BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 60, No. 1, 2012
DOI: 10.2478/v10175-012-0024-7

Generalized feedback stability for periodic linear time-varying, discrete-time systems

P. ORŁOWSKI*

Department of Control and Measurements, West Pomeranian University of Technology, 37 Sikorskiego St., 70-313 Szczecin, Poland

Abstract. The paper proposes a novel method for feedback stability evaluation for linear time-varying, discrete-time control systems. It is assumed that the time-varying system can be described by the general discrete-time, time-varying state space model and by the equivalent linear input-output (transfer) operator. The method extends feedback stability concepts for systems given in a general linear time-varying, discrete-time form, not only in the Lurie form. In the paper selected short-time stability concepts are employed for inference about feedback stability of systems defined on an infinite time-horizon. Theoretical considerations are complemented by numerical examples.

Key words: discrete-time systems, time-varying systems, non-stationary systems, feedback stability, frequency analysis.

1. Introduction

Many control design methods for linear time-invariant systems take advantage of closed-loop stability. The time-invariant character of the system is required for the frequency domain transformations (Laplace, Fourier and $\mathcal{Z}$ transform). The classical approach for discrete-time systems is based on the transformation to the $\mathcal{Z}$ domain. Until now there have existed some extensions for nonlinear [1–3] and time-varying systems feedback stability analysis. The main approach is based either on Lyapunov methods [4–6], or on extension of the Nyquist theory proposed by Popov [7–9] Yakubovich and Kalman [10–12]. These methods are applicable for a class of time-varying systems called Lurie systems, see Fig. 1.

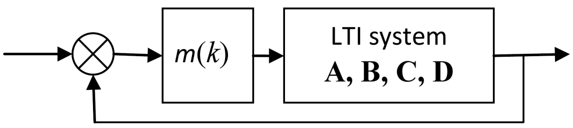

Fig. 1. Feedback control loop for Lurie system

The Lurie system can be described by a cascade of a static element, including nonlinear and time-varying connected with linear time-invariant dynamical system [4–16]. The main limitation of the method in respect to applications for time-varying systems analysis is assumed, by analogy to the Hammerstein model form of the system, e.g.: static time-varying element connected with dynamic, linear time invariant part.

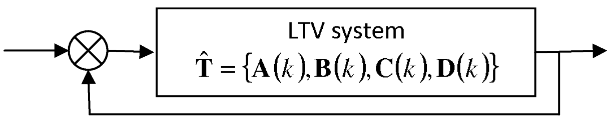

Fig. 2. Feedback control loop for general dynamical, time-varying system $\hat{\mathbf{T}}$ (gain m accounted in input matrix $\mathbf{B}(k)$)

The main aim of the paper is to formulate a theorem and propose rules applicable for the feedback stability analysis for linear time-varying, discrete-time systems.

The feedback system under consideration is a time-varying system with feedback loop shown in Fig. 2. The linear time-varying approach is of relevant interest in an adaptive control, multi-model design with improved transient performances and switching operations in piecewise affine systems. In order to describe the dynamics of time-varying discrete-time systems, one can use difference equations with time-dependent coefficients or a generalized description employing state equations with time-dependent matrices in following form:

$$\mathbf{x}_p(k+1) = \mathbf{A}(k)\mathbf{x}_p(k) + \mathbf{B}(k)\mathbf{v}_p(k), \tag{1}$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{v}(k), \quad \mathbf{x}(k_0) = \mathbf{x}_0, \tag{2}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is nominal state, $\mathbf{v}(k) \in \mathbb{R}^m$ is the nominal control, $\mathbf{y}(k) \in \mathbb{R}^p$ is the nominal output and $\mathbf{A}(k) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(k) \in \mathbb{R}^{n\times m}$, $\mathbf{C}(k) \in \mathbb{R}^{p\times n}$, $\mathbf{D}(k) \in \mathbb{R}^{p\times m}$ are system matrices, $k \in \{k_0, ..., k_0+N-1\}$ where $k_0 \in \mathbb{Z}$ and N is length of the time horizon. For infinite time horizon $N = \infty$. For single input single output systems $m = 1$, $p = 1$.

Stability of an linear time-varying (LTV) system without feedback was considered in e.g. [20–24]. The problems are slightly different from those for LTI systems. It is well-known that unforced piecewise constant linear systems, which associated matrix of dynamics takes values in a set of strictly Hurwitzian matrices, are not guaranteed to be exponentially stable [6, 25–27]. Instability can occur when an infinite number of switches between elements of that set are performed. A surprising result is that time-varying systems with constant and strictly stable eigenvalues may be unstable if the parameters of the dynamics matrix do not vary at a sufficiently small slope [27, 28].

*e-mail: orzel@zut.edu.pl

2. Operators notation

In order to describe the dynamics of time-varying discrete-time systems, one can employ state space equations with time-dependent matrices given by Eqs. (1)–(2). Alternatively, the model may be described by means of operators. Equations (1)–(2) can be converted into following operators form:

$$\widehat{\mathbf{y}} = \widehat{\mathbf{C}}\widehat{\mathbf{N}}\mathbf{x}_0 + \left(\widehat{\mathbf{C}}\widehat{\mathbf{L}}\widehat{\mathbf{B}} + \widehat{\mathbf{D}}\right)\widehat{\mathbf{v}} = \widehat{\mathbf{C}}\widehat{\mathbf{N}}\mathbf{x}_0 + \widehat{\mathbf{T}}\widehat{\mathbf{v}}. \quad (3)$$

