}{2}(Q-P)e^{ka}$
Carriage surface $x = a$	$B_{\delta x} = B_{ax}$ $\nu_o B_{\delta y} = \nu_{yy} B_{ay} + \nu_{yx} B_{ax}$	
Inner layer surface $x = 0$	$\nu_{yy} B_y + \nu_{yx} B_x = 0$	

Table 5
Solution of the differential equations

Region	anisotropic layer – (index a)	air-gap – (index δ)
Solutions ($z = \beta r$)	$R(z) = a_a z^c I_{pB}(z) + b_a z^c K_{pB}(z)$	$R(r) = a_\delta r^p + b_\delta r^{-p}$
constants	a_a, b_a	a_δ, b_δ

Table 6
The boundary conditions for magnetic field

Boundary condition (Fig. 2a,b)	Field excited by stator currents	Constants for solutions $z^c I_{pB}(z) \to I_{pB}(z)$, $z^c K_{pB}(z) \to K_{pB}(z)$ (the generalized McDonalds functions)
Stator current mmf $r = R + g = R_g$	$\nu_o B_{\delta\alpha} = -\dfrac{1}{R_s}\dfrac{\partial \Theta_s}{\partial \alpha}$	$\underline{a_a = \Theta_s \nu_o^{-1}\{UR_s^p - WR_s^{-p}\}^{-1}}$ $\underline{b_a = -a_a S}$, $\underline{a_\delta = a_a U}$, $\underline{b_\delta = a_a W}$ $S = \dfrac{\nu_{\alpha\alpha} I'_{pB}(\beta R_a) + \nu_{\alpha r} I_{pB}(\beta R_a) w_a}{\nu_{\alpha\alpha} K'_{pB}(\beta R_a) + \nu_{\alpha r} K_{pB}(\beta R_a) w_a}$
Rotor outer surface $r = R$	$B_{\delta r} = B_{ar}$, $\nu_o B_{\delta\alpha} = \nu_{\alpha\alpha} B_{a\alpha} + \nu_{\alpha r} B_{ar}$	$w_a = ipR_a^{-1}$, $w = ipR^{-1}$ $\beta = \sqrt{i\omega\gamma/\nu_{\alpha\alpha}}$, $P = \dfrac{\beta\nu_{\alpha\alpha}}{p\nu_o}(I'_{pB}(\beta R) - SK'_{pB}(\beta R)) + \dfrac{\nu_{\alpha r} Q w}{p\nu_o}$
Inner layer surface $r = R - a = R_a$	$\nu_{\alpha\alpha} B_\alpha + \nu_{\alpha r} B_r = 0$	$Q = I_{pB}(\beta R) - SK_{pB}(\beta R)$ $U = (PR^{-p+1} + QR^{-p})/2$ $W = (-PR^{p+1} + QR^p)/2$

$$LZ := \left[\frac{\partial}{\partial r}\left(\beta \cdot dZ(\beta \cdot r)\right) + \frac{1 - 2\cdot c}{r}\cdot\frac{\partial}{\partial r}\left(Z(\beta\cdot r)\right)\right] \qquad RZ := \left(\beta^2 + \frac{p^2}{r^2}\cdot\frac{\nu r}{\nu\alpha}\right)\cdot Z(\beta\cdot r) \qquad \frac{LZ}{RZ} = 1.00000$$

$$LD := \left[\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}D(r)\right) + \frac{1}{r}\cdot\frac{\partial}{\partial r}D(r)\right] \qquad RD := \frac{p\delta^2}{r^2}\cdot D(r) \qquad \frac{LD}{RD} = 1.00000$$

Fig. 3. Accuracy for partial differential equations solutions – cylindrical problem (Mathcad™ program extract)

6. Force calculation – linear electromechanical converter

The analysis of electromagnetic field is the background for electromechanical converter force analysis. The obtained solution for magnetic field vector potential is used for force calculation. The magnetic anisotropy of media used for converter construction influences on electromagnetic force value. In some cases appears the difference between forces values evaluated by means of Maxwell's and Lorentz's methods. The example derived below presents the influence of anisotropy on total force value. Total force is evaluated by means of coenergy method for checking the accuracy of total force calculation.

