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AUTOMATICS

Representation of solutions for fractional differential-algebraic systems with delays

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Abstract. Linear stationary fractional differential-algebraic systems with delay are studied. The differential operators are taken in the Caputo sense with its initial condones. An exponential growth is proved for these systems. The obtained result allows one to apply the Laplace transform for investigation of stationary systems and, as a consequence, to obtain analytical representation of solutions in the form of series in power of solutions to the determining equations. The results are illustrated by an example.

Key words: fractional differential equations, determining equations, differential-algebraic systems.

1. Introduction

Many papers and books have been recently devoted do the various types of fractional systems [1–3]. This paper deals with linear stationary fractional differential-algebraic systems with delays (FDAD systems), with some equations being fractional differential (we introduce the Caputo Fractional Derivative), the other-difference, with some variables being continuous the other-piecewise continuous. We apply fractional differential calculus to our investigations especially dealing with the Laplace transform. For this we prove exponential growth of solutions. We introduce the determining equations the same as for differential-algebraic systems (for example see [4] or [5]). We use the fractional Laplace inverse formulas to obtain our results.

2. Preliminaries

In this paper we consider algebraic-differential delay systems with the frictional derivative, for the classical first derivative such systems are a special case of descriptor (singular) systems with aftereffect

$$\frac{d}{dt}\left(\int_{-h}^{0} d_sG(t,s)x(t+s) + \int_{-h}^{0} d_sQ(t,s)u(t+s) + F_1(t)\right) = \int_{-h}^{0} d_sA(t,s)x(t+s) + \int_{-h}^{0} d_sB(t,s)u(t+s) + F_2(t),$$

where an n-vector function $x(\cdot)$ describes the behavior in time of the object (process) to be modeled, $u(\cdot)$ is an r-vector function specifying the input influence (control), the n-vector functions $F_1(\cdot)$ and $F_2(\cdot)$ specify perturbations, the entries of the matrix functions $G(t,s)$, $Q(t,s)$, $A(t,s)$ and $B(t,s)$ of the corresponding size have a bounded variation with respect to the second argument on $[-h,0]$ and $h>0$ is the value of the aftereffect.

In the stationary case of this equation with an operator $G : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n$ atomic at zero, the study of the existence, uniqueness, exponential estimate, and stability of solutions as well as their representation by the variation-of-constants formula can be found in [6, 7]. All these problems remain open in the general case for nonatomic operators.

Now we consider a special case of the above-represented schemes, namely, linear algebraic-differential systems with delay in the state.

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t),\ t>0, \\ x_2(t) &= A_{21}x_1(t) + A_{22}x_2(t-h) + B_2u(t),\ t \geq 0, \end{aligned} \tag{1}$$

For the above system see [8].

In this paper, we concentrate on the stationary FDAD system in the following form:

$$\begin{aligned} ({}^CD_t^{\alpha}x_1)(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t),\ t>0, \\ x_2(t) &= A_{21}x_1(t) + A_{22}x_2(t-h) + B_2u(t),\ t \geq 0, \end{aligned} \tag{2}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in \mathbb{R}^r$, $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{r\times n_1}$, $B_2 \in \mathbb{R}^{r\times n_2}$ are constant (real) matrices, $0<h$ is a constant delay. We regard an absolute continuous n_1-vector function $x_1(\cdot)$ and a piecewise continuous n_2-vector function $x_2(\cdot)$ as a solution of System (2) if they satisfy the equation $(2)_1$ for almost all $t>0$ and $(2)_2$ for all $t \geq 0$.

System (2) should be completed with initial conditions:

$$\begin{aligned} x_1(+0) = x_0, &\quad [({}^CD_t^{\alpha-1}x_1)(t)]_{t=0} = x_0, \\ x_2(\tau) = \psi(\tau), &\quad \tau \in [-h,0), \end{aligned} \tag{3}$$

where

$$x_0 \in \mathbb{R}^{n_1}; \quad \psi \in PC([-h,0),\mathbb{R}^{n_2})$$

and

$$PC([-h,0),\mathbb{R}^{n_2})$$

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denotes the set of piecewise continuous n_2-vector-functions in $[-h, 0]$. Observe that $x_2(t)$ at $t = 0$ is determined from Eq. $(2)_2$.