In order to make the system (3) equivalent to the system (1)–(2), operators $\widehat{\mathbf{T}} = \widehat{\mathbf{C}}\widehat{\mathbf{L}}\widehat{\mathbf{B}} + \widehat{\mathbf{D}}$ and $\widehat{\mathbf{C}}\widehat{\mathbf{N}}$ must be defined in the following way:

$$\widehat{\mathbf{L}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \phi_{k_0+1}^{k_0+1} & \mathbf{I} & \mathbf{0} & \vdots \\ \vdots & \ddots & \mathbf{I} & \mathbf{0} \\ \phi_{k_0+1}^{k_0+N-1} & \cdots & \phi_{k_0+N-1}^{k_0+N-1} & \mathbf{I} \end{bmatrix}, \quad (4)$$

$$\widehat{\mathbf{N}} = \begin{bmatrix} \phi_{k_0}^{k_0} \\ \vdots \\ \phi_{k_0}^{k_0+N-1} \end{bmatrix}, \quad (5)$$

$$\widehat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(k_0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}(k_0+N-1) \end{bmatrix}, \quad (6)$$

$$\widehat{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(k_0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}(k_0+N-1) \end{bmatrix}, \quad (7)$$

where $\phi_i^k = \mathbf{A}(k)\,\mathbf{A}(k-1)\ldots\mathbf{A}(i)$. In order to analyze the stability of the system, one has to know operators $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ which can be expressed with the help of the above operators. Operator $\widehat{\mathbf{N}}$ can be neglected when initial conditions are zero.

Operators of state $\widehat{\mathbf{x}}$, output $\widehat{\mathbf{y}}$ and input $\widehat{\mathbf{v}}$ have the following notations:

$$\widehat{\mathbf{x}} = \left[\mathbf{x}^T(k_0+1)\cdots\mathbf{x}^T(k_0+N)\right]^T, \quad (8)$$

$$\widehat{\mathbf{y}} = \left[\mathbf{y}^T(k_0+1)\cdots\mathbf{y}^T(k_0+N)\right]^T, \quad (9)$$

$$\widehat{\mathbf{v}} = \left[\mathbf{v}^T(k_0+1)\cdots\mathbf{v}^T(k_0+N)\right]^T. \quad (10)$$

For single input single output systems, considered in the paper input and output vectors $\mathbf{v}(k)$ and $\mathbf{y}(k)$ respectively, in each time instant k are in fact scalars. In such case input $\widehat{\mathbf{v}}$ and output $\widehat{\mathbf{y}}$ operators have the same dimensions and the transfer operator $\widehat{\mathbf{T}}$ defined by Eqs. (1)–(3) is described by a square matrix. For systems defined on finite time horizon all operators are represented by finite dimensional matrices and signals by finite dimensional vectors. Moreover, the input-output operator is a compact, Hilbert-Schmidt operator from l_2 into l_2 and actually maps bounded signals $v \in \mathcal{M} = l_2[k_0, k_0+N]$ into the signals $y \in \mathcal{P}$.

The norm of a sequence in the Hilbert-space is understood as Euclidean norm:

$$\|\widehat{\mathbf{v}}\| = \|\widehat{\mathbf{v}}\|_2 = \sqrt{\langle\widehat{\mathbf{v}},\widehat{\mathbf{v}}\rangle} = \sum_k \mathbf{v}^T(k)\,\mathbf{v}(k) = \widehat{\mathbf{v}}^T\widehat{\mathbf{v}}. \quad (11)$$

The ∞-norm of a sequence in the bounded sequences space is understood as:

$$\|\widehat{\mathbf{v}}\|_\infty = \max_k(|v(k)|). \quad (12)$$

Norms of operators are defined in the following way:

$$\left\|\widehat{\mathbf{T}}\right\| = \left\|\widehat{\mathbf{T}}\right\|_2 = \sup_{\widehat{\mathbf{v}}\neq 0}\frac{\left\|\widehat{\mathbf{T}}\widehat{\mathbf{v}}\right\|_2}{\|\widehat{\mathbf{v}}\|_2}, \quad (13)$$

$$\left\|\widehat{\mathbf{T}}\right\|_\infty = \sup_{\widehat{\mathbf{v}}\neq 0}\frac{\left\|\widehat{\mathbf{T}}\widehat{\mathbf{v}}\right\|_\infty}{\|\widehat{\mathbf{v}}\|_\infty}. \quad (14)$$

The input/output operator $\widehat{\mathbf{T}}$ can be alternatively defined also using a set of impulse responses of a time-varying system taken at different times, e.g. for SISO system it may be written:

$$\widehat{\mathbf{T}} = \begin{bmatrix} h(k_0,k_0) & 0 & \cdots & 0 & 0 \\ h(k_0+1,k_0) & h(k_0+1,k_0+1) & \cdots & \vdots & \vdots \\ h(k_0+2,k_0) & h(k_0+2,k_0+1) & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & h(k_0+N-2,k_0+N-2) & 0 \\ h(k_0+N-1,k_0) & \cdots & \cdots & h(k_0+N-1,k_0+N-2) & h(k_0+N-1,k_0+N-1) \end{bmatrix} \quad (15)$$

Table 1
Fundamental operations on time-varying discrete-time systems described using operator notation and state-space description