6.1. Linear motor. Linear motors are used in electrical traction and in robotics technologies. The simplified model and dimensions of the analysed model are presented in Fig. 1. The magnetomotive force of rail (that leads to the boundary condition at $x = a + g$) is monoharmonic in space and in time.

After evaluating the magnetic field potential distribution both the magnetic flux density components and the electromagnetic torque components can be evaluated, analytically. Maxwell stress tensor leads to total electromagnetic force by means of well-known formula

$$F = \nu_o \int_{\partial V} B_x B_y dS. \tag{41}$$

The electromagnetic force can be evaluated by Lorentz's force density as follows

$$F_L = \int_V j_z B_r dV, \tag{42}$$

where V is volume of conductive rail ($V = alY$).

The difference between these two forces – called anisotropy force Eq. (6) – is equal to

$$F_M = \int_V \frac{1}{2}(\nu_{vu} - \nu_{uv}) B_v \frac{\partial B_u}{\partial x} dV, \tag{43}$$

hence

$$F_M = \frac{1}{2}(\nu_{xy} - \nu_{yx}) \int_V \left(B_x \frac{\partial B_y}{\partial y} - B_y \frac{\partial B_x}{\partial y}\right) dV. \tag{44}$$

and disappears if the region is either isotropic or anisotropic while $\nu_{xy} = \nu_{yx}$.

Table 7
Exemplary force values for three cases of magnetic anisotropy

The case	The magnetic reluctivities	Lorentz force ratio F_L/F [-]	The anisotropy force ratio F_M/F [-]	The total force value F [N]
A	$\nu_{xy} = 0.2\nu_0 > \nu_{yx} = 0.1\nu_0$	1.128	-0.128	359
B	$\nu_{xy} = 0.1\nu_0 = \nu_{yx} = 0.1\nu_0$	1.000	0.000	362.5
C	$\nu_{xy} = 0.1\nu_0 < \nu_{yx} = 0.2\nu_0$	0.901	0.099	374

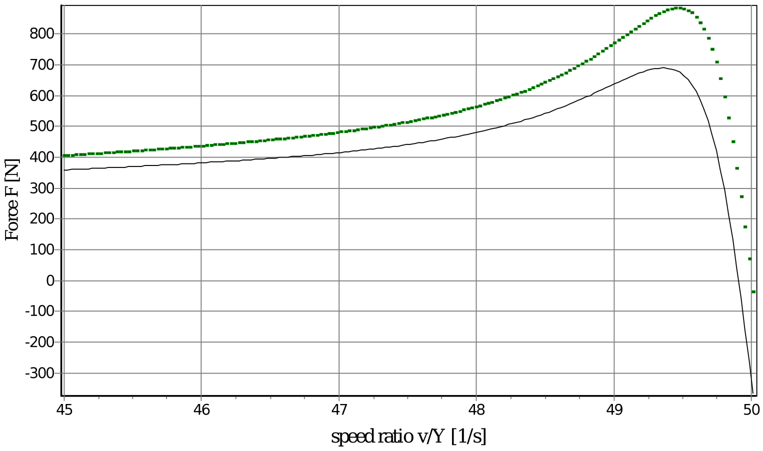

Fig. 4. Force F [N] vs. relative speed v/Y [1/s] calculated for anisotropic carriage by C++ program (solid line for Maxwell and coenergy methods, dots for Lorentz method for the case A of Table 7)

The total force is calculated with the help of coenergy method as follows

$$F = \left.\frac{\partial W_C}{\partial y}\right|_{j=\text{const}} = \int\limits_V \left(\vec{j}\,\frac{\partial \vec{A}}{\partial y} + \vec{B}\,\frac{\partial \vec{H}}{\partial y} \right) dV. \tag{45}$$

The coenergy method gives the same result as Maxwell stress method for all anisotropy cases.