Let us introduce the following notation:

${}^C D_t^\alpha$ is the left-sided Caputo fractional derivatives of order α defined by

$$({}^C D_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\left(\frac{d}{d\tau} f(\tau)\right)}{(t-\tau)^\alpha} d\tau,$$

where $0 < \alpha \leq 1$, $\alpha \in \mathbb{R}$ and $\Gamma(t) = \int_0^\infty e^{-\tau} \tau^{t-1} d\tau$ is the Euler gamma function. Similarly we define the fractional integral I_t^α

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

(see [1] for more details).

$T_t = \lim_{\epsilon \to +0} \left[\frac{t-\epsilon}{h}\right]$, where the symbol $[z]$ means entire part of the number z; I_n is the identity n by n matrix.

3. Evaluation of solutions

Lemma 3.1. [6] If $c \geq 0$, $f(t) \geq 0$, $g(t) \geq 0$ are continuous and

$$f(t) \leq c + \int_0^t f(\tau) g(\tau) d\tau, \ t > 0,$$

then

$$f(t) \leq c \ \exp\left(\int_0^t g(\tau) d\tau\right), \ t > 0.$$

Now, we can formulate the following.

Theorem 3.2. For each solution to system (2) corresponding to initial condition (3), where $\max_{t \in [-h,0]} \|\psi(t)\| = M_1$, and control $u(\cdot)$ whose growth rate does not exceed an exponential one, i.e. $\|u(t)\| \leq M_2 e^{\sigma t}$, $t \geq 0$ (M_2, σ are positive constants), positive numbers L and γ can be found such that $\|x_1(t)\| \leq L e^{\gamma t}$, $\|x_2(t)\| \leq L e^{\gamma t}$, $t \geq 0$, where L and γ may depend only on M_1, M_2, σ and the parameters of the system.

Proof. Let us put

$$x_1(\tau) = 0, \quad x_2(\tau - h) = 0, \quad \text{for } \tau < 0. \tag{4}$$

Multiplying $(2)_1$ by $e^{-\beta t}$, where β is an arbitrary positive number, we obtain

$$\begin{aligned} e^{-\beta t}({}^C D_t^\alpha x_1)(t) &= A_{11} e^{-\beta t} x_1(t) + \\ &+ A_{12} e^{-\beta t} x_2(t) + B_1 e^{-\beta t} u(t). \end{aligned} \tag{5}$$

Taking β such that $e^{-\beta t} < 1$, we have

$$\begin{aligned} ({}^C D_t^\alpha x_1)(t) &\leq A_{11} e^{-\beta t} x_1(t) + \\ &+ A_{12} e^{-\beta t} x_2(t) + B_1 e^{-\beta t} u(t). \end{aligned} \tag{6}$$

Solving the equation $(2)_2$ "step-wise", we obtain the representation

$$x_2(t) = \sum_{k=0}^{T_t} (A_{22})^k A_{21} x_1(t - kh) +$$
$$+ (A_{22})^{T_t+1} \psi(t - T_t h - h) + \sum_{k=0}^{T_t} (A_{22})^k B_2 u(t).$$

Substituting this equality into (5) and integrating by I_t^α with respect to τ from 0 to t, we obtain the relation

$$\begin{aligned} x_1(t) \leq x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} (A_{11}) e^{-\beta\tau} x_1(\tau) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} \sum_{k=0}^{T_\tau} (A_{22})^k A_{21} e^{-\beta\tau} x_1(\tau - kh) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22})^{T_\tau+1} e^{-\beta\tau} \psi(\tau - T_\tau h - h) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} \sum_{k=0}^{T_\tau} (A_{22})^k B_2 e^{-\beta\tau} u(\tau - kh) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} B_1 e^{-\beta\tau} u(\tau) d\tau. \end{aligned} \tag{7}$$