	Transfer operator	Linear time-varying state-space model
Equivalence of realizations $\widehat{\mathbf{T}}, \widehat{\mathbf{T}'}$	$\widehat{\mathbf{T}} = \widehat{\mathbf{T}'}$ $\mathbf{T}(i,j) = \mathbf{T}'(i,j)$	$\underset{i,j}{\forall} \begin{bmatrix} \mathbf{C}(i)\mathbf{A}(i-1)\cdots\mathbf{A}(j+1)\mathbf{B}(j) = \\ = \mathbf{C}'(i)\mathbf{A}'(i-1)\cdots\mathbf{A}'(j+1)\mathbf{B}'(j) \\ \mathbf{D}(i) = \mathbf{D}'(i) \end{bmatrix}$
Sum of realizations $\widehat{\mathbf{T}}_1, \widehat{\mathbf{T}}_2$	$\widehat{\mathbf{T}} = \widehat{\mathbf{T}}_1 + \widehat{\mathbf{T}}_2$ $\widehat{\mathbf{T}}_1 = \mathcal{L}(\mathcal{M},\mathcal{P})$ $\widehat{\mathbf{T}}_2 = \mathcal{L}(\mathcal{M},\mathcal{P})$	$\mathbf{A}(k) = \begin{bmatrix} \mathbf{A}_1(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2(k) \end{bmatrix}, \quad \mathbf{B}(k) = \begin{bmatrix} \mathbf{B}_1(k) \\ \mathbf{B}_2(k) \end{bmatrix}$ $\mathbf{C}(k) = \begin{bmatrix} \mathbf{C}_1(k) & \mathbf{C}_2(k) \end{bmatrix}, \quad \mathbf{D}(k) = [\mathbf{D}_1(k) + \mathbf{D}_2(k)]$
Cascade of realizations $\widehat{\mathbf{T}}_1, \widehat{\mathbf{T}}_2$	$\widehat{\mathbf{T}} = \widehat{\mathbf{T}}_2\widehat{\mathbf{T}}_1$ $\widehat{\mathbf{T}}_1 = \mathcal{L}(\mathcal{M},\mathcal{P}_1)$ $\widehat{\mathbf{T}}_2 = \mathcal{L}(\mathcal{P}_1,\mathcal{P}_2)$	$\mathbf{A}(k) = \begin{bmatrix} \mathbf{A}_2(k) & \mathbf{B}_2(k)\mathbf{C}_1(k) \\ \mathbf{0} & \mathbf{A}_1(k) \end{bmatrix}, \quad \mathbf{B}(k) = \begin{bmatrix} \mathbf{B}_2(k)\mathbf{D}_1(k) \\ \mathbf{B}_1(k) \end{bmatrix}$ $\mathbf{C}(k) = \begin{bmatrix} \mathbf{C}_2(k) & \mathbf{D}_2(k)\mathbf{C}_1(k) \end{bmatrix}, \quad \mathbf{D}(k) = [\mathbf{D}_2(k)\mathbf{D}_1(k)]$
Inverse realization $\widehat{\mathbf{T}}^{-1}$	$\widehat{\mathbf{T}}^{-1} = \mathcal{L}(\mathcal{P},\mathcal{M})$ $\widehat{\mathbf{v}} = \widehat{\mathbf{T}}^{-1}\widehat{\mathbf{T}}\widehat{\mathbf{v}} \Leftrightarrow \widehat{\mathbf{T}}^{-1}\widehat{\mathbf{T}} = \widehat{\mathbf{I}}_{\mathcal{M}}$ $\widehat{\mathbf{y}} = \widehat{\mathbf{T}}\widehat{\mathbf{T}}^{-1}\widehat{\mathbf{y}} \Leftrightarrow \widehat{\mathbf{T}}\widehat{\mathbf{T}}^{-1} = \widehat{\mathbf{I}}_{\wp}$	$\mathbf{A}'(k) = \mathbf{A}(k) - \mathbf{B}(k)\mathbf{D}^{-1}(k)\mathbf{C}(k)$ $\mathbf{B}'(k) = \mathbf{B}(k)\mathbf{D}^{-1}(k)$ $\mathbf{C}'(k) = -\mathbf{D}^{-1}(k)\mathbf{C}(k), \ \mathbf{D}'(k) = \mathbf{D}^{-1}(k)$
Feedback connection of two realizations $\widehat{\mathbf{T}}_1$ (forward) and $\widehat{\mathbf{T}}_2$(backward)	$\widehat{\mathbf{T}} = \left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)^{-1}\widehat{\mathbf{T}}_1$ $\widehat{\mathbf{T}}_1 = \mathcal{L}(\mathcal{M},\mathcal{P})$ $\widehat{\mathbf{T}}_2 = \mathcal{L}(\mathcal{P},\mathcal{M})$	$\mathbf{A}(k) = \begin{bmatrix} \mathbf{A}_1(k) & \mathbf{0} \\ \mathbf{B}_2(k)\mathbf{C}_1(k) & \mathbf{A}_2(k) \end{bmatrix} +$ $\begin{bmatrix} \mathbf{B}_1(k) \\ \mathbf{B}_2(k)\mathbf{D}_1(k) \end{bmatrix} \mathbf{D}_{i21}(k) \begin{bmatrix} \mathbf{D}_2(k)\mathbf{C}_1(k) & \mathbf{C}_2(k) \end{bmatrix}$ $\mathbf{B}(k) = \begin{bmatrix} \mathbf{B}_1(k) \\ \mathbf{B}_2(k)\mathbf{D}_1(k) \end{bmatrix} \mathbf{D}_{i21}(k)$ $\mathbf{C}(k) = \mathbf{D}_1(k)\mathbf{D}_{i21}(k) \begin{bmatrix} \mathbf{D}_2(k)\mathbf{C}_1(k) & \mathbf{C}_2(k) \end{bmatrix}$ $\mathbf{D}(k) = \mathbf{D}_1(k)\mathbf{D}_{i21}(k)$ $\mathbf{D}_{i21}(k) = (\mathbf{I} - \mathbf{D}_2(k)\mathbf{D}_1(k))^{-1}$

where $h(k_1, k_0)$ is the response of the system to the Kronecker delta $\delta(k - k_0)$ at time k_1 (after $k_1 - k_0$ samples). In the case of a nonzero input-output delay operator, $\widehat{\mathbf{D}} = \mathbf{0}$ and all diagonal entries of $\widehat{\mathbf{T}}$ are equal to zero.