Exemplary, for model of linear motor with data $\gamma = 10 \cdot 10^6$ S/m (rail conductivity), $a = 0.08$ m (conductive rail width), $l = 0.1$ m (width of rail), $g = 0.005$ m (air-gap width), $\Theta = 10000$ A (magnetomotive force first harmonic), $k = 2\pi/Y = 10$ m^{-1} (propagation constant), $s = i2\pi \cdot 5$ Hz (magnetic field pulsation), $\nu_{xx} = \nu_x = \nu_o/2$ (x-axis reluctivity), $\nu_{yy} = \nu_y = \nu_o/10$ (y-axis reluctivity) and different anisotropy reluctivities ν_{xy}, ν_{yx} (Table 7) forces values have been obtained with the help of the presented magnetic field analysis. Forces are calculated for chosen model of linear motor basing on C++ program – Fig. 4.

The table presents results of force calculations for three chosen cases of reluctivity anisotropy. It can be pointed out that only for symmetric reluctivity matrix ($\nu_{xy} = \nu_{yx}$) – Table 7, the case B, the both Lorentz's and Maxwell's methods give the same results. Moreover, the Table 7 shows that for magnetic anisotropy at $\nu_{xy} < \nu_{yx}$ the total force is greatest than Lorentz force.

6.2. Cylindrical motor. Cylindrically shaped motors are used in industry. The rotating cylindrically shaped induction motors with solid rotor are commonly used in robotics as actuators – simplified model is presented in Figs. 2a, b. The exemplary cylindrical induction motor with anisotropic and cylindrical rotor is considered.

The analytical solution of the cylindrical motor can be presented in terms of separated function $R(r)$ and $S(\alpha)$ obtained with the help of variable separation method presented. For monoharmonic magnetomotive force of stator the both magnetic field distribution and electromagnetic torque are calculated. The data for analysis are presented in Fig. 5.

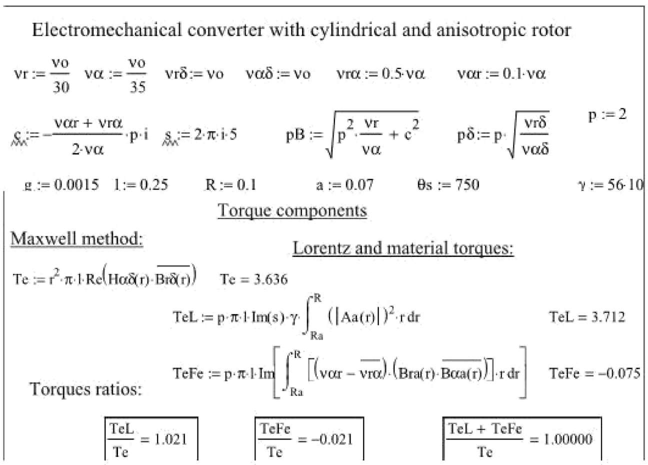

Fig. 5. Torque calculations for cylindrical converter – Mathcad program (for the case A)

For the evaluated magnetic field distribution the electromagnetic torque has been calculated. The electromagnetic torque's two components are presented: Lorentz T_{eL} and material T_{eM} that together constitute the total electromagnetic torque T_e. For cylindrical co-ordinate system ($L_r = 1$, $L_\alpha = r$, $L_z = r$) for horizontal force component $u = \alpha$, the Eq. (2) describes the electromagnetic force density as follows

$$f_\alpha = -\frac{1}{L_\alpha}\mathrm{div}(-L_\alpha H_\alpha \vec{B} + \vec{i}_\alpha e), \tag{46}$$

that leads to the total electromagnetic torque

$$T_e = \nu_o \int_{\partial V} r B_r B_\alpha dS = \nu_o r^2 l \int_0^{2\pi} B_r B_\alpha d\alpha, \tag{47}$$

where l is converter rotor length, r is the radius of surface situated in the air-gap.

The Lorentz's force leads to the torque as follows

$$T_{eL} = \int_V r j_z B_r dV, \tag{48}$$

where V is the volume of conductive layer on rotor (which spreads between radii: R-a and R).

The difference between these two forces – called anisotropy torque – is equal to

$$T_{eM} = \int_V \frac{1}{2}(\nu_{r\alpha} - \nu_{\alpha r}) B_r \frac{\partial B_\alpha}{r\partial\alpha} dV, \tag{49}$$

hence

$$T_{eM} = \frac{1}{2}(\nu_{r\alpha} - \nu_{\alpha r}) \int_V \left(B_r \frac{\partial B_\alpha}{\partial\alpha} - B_\alpha \frac{\partial B_r}{\partial\alpha} \right) d\alpha dr dz. \tag{50}$$

and disappears if the region is either isotropic or anisotropic while $\nu_{r\alpha} = \nu_{\alpha r}$.