By (4) and (7), we have

$$\begin{aligned} x_1(t) \leq x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} (A_{11}) e^{-\beta\tau} x_1(\tau) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{T_t} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22})^k A_{21} e^{-\beta\tau} x_1(\tau - kh) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{T_t - 1} \int_{kh}^{(k+1)h} \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22})^{k+1} \cdot \\ &\cdot e^{-\beta\tau} \psi(\tau - kh - h) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{T_t h}^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22})^{T_\tau+1} \cdot \\ &\cdot e^{-\beta\tau} \psi(\tau - T_\tau h - h) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{T_t} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22})^k B_2 e^{-\beta\tau} u(\tau - kh) d\tau + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} B_1 e^{-\beta\tau} u(\tau) d\tau. \end{aligned} \tag{8}$$

Then we obtain

$$x_1(t) \le x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} (A_{11}) e^{-\beta\tau} x_1(\tau) d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{t-kh} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22} e^{-\beta h})^k A_{21} e^{-\beta\tau} x_1(\tau) d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{T_t - 1} \int_{-h}^0 \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22} e^{-\beta h})^{k+1} e^{-\beta\tau} \psi(\tau) d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{-h}^{t - T_t h - h} \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22} e^{-\beta h})^{T_\tau + 1} e^{-\beta\tau} \psi(\tau) d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{T_t} \int_0^{t-kh} \frac{1}{(t-\tau)^{1-\alpha}} A_{12} (A_{22} e^{-\beta h})^k B_2 e^{-\beta\tau} u(\tau) d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} B_1 e^{-\beta\tau} u(\tau) d\tau. \tag{9}$$

Evaluating (9) in the norm, we have

$$\|x_1(t)\| \le \|x_0\| +$$
$$\frac{1}{\Gamma(\alpha)} \left(\| |A_{12}\| \sum_{k=0}^{+\infty} \|A_{22} e^{-\beta h}\|^k \|A_{21}\| + \|A_{11} - \beta I_{n_1}\| \right) \cdot$$
$$\cdot \int_0^t \| \frac{e^{-\beta\tau}}{(t-\tau)^{1-\alpha}} \| \|x_1(\tau)\| d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{-h}^0 \|A_{12}\| \sum_{k=0}^{+\infty} \|A_{22} e^{-\beta h}\|^k .$$
$$\cdot \|A_{22}\| \|\psi(\tau)\| e^{-\beta h} \| \frac{e^{-\beta\tau}}{(t-\tau)^{1-\alpha}} \| d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \left(\|A_{12}\| \sum_{k=0}^{+\infty} \|A_{22} e^{-\beta h}\|^k \|B_2\| + \|B_1\| \right) \cdot$$
$$\cdot \int_0^t \|u(\tau)\| \| \frac{e^{-\beta\tau}}{(t-\tau)^{1-\alpha}} \| d\tau, \quad t > 0. \tag{10}$$

Taking $\beta > 0$ such that

$$\|A_{22} e^{-\beta h}\| < 1, \qquad \beta - \sigma > 0,$$
$$\int_0^t \| \frac{e^{(\sigma-\beta)\tau}}{(t-\tau)^{1-\alpha}} \| d\tau < 1,$$

we have

$$\|x_1(t)\| \le \|x_0\| +$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{-h}^0 \frac{\|A_{12}\|}{1 - \|A_{22} e^{-\beta h}\|} \|A_{22}\| M_1 e^{-\beta h} e^{-\beta\tau} d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \left(\frac{\|A_{12}\| \|A_{21}\|}{1 - \|A_{22} e^{-\beta h}\|} \| + \|A_{11}\| \right) \int_0^t \|x_1(\tau)\| d\tau +$$
$$+ \frac{1}{\Gamma(\alpha)} \left(\frac{\|A_{12}\| \|B_2\|}{1 - \|A_{22} e^{-\beta h}\|} + \|B_1\| \right) M_2, \quad t > 0. \tag{11}$$