Employing operator's description of the system one can simplify a notation and easily express some operations on the system. Selected operations described using transfer operator notation with their state-space equivalents are given in Table 1.

3. Stability concepts on finite time horizon

Short-time stability is a stability concept for time-varying systems which differ from well known concepts defined for time-invariant systems. Quoting d'Angelo [35]: "*Short-time stability* deals with determining whether a system response lies within *specified* bounds over *specified* intervals of time when the inputs (initial conditions) are within specified bounds (...) Such concept can be of greater practical use since in many systems, the approximate operational time interval of interest is often known in advance and, due to physical limitations, inputs and outputs greater than some finite values are essentially *unbounded*". Definitions and detailed description of the Short-Time Stability or Finite-Time Stability and related concepts: Short-Time Nonresonance, Short-Time Boundedness one can find in many papers and books related to time-varying systems e.g. [35–45].

Short-Time Stability defined in [35] corresponds to classical internal stability (non-driven systems), whereas Short-Time Nonresonance corresponds to classical input-output stability (driven systems). The strongest concept is Short-Time Boundedness containing both Short-Time Nonresonance and Short-Time Stability. The definitions below are and modified versions of Short-Time Stability and Nonresonance from [35].

Definition 1. The linear time-varying system defined on the sample interval $k \in \{k_0, ..., k_0 + N - 1\}$ with finite N, characterized by the following operator equation:

$$\widehat{\mathbf{x}} = \widehat{\mathbf{N}}\mathbf{x}_0 \tag{16}$$

is said to be internally short-time stable with respect to ε_0, N, $C_{\widehat{\mathbf{x}}}$ if

$$\|\mathbf{x}_0\|_\infty \leq \varepsilon_0 \tag{17}$$

implies that

$$\|\widehat{\mathbf{x}}\|_\infty \leq C_{\widehat{\mathbf{x}}} \tag{18}$$

on the interval $[k_0, k_0 + N - 1]$.

Theorem 1. The linear time-varying system described by the transfer operator $\widehat{\mathbf{T}}$ is internally short-time stable on the interval $[k_0, k_0 + N - 1]$ with respect to ε_0, N, $C_{\widehat{\mathbf{x}}}$ if the norm of the natural response operator satisfies the following condition

$$\left\|\widehat{\mathbf{N}}\right\|_\infty \leq \frac{C_{\widehat{\mathbf{x}}}}{\varepsilon_0}. \tag{19}$$

Proof.

Taking both side norms from Eq. (16) and applying triangle inequality for norms we have

$$\|\widehat{\mathbf{x}}\|_\infty \leq \left\|\widehat{\mathbf{N}}\right\|_\infty \|\mathbf{x}_0\|_\infty \leq \left\|\widehat{\mathbf{N}}\right\|_\infty \varepsilon_0.$$

Norm of state trajectory $\|\widehat{\mathbf{x}}\|_\infty$ is less then constant $C_{\widehat{\mathbf{x}}}$ if the following condition is satisfied

$$\left\|\widehat{\mathbf{N}}\right\|_{\infty}\varepsilon_0 \le C_{\widehat{\mathbf{x}}}.$$

This finishes the proof.

Definition 2. The linear time-varying system defined on the sample interval $k \in \{0, ..., N-1\}$ with finite N, characterized by the following operator equation:

$$\widehat{\mathbf{y}} = \widehat{\mathbf{T}}\widehat{\mathbf{v}} \tag{20}$$

is said to be input-output short-time stable with respect to $\varepsilon_{\widehat{\mathbf{v}}}$, N, $C_{\widehat{\mathbf{y}}}$ if

$$\|\widehat{\mathbf{v}}\|_{\infty} \le \varepsilon_{\widehat{\mathbf{v}}} \tag{21}$$

implies that

$$\|\widehat{\mathbf{y}}\|_{\infty} \le C_{\widehat{\mathbf{y}}} \tag{22}$$

on the interval $[k_0, k_0+N-1]$.

Theorem 2. The linear time-varying system described by transfer operator $\widehat{\mathbf{T}}$ is input-output short-time stable on the interval $[k_0, k_0+N-1]$ with respect to $\varepsilon_{\widehat{\mathbf{v}}}$, N, $C_{\widehat{\mathbf{y}}}$ if the norm of the transfer operator satisfy following condition

$$\left\|\widehat{\mathbf{T}}\right\|_{\infty} \le \frac{C_{\widehat{\mathbf{y}}}}{\varepsilon_{\widehat{\mathbf{v}}}}. \tag{23}$$

Proof.

Taking both side norms from Eq. (20) and applying triangle inequality for norms we have

$$\|\widehat{\mathbf{y}}\|_{\infty} \le \left\|\widehat{\mathbf{T}}\right\|_{\infty}\|\widehat{\mathbf{v}}\|_{\infty} \le \left\|\widehat{\mathbf{T}}\right\|_{\infty}\varepsilon_{\widehat{\mathbf{v}}}.$$

Norm of the output trajectory $\|\widehat{\mathbf{y}}\|_{\infty}$ is less then constant $C_{\widehat{\mathbf{y}}}$ if the following condition is satisfied

$$\left\|\widehat{\mathbf{T}}\right\|_{\infty}\varepsilon_{\widehat{\mathbf{v}}} \le C_{\widehat{\mathbf{y}}}.$$

This finishes the proof.

The input-output short-time stability of the system can be judged from a norm of the finite dimensional transfer operator of the system whereas internal short-time stability can be judged from a norm of the natural response operator $\widehat{\mathbf{N}}$.

4. Extension to stability on infinite time horizon

As the system transfer operator $\widehat{\mathbf{T}}$ simply define relation between input and output $\widehat{\mathbf{y}} = \widehat{\mathbf{T}}\widehat{\mathbf{v}}$ some important system properties i.e. stability follows directly from fundamental properties of the operator $\widehat{\mathbf{T}}$, for example stability.