For example, for model of cylindrical motor having data $\gamma = 56 \cdot 10^6$ S/m (rotor conductivity), $a = 0.07$ m (conductive rotor layer width), $R = 0.1$ m (rotor outer radius), $l = 0.25$ m (rotor length), $g = 0.0015$ m (air-gap width), $\Theta_1 = 750$ A (magnetomotive force first harmonic), $p = 2$ (pair pole number), $s = i2\pi{\cdot}5$ Hz (rotor current pulsation) $\nu_{rr} = \nu_r = \nu_o/30$ (radial reluctivity), $\nu_{\alpha\alpha} = \nu_\alpha = \nu_o/35$ (tangential reluctivity) and different anisotropy reluctivities $\nu_{r\alpha}$, $\nu_{\alpha r}$ (see Table 8) torques values have been obtained with the help of the presented magnetic field analysis (Figs. 5–7).

Table 8 presents results of torque calculations for three chosen cases of reluctivity anisotropy. It can be pointed out that only for symmetric reluctivity matrix ($\nu_{r\alpha} = \nu_{\alpha r}$) – Table 8 for the case B both Lorentz's and Maxwell's methods give the same results for torque.

The total torque values has been calculated also using magnetic coenergy W_c as follows

$$T_e = \left.\frac{\partial W_C}{\partial\alpha}\right|_{j=\mathrm{const}} = \int_V \left(\vec{j}\frac{\partial\vec{A}}{\partial\alpha} + \vec{B}\frac{\partial\vec{H}}{\partial\alpha} \right) dV, \tag{51}$$

in order to confirm the torque value evaluated with the help of Eq. (2) proved, previously. The both Maxwell and coenergy methods give the same results.

Exemplary calculation procedure is shown in Fig 6 for the data of case C in Table 8. Torques calculated by means of both Maxwell and co-energy methods are equal.

The torque analysis can be also repeated for spherically-shaped electromechanical converter [32]. The presented analytical way of analysis could be applied for synchronous motors [39, 40].

Table 8
Exemplary torque values for three cases of magnetic anisotropy

The case	The magnetic reluctivities	Lorent'z torque ratio T_L/T_e [-]	The anisotropy torque ratio T_M/T_e [-]	The total torque value T_e [Nm]
A	$\nu_{r\alpha} = 0.5\nu_\alpha > \nu_{\alpha r} = 0.1\nu_\alpha$	1.021	−0.021	3.636
B	$\nu_{r\alpha} = 0.5\nu_\alpha = \nu_{\alpha r} = 0.5\nu_\alpha$	1.000	0.000	3.643
C	$\nu_{r\alpha} = 0.1\nu_\alpha < \nu_{\alpha r} = 0.5\nu_\alpha$	0.980	0.020	3.648

$$\mathrm{j(r)} := -\mathrm{s}\cdot\gamma\cdot\mathrm{Aa(r)} \qquad \delta\mathrm{Wj}(\Delta\alpha) := \pi\cdot\mathrm{l}\cdot\mathrm{Re}\left[\int_{\mathrm{Ra}}^{\mathrm{R}} \left(\mathrm{j(r)}\cdot\overline{\left(-\mathrm{i}\cdot\mathrm{p}\;\mathrm{Aa(r)}\cdot\mathrm{e}^{-\mathrm{i}\cdot\mathrm{p}\cdot\Delta\alpha}\right)}\right)\cdot\mathrm{r}\,\mathrm{dr}\right] \qquad \Delta\alpha := 10^{-7}$$