Hence, we obtain

$$\|x_1(t)\| \le \omega_\beta + N_\beta \int_0^t \|x_1(\tau)\| d\tau, \tag{12}$$

where

$$N_\beta = \frac{1}{\Gamma(\alpha)} \left(\frac{\|A_{12}\| \|A_{21}\|}{1 - \|A_{22} e^{-\beta h}\|} \| + \|A_{11}\| \right),$$
$$\omega_\beta = \|x_0\| + \frac{1}{\Gamma(\alpha)} \frac{\|A_{12}\|}{1 - \|A_{22} e^{-\beta h}\|} \|A_{22}\| M_1 \frac{(1 - e^{-\beta h})}{\beta} +$$
$$\frac{1}{\Gamma(\alpha)} \left(\frac{\|A_{12}\| \|B_2\|}{1 - \|A_{22} e^{-\beta h}\|} + \|B_1\| \right) M_2 \le K_\beta,$$
$$K_\beta = \|x_0\| + \frac{1}{\Gamma(\alpha)} \frac{\|A_{12}\|}{1 - \|A_{22} e^{-\beta h}\|} \|A_{22}\| M_1 \frac{1}{\beta} +$$
$$\frac{1}{\Gamma(\alpha)} \left(\frac{\|A_{12}\| \|B_2\|}{1 - \|A_{22} e^{-\beta h}\|} + \|B_1\| \right) M_2.$$

By Lemma 1, we have

$$\|x_1(t)\| \le K_\beta e^{N_\beta t}.$$

By the substitution $\gamma = N_\beta$, $L_1 = K_\beta$, we obtain

$$\|x_1(t)\| \le L_1 e^{\gamma t}. \tag{13}$$

We proof evaluation of $x_2(t)$, $t \ge 0$ by the induction on intervals: $t \in [((k-1)h, kh)$, $k = 1, 2, \ldots$. We define L such that

$$L \ge \max \left\{ L_1, \|A_{21}\| L_1 + \|A_{22}\| M_1 + \right.$$
$$\left. + \|B_2\| M_2, \frac{\|A_{21}\| L_1 + \|B_2\| M_2}{1 - \|A_{22} e^{-\gamma h}\|} \right\} \tag{14}$$

then we have

$$L \ge \|A_{21}\| L_1 + L \|A_{22}\| e^{-\gamma h} + \|B_2\| M_2.$$

First, for $k = 1$, $t \in [0, h)$, we obtain

$$\|x_2(t)\| \le \|A_{21}\| \|x_1(t)\| +$$
$$+ \|A_{22}\| \max_{t \in [-h, 0]} \|\psi(t)\| + \|B_2\| \|u(t)\| \le$$
$$+ \le \|A_{21}\| L_1 e^{\gamma t} + \|A_{22}\| M_1 + \|B_2\| M_2 e^{\sigma t} \le$$
$$+ \le (\|A_{21}\| L_1 + \|A_{22}\| M_1 + \|B_2\| M_2) e^{\gamma t} \le L e^{\gamma t}$$

then

$$\|x_2(t)\| \le Le^{\gamma t}, \tag{15}$$

is true for $k = 1$, $t \in [0, h)$. Assuming that (15) holds for $i = 1; \dots; k-1$, $t \in [0, (k-1)h)$ let us prove it holds true for $k = p$, $t \in [(k-1)h, kh)$ i.e.,

$$\begin{gathered}\|x_2(t)\| \le \|A_{21}\|\|x_1(t)\|+ \\ +\|A_{22}\|\|x_2(t-h)\| + \|B_2\|\|u(t)\| \le \\ + \le \|A_{21}\|L_1e^{\gamma t} + \|A_{22}\|Le^{\gamma(t-h)} + \|B_2\|M_2e^{\sigma t} \le \\ + \le \left(\|A_{21}\|L_1 + \|A_{22}\|Le^{-\gamma h} + \|B_2\|M_2\right)e^{\gamma t} \le Le^{\gamma t}.\end{gathered}$$

This completes the proof.