This system is called *internally stable* if the evolution family φ_i^k is exponentially stable, that is, $\left\|\varphi_i^k\right\| \le Me^{-\beta(k-i)}$ for some constants $\beta > 0$, $M > 0$ and all $k \ge i$.

The system (1)–(2) is *input-output stable* if $\widehat{\mathbf{T}}$ is a bounded operator from $\mathcal{M}$ to $\mathcal{P}$.

Corollary 1. If the norm $\left\|\widehat{\mathbf{T}}\right\|$ of operator $\widehat{\mathbf{T}}$ defined on infinite time horizon is finite it can be said that the system described by transfer operator $\widehat{\mathbf{T}}$ is input-output stable. Infinite norm of the transfer operator implies the system is input-output unstable.

Practical applicability of above result depends whether the norm of transfer operator defined on infinite time horizon can be computed or estimated. There are exists some practical methods applicable for some classes of systems.

Norm of transfer operator defined on infinite time horizon can be also computed for periodic linear time-varying systems employing lifting technique. The paper [32] is an overview and comparison of techniques which allows to rewrite time-varying systems using time-invariant representation with increased but finite dimensions. Norm of the transfer operator for such system can be computed in similar way as for linear time-invariant systems. More description for the lifting technique for periodic time-varying systems can be found in [30–34, 37].

It has been shown in [29] that a computationally efficient system norm estimate can be obtained using the running finite-time horizon approach. In particular: if the norm of a transfer operator defined on infinite time horizon is finite $\left\|\widehat{\mathbf{T}}\right\| = c$ then there exists a limit c such that

$$\lim_{N\to\infty}\left\|\widehat{\mathbf{T}}_{[N]}\right\| = c \tag{24}$$

additionally

$$\underset{N\in Z}{\forall}\ \left\|\widehat{\mathbf{T}}_{[N-1]}\right\| \le \left\|\widehat{\mathbf{T}}_{[N]}\right\|. \tag{25}$$

For large enough lengths of the time horizon finite time horizon norm is an approximation of the infinite time horizon norm, i.e.:

$$\underset{N\ge N_0}{\forall}\left\|\widehat{\mathbf{T}}_{[N]}\right\| \cong \left\|\widehat{\mathbf{T}}\right\|. \tag{26}$$

It was stated in [29] that minimal length of the time horizon required for computations is dependent both on the dominant time constant of the system and the variability period of the system matrices. The method based on finite time horizon estimation can be easily applied for systems with fractional values of the switching parameter ε, i.e. for systems with fractional variability periods and other linear time-varying systems. i.e. almost periodic and almost constant systems.

5. Feedback stability

One of the most important tasks in the analysis and synthesis of control systems is stability testing of the feedback system. The stability test can return two results only. The results classify systems into two classes stable and unstable. Sometimes instead of such dichotomic classification one prefer to use some continuous measure. Classical frequency methods applicable for linear time-invariant systems allows to compute additional stability measures called as stability margins (gain margin and phase margin). Such measures are successfully employed in analysis and simplified synthesis of control systems. For linear time-invariant systems stability margins are defined and determined using frequency diagrams: Nyquist or Bode.

Generalisation of the concept of stability margin and method for feedback stability testing for linear time-varying systems can be done using introduced operators notation.

Theorem 3. Let an open loop linear time-varying system is described by transfer operator $\widehat{\mathbf{T}}$. The feedback system from Fig. 2 is stable if the norm of open loop transfer operator satisfies the condition

$$\left\|\widehat{\mathbf{T}}\right\| < 1. \tag{27}$$

Proof.

According to Table 1 feedback system depicted in Fig. 2 with $\widehat{\mathbf{T}}_1 = \widehat{\mathbf{T}}$, $\widehat{\mathbf{T}}_2 = \widehat{\mathbf{I}}$ can be described by equivalent feedback operator:

$$\widehat{\mathbf{T}}_F = \left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}\right)^{-1} \widehat{\mathbf{T}}.$$

Taking account Corollary 1 the feedback system is stable if and only if the norm $\left\|\widehat{\mathbf{T}}_F\right\|$ of equivalent feedback transfer operator is finite.

Let us recall the condition stated in the Theorem 2

$$\left\|\widehat{\mathbf{T}}\right\| < 1.$$

Then following inequality holds

$$\left\|\widehat{\mathbf{I}} - \widehat{\mathbf{T}}\right\| > 0.$$

The term $\left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}\right)$ is invertible with finite norm $\left\|\left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}\right)^{-1}\right\|$. Then the norm $\left\|\widehat{\mathbf{T}}_F\right\|$ is finite and bounded by

$$\left\|\widehat{\mathbf{T}}_F\right\| \leq \left\|\left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}\right)^{-1}\right\| \left\|\widehat{\mathbf{T}}\right\|.$$

What proves stability of the feedback system.

Consequently similar stability condition can be proved system connected with two transfer operators $\widehat{\mathbf{T}}_1$, $\widehat{\mathbf{T}}_2$. According to Table 1 equivalent feedback operator is given by $\widehat{\mathbf{T}}_F = \left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)^{-1} \widehat{\mathbf{T}}_1$.