$$\delta\mathrm{Wa}(\Delta\alpha) := \pi\cdot\mathrm{l}\cdot\mathrm{Re}\left[\left[\int_{\mathrm{Ra}}^{\mathrm{R}} \overline{\left(-\mathrm{i}\cdot\mathrm{p}\cdot\mathrm{e}^{-\mathrm{i}\cdot\mathrm{p}\cdot\Delta\alpha}\right)}\cdot\left(\overline{\mathrm{Hra(r)}}\cdot\mathrm{Bra(r)} + \overline{\mathrm{H\alpha a(r)}}\cdot\mathrm{B\alpha a(r)}\right)\cdot\mathrm{r}\,\mathrm{dr}\right]\right] \qquad \delta\mathrm{W}(\Delta\alpha) := \delta\mathrm{Wj}(\Delta\alpha) + \delta\mathrm{Wa}(\Delta\alpha)$$

$$\mathrm{Te\_coenergy} := \delta\mathrm{W}(\Delta\alpha) \qquad \boxed{\frac{\mathrm{Te\_coenergy}}{\mathrm{Te}} = 1.00000} \qquad \mathrm{Te\_coenergy} = 3.648$$

Fig. 6. Maxwell's and co-energy methods comparison – Mathcad program (the case C, Table 8)

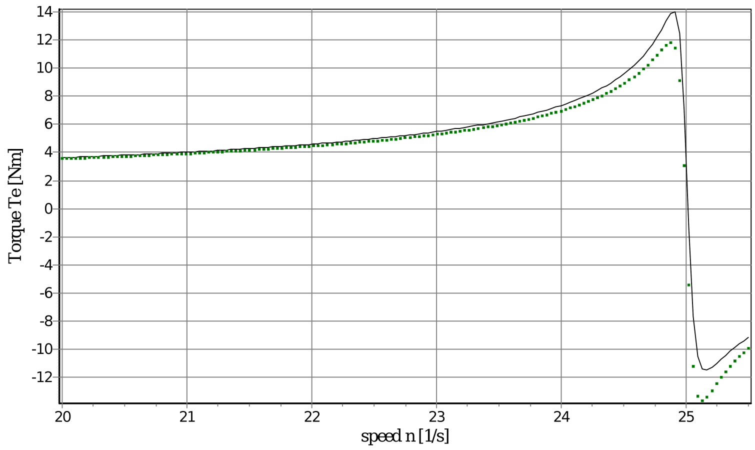

Fig. 7. Torque T_e [Nm] vs. speed n [1/s] calculated for anisotropic rotor by C++ program (solid line for Maxwell and coenergy methods, dots for Lorentz method for the case C of Table 8)

7. Conclusions

The paper presents electromagnetic force and torque analysis for different magnetic anisotropy of material used for building electromechanical converter. The force and torque are calculated by means of the Maxwell's, coenergy and Lorentz's methods. The accuracy for force and torque calculations has been checked by coenergy method.

For example, the analytical electromagnetic field analyses for linear and cylindrical electromechanical converter models are provided. For both models of electromechanical converters the variable separation variable method leads to analytical solutions for electromagnetic field distribution and further force and torque values.

The mathematical proof for the main Eq. (2), that describes force density, has been provided basing on vector notation (Appendix).

The statement proved and analyses carried out for linear and cylindrical electromechanical converters have brought out to the issues, such as

– it is possible to express the total Lorentz force by means of surface integral for isotropic and magnetically anisotropic regions with symmetrical reluctivity matrix

$$F = F_L \Leftrightarrow \qquad \nu_{xy} = \nu_{yx},$$

– it is possible to express the total Lorentz torque by means of surface integral for isotropic and magnetically anisotropic regions with symmetrical reluctivity matrix

$$T_e = T_{eL} \Leftrightarrow \qquad \nu_{r\alpha} = \nu_{\alpha r},$$

– the Lorentz's and Maxwell's methods lead to the same results for regions that are magnetically anisotropic, for normal anisotropy case, when region is homogenous, there is no hysteresis phenomenon and no permanent magnets,
– the Lorentz's and Maxwell's methods lead to different results for medium with non-symmetrical reluctivity matrix Eq. (7),
– the difference between force/torque values for Maxwell's and Lorentz's methods is described by the so-called anisotropy component that volume density is given by Eq. (6).