4. Representation of solutions into series of determining equations solutions

Let us introduce the determining equations of system (2) (see [8] for more details).

$$\begin{gathered}X_{1,k}(t) = A_{11}X_{1,k-1}(t)+ \\ + A_{12}X_{2,k-1}(t) + B_1U_{k-1}(t), \\ X_{2,k}(t) = A_{21}X_{1,k}(t) + A_{22}X_{2,k}(t-h)+ \\ + B_2U_{k-1}(t), \quad k = 0, 1, \dots;\end{gathered} \tag{16}$$

with initial conditions

$$\begin{gathered}X_{1,k}(t) = 0, X_{2,k}(t) = 0 \quad \text{for} \quad t < 0 \quad \text{or} \quad k \le 0; \\ U_0(0) = I_n,\ U_k(t) = 0 \quad \text{for} \quad t^2 + k^2 \ne 0.\end{gathered}$$

Here, we establish some algebraic properties of $X_{1,k}$, $X_{2,k}$.

Proposition 4.1. The following identities hold:

$$\begin{gathered}\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k \cdot \\ \cdot \left(B_1 + A_{12}(I_{n_2} - \omega A_{22})^{-1}B_2\right) = \\ = \sum_{j=0}^{+\infty} X_{1,k+1}(jh)\omega^j, \quad k = 0, 1, \dots;\end{gathered} \tag{17}$$

$$\begin{gathered}(I_{n_2} - \omega A_{22})^{-1}A_{12}\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k \cdot \\ \cdot \left(B_1 + A_{12}(I_{n_2} - \omega A_{22})^{-1}B_2\right) = \\ \sum_{j=0}^{+\infty} X_{2,k+1}(jh)\omega^j, \quad k = 1, 2, \dots;\end{gathered} \tag{18}$$

$$(I_{n_2} - \omega A_{22})^{-1}B_2 = \sum_{j=0}^{+\infty} X_{2,0}(jh)\omega^j, \tag{19}$$

where $|\omega| < \omega_1$ and ω_1 is a sufficiently small real number.

Let us introduce the determining equations of homogenous system (2).

$$\begin{gathered}\widetilde{X}_{1,k}(t) = A_{11}\widetilde{X}_{1,k-1}(t) + A_{12}\widetilde{X}_{2,k-1}(t), \\ \widetilde{X}_{2,k}(t) = A_{21}\widetilde{X}_{1,k}(t) + A_{22}\widetilde{X}_{2,k}(t-h), \\ t \ge 0, \qquad k = 1, 2, \dots;\end{gathered} \tag{20}$$

with initial conditions

$$\begin{gathered}\widetilde{X}_{1,k}(t) = 0, \widetilde{X}_{2,k}(t) = 0 \quad \text{for} \quad t < 0 \quad \text{or} \quad k \le 0; \\ \widetilde{X}_{1,1}(0) = 0, \widetilde{X}_{1,1}(\tau) = 0 \quad \text{if} \quad \tau \ne 0.\end{gathered}$$

Similar to Proposition 1 we can formulate the following.

Proposition 4.2. The following identities hold:

$$\begin{gathered}\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k = \\ = \sum_{j=0}^{+\infty} \widetilde{X}_{1,k+1}(jh)\omega^j, \quad k = 1, 2, \dots;\end{gathered} \tag{21}$$

$$\begin{gathered}(I_{n_2} - \omega A_{22})^{-1}A_{12}\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k = \\ = \sum_{j=0}^{+\infty} \widetilde{X}_{2,k+1}(jh)\omega^j, \quad k = 1, 2, \dots;\end{gathered} \tag{22}$$

where $|\omega| < \omega_1$ and ω_1 is a sufficiently small real number.