Theorem 4. Let an open loop linear time-varying system is described by two transfer operators $\widehat{\mathbf{T}}_1, \widehat{\mathbf{T}}_2$. The feedback system from Fig. 3 described by transfer operator:

$$\widehat{\mathbf{T}}_F = \left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)^{-1} \widehat{\mathbf{T}}_1 \tag{28}$$

is stable if the norms satisfy following conditions

$$\left\|\widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right\| < 1, \tag{29}$$

$$\left\|\widehat{\mathbf{T}}_1\right\| \quad \text{is finite.} \tag{30}$$

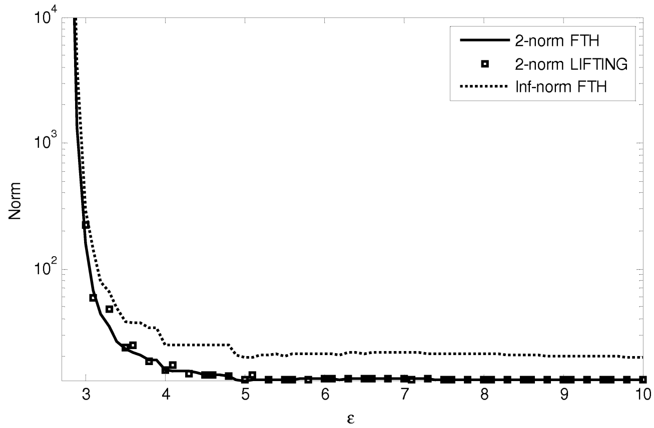

Fig. 3. Estimated norm of the LTV switching system without feedback loop vs. parameter ε

Proof.

According to Table 1 feedback system depicted in Fig. 3 can be described by equivalent feedback operator:

$$\widehat{\mathbf{T}}_F = \left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)^{-1} \widehat{\mathbf{T}}_1.$$

Taking account Corollary 1 the feedback system is stable if and only if the norm $\left\|\widehat{\mathbf{T}}_F\right\|$ of equivalent feedback transfer operator is finite.

Let us recall the condition stated in the Theorem 2

$$\left\|\widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right\| < 1.$$

Then following inequality holds

$$\left\|\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right\| > 0.$$

The term $\left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)$ is invertible with finite norm $\left\|\left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)^{-1}\right\|$. Then the norm $\left\|\widehat{\mathbf{T}}_F\right\|$ is finite and bounded by

$$\left\|\widehat{\mathbf{T}}_F\right\| \leq \left\|\left(\widehat{\mathbf{I}} - \widehat{\mathbf{T}}_1\widehat{\mathbf{T}}_2\right)^{-1}\right\| \left\|\widehat{\mathbf{T}}_1\right\|.$$

If the norm $\left\|\widehat{\mathbf{T}}_1\right\|$ is finite. What proves stability of the feedback system.

Inequalities (27) and (28) are sufficient conditions for stability of linear time-varying feedback systems.

Having defined conditions for feedback stability for time-varying systems we can introduce a new stability measure called norm margin related to concept of stability margin (gain margin for linear time-invariant systems).

Definition 3. The open-loop linear time-varying system is described by the transfer operator $\widehat{\mathbf{T}}$. The norm margin is defined by the following relation:

$$N_m = 20 \log \frac{1}{\left\|\widehat{\mathbf{T}}\right\|}. \tag{31}$$

The norm margin is given in decibels, where positive values represents expected distance to border stability of the feedback system.

The measure is defined using only sufficient condition for stability. Real distance to the border of stability for the feedback system can be equal or greater than the expected distance.

6. Numerical examples

The considered system is a special case of the LTV system whereas $\mathbf{A}(k)$ is the time-varying system matrix with invariant eigenvalues. The system is characterized by constant (time-invariant) eigenvalues of the system matrix despite changes in its entries. This idea is borrowed from [27, 28]. The additional parameter ε allows changes of the system with a degree of non-stationarity as well as the pole location. Eigenvalues of matrix $\mathbf{A}(k)$ are inside the unitary circle, but can be either stable or unstable with respect to switching in the structure of the system. The important property of the system is defined by the switching interval – parameter ε. System matrices (1)–(2) are the following:

$$\begin{aligned} \mathbf{A}(k) &= \mathbf{A}_\kappa, \qquad \mathbf{B}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}, \\ \mathbf{C}(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad \mathbf{D}(k) = 0, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \mathbf{A}_0 &= \begin{bmatrix} 2 & 1.2 \\ -2 & -1 \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} -1 & -2 \\ 1.2 & 2 \end{bmatrix}, \\ \mathbf{A}_2 &= \begin{bmatrix} -1 & 1.2 \\ -2 & 2 \end{bmatrix}, \qquad \mathbf{A}_3 = \begin{bmatrix} 2 & -2 \\ 1.2 & -1 \end{bmatrix}, \\ \kappa &= \mathrm{floor}\left(\mathrm{rem}\left(\frac{k}{\varepsilon}, 4\right)\right). \end{aligned} \tag{33}$$

The variable κ denotes rounding towards negative infinity (floor) of the remanent (signed remainder of k/ε after division by 4). Sampling periods are equal to $T_p = 0.04$. Eigenvalues of the matrix $\mathbf{A}(k)$ are independent of the parameter ε and equal to $0.5 \pm 0.3873\mathrm{i}$ for all k.

Value of the parameter ε have significant impact on the properties of the system. Small values $\varepsilon < 2.8$ implies unstable character of the system whereas large values $\varepsilon \geq 3$ results in stable, switching system. For $2.8 \leq \varepsilon < 3$ the system is close to the boundary of stability.

Figure 3 shows values of the estimated transfer operator 2-norm and inf-norm using norms computed on finite time horizon $\left\|\widehat{\mathbf{T}}_{[N]}\right\|_2$ and $\left\|\widehat{\mathbf{T}}_{[N]}\right\|_\infty$ vs. parameter ε. The length of the time horizon is chosen to satisfy following error condition: $\left|\left\|\widehat{\mathbf{T}}_{[N-10]}\right\| \Big/ \left\|\widehat{\mathbf{T}}_{[N]}\right\| - 1\right| \leq 0.02$. The condition is satisfied for $\varepsilon \geq 3$. Values of 2-norm estimated using lifting techniques are depicted by squares whereas values of 2-norm estimated using finite horizon methods are depicted by solid line. Dotted line depicts values estimated using finite time horizon techniques of the ∞-norm of the system transfer operator.