Appendix

Electromagnetic field forces

Electromagnetic field force volume density in curvilinear co-ordinate system can be presented with the help of the Maxwell equation

$$\mathrm{curl}\overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{D}}{\partial t}, \tag{A.1}$$

and Lorentz's force density in the form of

$$\overrightarrow{f}_L = \rho \overrightarrow{E} + \overrightarrow{j} \times \overrightarrow{B}, \tag{A.2}$$

that constitutes one component of total electromagnetic field force

$$\overrightarrow{f}_L = \overrightarrow{E}\mathrm{div}\overrightarrow{D} + (\mathrm{curl}\overrightarrow{H} - \dot{\overrightarrow{D}}) \times \overrightarrow{B}. \tag{A.3}$$

Furthermore,

$$\begin{aligned}\overrightarrow{f}_L = \mathrm{curl}\overrightarrow{H} \times \overrightarrow{B} + \overrightarrow{H}\mathrm{div}\overrightarrow{B} - \frac{\partial(\overrightarrow{D} \times \overrightarrow{B})}{\partial t}\\ + \overrightarrow{D} \times \mathrm{curl}\overrightarrow{E} + \overrightarrow{E}\mathrm{div}\overrightarrow{D},\end{aligned} \tag{A.4}$$

where it was added $\overrightarrow{H} \cdot \mathrm{div}\overrightarrow{B} = 0$.

Let us present constitutive relation in the general form of

$$H_u = \nu_{uw} B_w - \Delta I_u, \tag{A.5}$$

where reluctivities ν_{uw} could be asymmetrical, ΔI_u is u^{th} component of magnetisation vector (it can be defined either for permanent magnets or for hysteresis region or it denotes component independent from magnetic flux density).

The first and second components on the right-hand side of (A.4) can be written in the form of

$$\begin{aligned}\mathrm{curl}\overrightarrow{H} \times \overrightarrow{B} + \overrightarrow{H}\mathrm{div}\overrightarrow{B} = \overrightarrow{i}_u \mathrm{div}_u(\overrightarrow{\sigma}_{\mu u})\\ - \overrightarrow{\Delta}_\mu - \overrightarrow{N}_\mu - \overrightarrow{Q}_\mu - \overrightarrow{M}_\mu,\end{aligned} \tag{A.6a}$$

where it was denoted (L_u is Lame coefficient for u^{th} co-ordinate; no summation due to u)

$$\mathrm{div}_u(*) = \mathrm{L}_{|\mathrm{u}|}^{-1}\mathrm{div}(L_u(*)), \tag{A.6b}$$

$$\vec{\sigma}_{\mu u} = -H_u\vec{B} + \vec{i}_u e_\mu, \tag{A.6c}$$

$$e_\mu = \frac{1}{2}\vec{H}\vec{B}, \tag{A.6d}$$

$$\vec{N}_\mu = \frac{1}{2}B_u B_w \mathrm{grad}(\nu_{uw}), \tag{A.6e}$$

which is called non-homogenous force component

$$\vec{Q}_\mu = \frac{1}{2}\mathrm{grad}(B_u \Delta I_u) - B_u \mathrm{grad}\left(\Delta I_u\right), \tag{A.6f}$$

which is called hysteresis component (resulting from hysteresis phenomenon), and an auxiliary vector defined as follows

$$\vec{M}_\mu = \frac{1}{2}(\nu_{vu} - \nu_{uv})B_v \mathrm{grad}(B_u), \tag{A.6g}$$

which is called anisotropy component, and an auxiliary vector defined as follows

$$\vec{\Delta}_\mu = \frac{1}{2}B_v H_v \vec{i}_u \frac{\partial \ln(L_v^2/|L|)}{L_u \partial x_u}, \tag{A.6h}$$

for orthogonal curvilinear co-ordinate system u, v, w ($L = L_u L_v L_w$ multiplication of all Lame coefficients).