Theorem 4.3. The solution to system (2) with initial conditions (3) for $t \ge 0$ exists, is unique and can be represented by the following formulas:

$$\begin{gathered}x_1(t) = \sum_{k=0}^{+\infty} \sum_{\substack{i \\ t-ih>0}} X_{1,k+1}(ih) \cdot \\ \cdot \int_0^{t-ih} \frac{(t-\tau-ih)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))} u(\tau)d\tau + s_1(t, x_0, \psi),\end{gathered} \tag{23}$$

$$\begin{gathered}x_2(t) = \sum_{k=0}^{+\infty} \sum_{\substack{i \\ t-ih>0}} X_{2,k+1}(ih) \cdot \\ \cdot \int_0^{t-ih} \frac{(t-\tau-ih)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))} u(\tau)d\tau+ \\ + \sum_{\substack{i \\ t-ih>0}} X_{2,0}(ih)u(t-ih) + s_2(t, x_0, \psi),\end{gathered} \tag{24}$$

where $s_1(t, x_0, \psi)$, $s_2(t, x_0, \psi)$ – functions depending only on the initial data:

$$\begin{gathered}s_1(t, x_0, \psi) = \\ \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j)h>0}} \widetilde{X}_{1,k+1}(ih)A_{12}(A_{22})^{i+1} \cdot \\ \cdot \int_0^{t-(i+j)h} \frac{(t-\tau-(i+j)h)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))} \psi(\tau-h)d\tau+ \\ + \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha k}}{\Gamma(\alpha k+1)} \widetilde{X}_{1,k+1}(jh)x_0,\end{gathered}$$

$$s_2(t,x_0,\psi)=$$

$$\sum_{k=0}^{+\infty}\sum_{\substack{i,j\\ t-(i+j)h>0}}\widetilde{X}_{2,k+1}(ih)A_{12}(A_{22})^{i+1}\cdot$$

$$\cdot\int_0^{t-(i+j)h}\frac{(t-\tau-(i+j)h)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\psi(\tau-h)d\tau+$$

$$\sum_{k=0}^{+\infty}\sum_{\substack{j\\ t-jh>0}}\frac{(t-jh)^{\alpha k}}{\Gamma(\alpha k+1)}\widetilde{X}_{2,k+1}(ih)x_0+$$

$$+\sum_{i=0}^{+\infty}(A_{22})^{i+1}\psi(t-(i+1)h)),$$

where $\psi(\tau)\equiv 0$ for $\tau\notin[-h,0)$.

Proof. First we use the classical formula for the Laplace transformation of the fractional derivative of Eq. $(2)_1$

$$\int_0^\infty e^{-pt}({}^CD_t^\alpha x_1)(t)dt=$$

$$=p^\alpha\breve{x}_1(p)-p^{\alpha-1}[({}^CD_t^{\alpha-1}x_1)(t)]_{t=0}=$$

$$=p^\alpha\breve{x}_1(p)-p^{\alpha-1}x_0.$$

We apply the Laplace transform to system (2)

$$p^\alpha\breve{x}_1(p)-p^{\alpha-1}x_0= =A_{11}\breve{x}_1(p)+A_{12}\breve{x}_2(p)+B_1\breve{u}(p), \tag{25}$$

$$\breve{x}_2(p)=A_{21}\breve{x}_1(p)+A_{22}e^{-ph}\breve{x}_2(t)+ +A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+B_2\breve{u}(p), \tag{26}$$

where $\breve{x}_1(p)$, $\breve{x}_2(p)$, $\breve{u}(p)$ are Laplace transforms of functions $x_1(t)$, $x_2(t)$, $u(t)$ respectively. Solving (26), we obtain

$$\breve{x}_2(p)=\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{12}\breve{x}_1(p)+ +\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+ +\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}B_2\breve{u}(p), \tag{27}$$

$$\breve{x}_1(p)=\left(p^\alpha I_{n_1}-A_{11}-A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{21}\right)^{-1}\times$$
$$\times\left(A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+\right.$$
$$\left.+p^{\alpha-1}x_0+\left(B_1+A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}B_2\right)\breve{u}(p)\right)=$$
$$=\sum_{k=0}^{+\infty}\frac{1}{(p^\alpha)^{k+1}}\left(A_{11}+A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{21}\right)^k\times$$
$$\times\left(A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+\right.$$
$$\left.p^{\alpha-1}x_0+\left(B_1+A_{12}(I_{n_2}-\omega A_{22})^{-1}B_2\right)\breve{u}(p)\right). \tag{28}$$