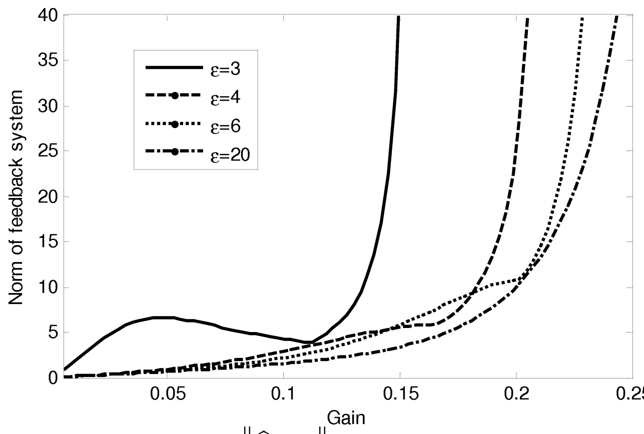

Fig. 4. Estimated norm $\left\|\widehat{\mathbf{T}}_{[200]}\right\|_2$ of the feedback LTV switching system vs. controller gain for parameter $\varepsilon = 3, 4, 6, 20$

In order to examine the influence of parameter ε on feedback system stability, an analysis has been carried out for 4 different values of ε equal to 3, 4, 6, and 20. Figure 4 shows norm of the LTV feedback control system vs. proportional controller gain. Corresponding norms $\left\|\widehat{\mathbf{T}}_{[200]}\right\|_2$ of LTV system, their reciprocals and values of norm margin are collected in Table 2. The reciprocal of the system norm $1\Big/\left\|\widehat{\mathbf{T}}\right\|_2$ represents supremum of the guaranteed stable controller gain for feedback loop system. For time-invariant systems controller gain equal to $1\Big/\left\|\widehat{\mathbf{T}}\right\|_2$ corresponds to boundary stable system, whereas for time-varying system the reciprocal of the system norm is only estimate where the conservatism depends mostly on the degree of time-variability [36] of the system. Relating Table 2 to Fig. 4 it can be concluded that small values of ε i.e. $\varepsilon = 3$ corresponds to large conservatism – real controller gain for boundary stable feedback system is equal to 0.153, about 24 times higher, while large values of ε i.e. $\varepsilon = 20$ corresponds to much more lower conservatism of the estimates – real controller gain for boundary stable feedback system is equal to 0.284 about 4 times higher. Finite time horizon system norm $\left\|\widehat{\mathbf{T}}_{[200]}\right\|_2$ for $\varepsilon = 2$ is assumed to be infinite since its value is larger than largest floating point number in Matlab.

Table 2
Norms of the LTV switching system and their reciprocals for different values of parameter ε

ε	2	3	4	6	20
$\left\|\widehat{\mathbf{T}}_{[200]}\right\|_2$	$\approx \infty (> 2^{1022})$	153.8	15.67	13.35	13.05
$1\Big/\left\|\widehat{\mathbf{T}}_{[200]}\right\|_2$	$\approx 0 (< 2^{-1022})$	0.0065	0.0638	0.0749	0.0766
Nm	–	−43.7 dB	−23.9 dB	−22.5 dB	−22.3 dB

7. Conclusions

The main achievement of the paper is the generalization of the feedback stability concept for linear time-varying discrete-time systems. The proposed method is applicable to a general class of linear periodic, almost periodic, almost constant time-varying systems.

A minimal length of the time horizon required for computations is dependent both on the dominant time constant of the system and the variability period of the system matrices. The method based on finite time horizon estimation can be easily applied for systems with fractional values of the switching parameter ε, i.e. for systems with fractional variability periods and other linear time-varying systems. i.e. almost periodic and almost constant systems.

The norm margin is only sufficient condition for stability. Real distance to the border of stability for the feedback system can be equal or greater than the expected distance.

REFERENCES

[1] R.W. Brockett and H.B. Lee, "Frequency domain instability criteria for time-varying and nonlinear systems", *Proc. IEEE* 55, 604–619 (1967).

[2] R.W. Brockett and J.W. Willems, " Frequency domain stability criteria", *IEEE Trans. Autom. Contr.* 11, 255–261, 407–413 (1966).

[3] H. Kwon and AE. Pearson, "On feedback stabilization of time-varying discrete linear systems", *IEEE Trans. Aut. Contr.* 23 (3), 479–481 (1966).

[4] R.E. Kalman, "Lyapunov functions for the problem of Lurie in automatic control", *Proc. Nat. Acad. Sci.* 49, 201–205 (1963).

[5] A. Polański, "Chosen problems of linear systems' stability with variable in time parameters", *Scientific Note Books of the Silesian University of Technology: Automatics* 128, CD-ROM (2000), (in Polish).

[6] R.N. Shorten and K.S. Narendra, "On the stability and existence of common Lyapunov functions for stable linear switching systems", *Proc. 37th IEEE Conf. on Decision and Control* 1, 3723–3724 (1998).

[7] V.M. Popov, "Absolute stability of nonlinear systems of automatic control", *Automation and Remote Control* 22, 857–875 (1961).

[8] V.M. Popov, *Hyperstability of Control Systems*, Springer-Verlag, Berlin, 1973.

[9] V.M. Popov, "The solution of a new stability problem for controlled systems", *Automation and Remote Control* 24, 1–23 (1963).

[10] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma", *Syst. Contr.* 28, 7–10 (1966).

[11] V.A. Yakubovich, "A frequency theorem for the case in which the state and control spaces are Hilbert spaces, with an applications to some problems in the synthesis of optimal controls", *Siberian J. of Mathematics* 15, 457–476 (1974).