The fourth and fifth component on the right-hand side of (A.4) can be rearranged in the same manner

$$\begin{aligned}\mathrm{curl}\vec{E}\times\vec{D} + \vec{E}\mathrm{div}\vec{D} = \vec{i}_u \mathrm{div}_u(\vec{\sigma}_{\varepsilon u}) \\ -\vec{\Delta}_\varepsilon - \vec{N}_\varepsilon - \vec{Q}_\varepsilon - \vec{M}_\varepsilon,\end{aligned} \tag{A.7}$$

where the constitutive relation (dielectric permittvities ε_{uw} could be asymmetrical) is introduced as follows

$$D_u = \varepsilon_{uw}E_w - \Delta P_u, \tag{A.8}$$

where ΔP_u is u^{th} component of electric polarisation vector, and

$$\vec{\sigma}_{\varepsilon u} = -E_u\vec{D} + \vec{i}_u e_\varepsilon. \tag{A.9a}$$

$$e_\varepsilon = \frac{1}{2}\vec{E}\vec{D}, \tag{A.9b}$$

$$\vec{N}_\varepsilon = -\frac{1}{2}E_u E_w \mathrm{grad}(\varepsilon_{uw}), \tag{A.9c}$$

$$\vec{Q}_\varepsilon = \frac{1}{2}\mathrm{grad}(\Delta P_u E_u) - E_u \mathrm{grad}\left(\Delta P_u\right), \tag{A.9d}$$

$$\vec{M}_\varepsilon = -\frac{1}{2}(\varepsilon_{vu} - \varepsilon_{uv})E_v \mathrm{grad}(E_u), \tag{A.9e}$$

and an auxiliary vector

$$\vec{\Delta}_\varepsilon = \frac{1}{2}D_v E_v \vec{i}_u \frac{\partial \ln(L_v^2/|L|)}{L_u \partial x_u}. \tag{A.9f}$$

Hence Eq. (A.4) takes the form of

$$\underline{\underline{\vec{f}_L = -\frac{\partial(\vec{D}\times\vec{B})}{\partial t} - \vec{i}_u \mathrm{div}_u\left(\vec{\sigma}_u\right) - \vec{\Delta} - \vec{N} - \vec{Q} - \vec{M},}} \tag{A.10}$$

where

$$\vec{\sigma}_u = -E_u\vec{D} - H_u\vec{B} + \vec{i}_u e, \tag{A.11a}$$

$$e = e_\varepsilon + e_\mu = \frac{1}{2}\vec{E}\vec{D} + \frac{1}{2}\vec{H}\vec{B}, \tag{A.11b}$$

$$\vec{N} = \vec{N}_\varepsilon + \vec{N}_\mu = -\frac{1}{2}E_u E_w \mathrm{grad}(\varepsilon_{uw}) + \frac{1}{2}B_u B_w \mathrm{grad}(\nu_{uw}), \tag{A.11c}$$

$$\vec{Q} = \vec{Q}_\varepsilon + \vec{Q}_\mu. \tag{A.11d}$$

$$\vec{\Delta} = \left(\frac{1}{2}D_v E_v + \tfrac{1}{2}B_v H_v\right)\vec{i}_u \frac{\partial \ln(L_v^2/|L|)}{L_u \partial x_u}, \tag{A.11e}$$

where anisotropy component is equal to

$$\vec{M} = \frac{1}{2}(\nu_{vu} - \nu_{uv})B_v \mathrm{grad}(B_u) - \frac{1}{2}(\varepsilon_{vu} - \varepsilon_{uv})E_v \mathrm{grad}(E_u), \tag{A.11f}$$

and introducing Maxwell force density given by Maxwell stress tensor (given also by coenergy) in the form of

$$\vec{f} = -\vec{i}_u \mathrm{div}_u\left(\vec{\sigma}_u\right) - \vec{\Delta}, \tag{A.11g}$$

it can be written

$$\underline{\underline{\vec{f} = -\vec{i}_u \mathrm{div}_u\left(\vec{\sigma}_u\right) - \vec{\Delta} = \vec{f}_L + \vec{f}_P + \vec{N} + \vec{Q} + \vec{M},}} \tag{A.12}$$

where Maxwell stress tensor equals to

$$\vec{\sigma}_u = -H_u\vec{B} - E_u\vec{D} + \vec{i}_u e, \tag{A.13}$$

and Poynting force density (of electromagnetic field momentum [2, 3, 25])

$$\vec{f}_P = \frac{\partial(\vec{D}\times\vec{B})}{\partial t}. \tag{A.14}$$

The presented above relations take the simpler form for the case when the magnetic field is dominant.

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