Applying Propositions 4.1 and 4.2 to (27) and (28) we obtain

$$\breve{x}_1(p)=$$
$$\sum_{k=0}^{+\infty}\frac{1}{(p^\alpha)^{k+1}}\sum_{j=0}^{+\infty}e^{-jph}\widetilde{X}_{1,k+1}(jh)A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}\cdot$$
$$\cdot A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+$$
$$\sum_{k=0}^{+\infty}\frac{1}{p^{\alpha k+1}}\sum_{j=0}^{+\infty}e^{-jph}\widetilde{X}_{1,k+1}(jh)x_0+$$
$$+\sum_{k=0}^{+\infty}\frac{1}{(p^\alpha)^{k+1}}\sum_{j=0}^{+\infty}e^{-jph}X_{1,k+1}(jh)\breve{u}(p), \tag{29}$$

$$\breve{x}_2(p)=$$
$$\sum_{k=0}^{+\infty}\frac{1}{(p^\alpha)^{k+1}}\sum_{j=0}^{+\infty}e^{-jph}\widetilde{X}_{2,k+1}(jh)\cdot$$
$$\cdot A_{12}\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+$$
$$+\sum_{k=0}^{+\infty}\frac{1}{p^{\alpha k+1}}\sum_{j=0}^{+\infty}e^{-jph}\widetilde{X}_{2,k+1}(jh)x_0+$$
$$+\sum_{k=0}^{+\infty}\frac{1}{(p^\alpha)^{k+1}}\sum_{j=0}^{+\infty}e^{-jph}\widetilde{X}_{2,k+1}(jh)\breve{u}(p)+ \tag{30}$$
$$+\left(I_{n_2}-A_{22}e^{-ph}\right)^{-1}A_{22}e^{-ph}\int_{-h}^0 e^{-p\tau}\psi(\tau)d\tau+$$
$$+\sum_{j=0}^{+\infty}e^{-jph}\widetilde{X}_{2,0}(jh)\breve{u}(p).$$

By the inverse Laplace transform the proof of theorem 4.1 is complete.

5. Example

Let us consider the following system:

$$({}^C D_t^\alpha x_1)(t) = \begin{bmatrix}1\end{bmatrix} x_1(t) + \\ + \begin{bmatrix}0 & -1\end{bmatrix} x_2(t) + \begin{bmatrix}1\end{bmatrix} u(t), \qquad t > 0,$$

$$x_2(t) = \begin{bmatrix}0\\1\end{bmatrix} x_1(t) + \\ + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} x_2(t-h) + \begin{bmatrix}1\\0\end{bmatrix} u(t), \qquad t \geq 0,$$

with initial conditions and control:

$$x_0 \in \mathbb{R}, \quad \psi(\tau) \in \mathbb{R}^2, \quad \tau \in [-h, 0), \quad u(t) \equiv 0.$$

First compute the solutions of the determining system:

$$\widetilde{X}_{1,1}(0) = 1, \quad \widetilde{X}_{1,2}(0) = A_{11} + A_{12}A_{21} = 0,$$
$$\widetilde{X}_{1,k}(0) = 0, \quad k \geq 2, \quad \widetilde{X}_{1,k}(t) = 0, \quad k \geq 1$$

and $t \geq 0$, by Theorem 1 we may compute $x_1(t)$.

$$\widetilde{X}_{2,1}(0) = \begin{bmatrix}0\\1\end{bmatrix},$$

$$\widetilde{X}_{2,2}(0) = A_{21}\widetilde{X}_{1,2}(0) + A_{22}\widetilde{X}_{2,2}(-h) = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\widetilde{X}_{2,k}(0) = \begin{bmatrix}0\\0\end{bmatrix}, \qquad k \geq 2.$$