[12] V.A. Yakubovich, "The frequency theorem in control theory", *Siberian J. of Mathematics* 14, 384–419 (1973).

[13] P. Grabowski, *Stability of Lurie's Systems,* AGH Publishing House, Kraków, 1999, (in Polish).

[14] A. Halanay and V. Ionescu, "Generalized discrete-time Popov-Yakubovich theory", *Systems & Control Letters* 20 (1), 1–6 (1993).

[15] V. Ionescu and M. Weiss, "Continuous and discrete-time Riccati theory: a Popov function approach", *Linear Algebra and its Applications* 193, 173–209 (1993).

[16] D. Jonson, "A Popov criterion for systems with slowly time-varying parameters", *IEEE Trans. Aut. Contr.* 44, 844–846 (1999).

[17] P. Orlowski, "Methods for stability evaluation for linear time varying, discrete-time systems on finite time horizon", *Int. J. Control* 79 (3), 249–262 (2006).

[18] P. Orlowski, "An extension of Nyquist feedback stability for linear time-varying, discrete-time systems", *Int. J. Factory Automation, Robotics and Soft Computing* 2, 51–56 (2007).

[19] P. Orlowski, "Feedback stability of discrete-time, linear time-varying systems", *Proc. 16th Mediterranean Conf. on Control and Automation* 1, 1002–1007 (2008).

[20] K. Liu, "*Extension of modal analysis to linear time-varying systems", J. Sound and Vibration* 226, 149–167 (1999).

[21] S. Shokoohi and L. Silverman, "Linear time-variant systems: stability of reduced models", *Automatica* 20, 59–67 (1987).

[22] D.O. Anderson, "Internal and external stability of linear time varying systems", *SIAM J. Control and Optimization* 20, 408–413 (1982).

[23] P. Iglesias, "Input-output stability of sampled-data linear time-varying systems", *IEEE Trans. Aut. Contr.* 40 (9), 1647–1650 (1995).

[24] H.S. Han and J.G. Lee, "Necessary and sufficient conditions for stability of time-varying discrete interval matrices", *Int. J. Control* 59 (4), 1021–1029 (1994).

[25] D.E. Miller and E.J. Davison, "An adaptive controller which provides Lyapunov stability", *IEEE Trans. Automat. Control* 34, 599–609 (1989).

[26] K.S. Narendra and J. Balakrishnan, "Adaptive control using multiple models", *IEEE Trans. Automat. Control* 42, 171–187 (1989).

[27] M. De La Sen, "Robust stability of a class of linear time-varying systems", *IMA J. Mathematical Control and Information* 19, 399–418 (2002).

[28] H.K. Khalil, *Nonlinear Systems*, Prentice-Hall, New York, 1996.

[29] P. Orlowski, "Discrete-time, linear periodic time-varying system norm estimation using finite time horizon transfer operators", *Automatika* 51 (4), 325–332 (2010).

[30] R.A. Meyer and C.S. Burrus, "A unified analysis of multirate and periodically timevarying digital filters", *IEEE Trans. Circuits and Systems* 22, 162–168 (1975).

[31] D.S. Flamm, "A new shift-invariant representation of periodic linear systems", *Systems & Control Lett.* 17, 9–14 (1991).

[32] S. Bittanti and P. Colaneri, Invariant representations of discrete-time periodic systems, *Automatica* 36, 1777–1793 (2000).

[33] A. Varga, "Computation of transfer function matrices of generalized state-space models", *Int. J. Control* 50, 2543–2561 (1989).

[34] A.J. Laub, "Efficient multivariable frequency response computations", *IEEE Trans. Autom. Control* 26, 407–408 (1981).

[35] H. D'Angelo, *Linear Time-Varying Systems*, Allyn and Bacon, Boston, 1970.

[36] P. Orlowski, "Determining the degree of system variability for time-varying discrete-time systems", *Proc. 16th IFAC World Congress* 1, CD-ROM (2005).

[37] B. Bamieh, J.B. Pearson, B.A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applica-

tions to sampled-data control", *Systems & Control Letters* 17 (2), 79–88 (1991).

[38] F. Amato, M. Ariola, and P. Dorato, "Finite-time control of linear systems subject to parametric uncertainties and disturbances", *Automatica* 37 (9), 1459–1463 (2001).

[39] A. Davari and R.K. Ramanathaiah, "Short-time stability analysis of time-varying linear systems", *Proc. 26th Southeastern Symposium on System Theory* 1, 302–304 (1994).

[40] P. Dorato, "Short time stability in linear time-varying systems", *Proc. IRE Int. Convention Record* 4, 83–87 (1961).

[41] P. Dorato, C.T. Abdallah, and D. Famularo, "Robust finite-time stability design via linear matrix inequalities", *Proc. 36th IEEE Conf. on Decision and Control* 2, 1305–1306 (1997).

[42] P. Dorato, "An overview of finite-time stability", in: *Current Trends in Nonlinear Systems and Control* eds.: L. Menini, L. Zaccarian and C.T. Abdallah, pp. 185–194, Birkhäuser, Boston, 2006.

[43] S. Mastellone, P. Dorato, and C.T. Abdallah, "Finite-time stability for nonlinear networked control systems", in: *Current Trends in Nonlinear Systems and Control* eds.: L. Menini, L. Zaccarian and C.T. Abdallah, pp. 535–553, Birkhäuser, Boston, 2006.

[44] D.Y. Rew, M.J. Tahk, and H. Cho, "Short-time stability of proportional navigation guidance loop", *IEEE Trans. Aerospace and Electronic Systems* 32 (3), 1107–1115, (1996).

[45] L. Weiss and E.F. Infante, "Finite time stability under perturbing forces and on product spaces", *IEEE Trans. Autom. Contr.* 12, 54–59 (1967).