For $t = h$ we get

$$\widetilde{X}_{2,1}(h) = A_{21}\widetilde{X}_{1,1}(h) + A_{22}\widetilde{X}_{2,1}(0) = \begin{bmatrix}1\\0\end{bmatrix},$$

$$\widetilde{X}_{2,2}(h) = A_{21}\widetilde{X}_{1,2}(h) + A_{22}\widetilde{X}_{2,2}(0) = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\widetilde{X}_{2,k}(h) = \begin{bmatrix}0\\0\end{bmatrix}, \qquad k \geq 2,$$

$$\widetilde{X}_{2,1}(2h) = A_{21}\widetilde{X}_{1,1}(2h) + A_{22}\widetilde{X}_{2,1}(h) = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\widetilde{X}_{2,k}(jh) = \begin{bmatrix}0\\0\end{bmatrix}, \qquad k \geq 1$$

and $j \geq 2$ by Theorem 1 we may compute $x_2(t)$.

Solving our system "step by step" we get:
for $t \in [0, h]$

$$({}^C D_t^\alpha x_1)(t) = \begin{bmatrix}1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & -1\end{bmatrix} x_2(t) = 0,$$
$$x_1(t) \equiv x_0,$$
$$x_2(t) = \begin{bmatrix}0\\1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}\psi_1(t-h)\\\psi_2(t-h)\end{bmatrix} = \begin{bmatrix}\psi_2(t-h)\\x_1(t)\end{bmatrix},$$
$$x_2(t) = \begin{bmatrix}\psi_2(t-h)\\x_0\end{bmatrix}.$$

for $t \in [h, 2h]$

$$({}^C D_t^\alpha x_1)(t) = \begin{bmatrix}1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & -1\end{bmatrix} x_2(t) = 0,$$
$$x_1(t) \equiv x_0,$$
$$x_2(t) = \begin{bmatrix}0\\1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}\psi_2(t-h)\\x_0\end{bmatrix} = \begin{bmatrix}x_0\\x_0\end{bmatrix},$$
$$x_2(t) = \begin{bmatrix}x_0\\x_0\end{bmatrix}.$$

for $t \geq 2h$

$$({}^C D_t^\alpha x_1)(t) = \begin{bmatrix}1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & -1\end{bmatrix} x_2(t) = 0,$$
$$x_1(t) \equiv x_0,$$
$$x_2(t) = \begin{bmatrix}0\\1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_0\\x_0\end{bmatrix} = \begin{bmatrix}x_0\\x_0\end{bmatrix},$$
$$x_2(t) = \begin{bmatrix}x_0\\x_0\end{bmatrix}.$$

Finally, we see that these two methods give the same solution.

6. Conclusions

Thus, in this work the exponential estimate for the growth rate of solutions to the stationary FDAD systems is proved. This allows one to apply the Laplace transform to such systems. This provides considerable simplification of the representation of solutions to the FDAD systems. The solutions found in it have important applications in qualitative control theory in FDAD systems. This will be discussed in next papers.

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REFERENCES

[1] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, 4th ed., Springer, Berlin, 2007.

[2] T. Kaczorek, "Asymptotic stability of positive fractional 2D linear systems", *Bull. Pol. Ac.: Tech.* 57 (3), 289–292 (2009).

[3] T. Kaczorek, "Positive fractional 2D hybrid linear systems", *Bull. Pol. Ac.: Tech.* 56 (3), 273–277 (2008).

[4] V.M. Marchenko and O.N. Poddubnaya, "Linear stationary differential-algebraic systems: I. solutions representation", *J. Computer and Systems Sciences Int.* 5, 699–713 (2006).

[5] V.M. Marchenko and O.N. Poddubnaya, "Relative controllability of stationary hybrid systems", *Proc. IEEE Methods and Models in Automation and Robotics* 1, 267–272 (2004).

[6] R. Bellman and K.L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.

[7] J.K. Hale and Verduyn S.M. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[8] V.M. Marchenko, O.N. Poddubnaya, and Z. Zaczkiewicz, "On the observability of linear differential-algebraic systems with delays", *IEEE Trans. Automat. Control.* 51 (8), 1387–1392 (2006).