Positive realizations with reduced numbers of delays for 2D continuous-discrete linear systems

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Abstract. A new method is proposed for determination of positive realizations with reduced numbers of delays of linear 2D continuous-discrete systems. Sufficient conditions for the existence of the positive realizations of a given proper transfer function are established. It is shown that there exists a positive realization with reduced numbers of delays if there exists a positive realization without delays but with a greater dimension. The proposed method is demonstrated on a numerical example.

Key words: computation, linear, 2D continuous-discrete, delay, state-space realization, sufficient conditions.

1. Introduction

Determination of the state space equations for a given transfer matrix is a classical problem, called the realization problem, which has been addressed in many papers and books [1–12]. It is well-known that to find a realization for a given transfer function [1, 11, 13, 14] firstly we have to find a state matrix for a given denominator of the transfer function. An overview on the positive realization problem is given in [1, 11, 13, 15]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [5, 16–19] and the positive realization problem for discrete-time systems with delays in [9, 10, 20]. The fractional positive linear systems have been addressed in [11, 21, 22]. The realization problem for fractional linear systems has been analyzed in [7] and for positive 2D hybrid systems in [8]. A method based on similarity transformation of the standard realization to the discrete positive system has been proposed in [5]. Conditions for the existence of positive stable realization with system Metzler matrix for transfer function has been established in [17]. The problem of determination of the set of Metzler matrices for given stable polynomials has been formulated and partly solved in [18]. A new modified state variable diagram method for determination of positive realizations with the reduced number of delays for given proper transfer matrices of continuous-time linear systems has been proposed in [23]. An extension of this method for discrete-time linear systems is given in [24].

In this paper a new method for determination of positive realizations with reduced number of delays for given proper transfer matrices of 2D linear continuous-discrete systems is proposed.

The paper is organized as follows. In Sec. 2 some preliminaries concerning 2D positive continuous-discrete linear systems with delays are given and the realization problem is formulated. Basic lemmas and the proposed method for single-input single-output systems are presented in Sec. 3. An extension of the proposed method for multi-input multi-output is given in Sec. 4. Concluding remarks are given in Sec. 5.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n \times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n \times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with non-negative off-diagonal entries), $\Re^{n \times m}(s, z, w, z^{-1})$ – the set of $n \times m$ rational matrices in s, z, w and z^{-1}, $\Re^{n \times m}[s, z, w, z^{-1}]$ – the set of $n \times m$ polynomial matrices in s, z, w and z^{-1}, I_n – the $n \times n$ identity matrix.

2. Preliminaries and the problem formulation

Consider the 2D continuous-discrete linear system with $\overline{q}_1$ delays in a continuous variable t (time) and $\overline{q}_2$ delays in the discrete variable i of the form

$$\dot{x}(t, i+1) = A_0 x(t, i) + A_1 \dot{x}(t, i)$$
$$+ A_2 x(t, i+1) + B_0 u(t, i)$$
$$+ \sum_{j=1}^{\overline{q}_1} (A_{j,0} x(t - jd, i) + B_{j,0} u(t - jd, i)) \qquad (1a)$$
$$+ \sum_{k=1}^{\overline{q}_2} (A_{0,k} x(t, i - k) + B_{0,k} u(t, i - k)),$$

$$y(t, i) = C x(t, i) + D u(t, i), \quad t \in \Re_+ = [0, +\infty],$$
$$i \in Z_+ = \{0, 1, ...\}, \qquad (1b)$$

where $\dot{x}(t, i) = \frac{\partial x(t, i)}{\partial t}$, $x(t, i) \in \Re^n$, $u(t, i) \in \Re^m$, $y(t, i) \in \Re^p$ are the state, input and output vectors and $A_0, A_1, A_2 \in \Re^{n \times n}, B_0 \in \Re^{n \times m}$, $A_{j,0} \in \Re^{n \times n}, B_{j,0} \in \Re^{n \times m}$, $j = 0, 1, ..., \overline{q}_1$; $A_{0,k} \in \Re^{n \times n}$, $B_{0,k} \in \Re^{n \times m}$, $k = 0, 1, ..., \overline{q}_2$; $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$, $d > 0$ is a delay.

In a special case some matrices $A_{j,0}, B_{j,0}$ and $A_{0,k}, B_{0,k}$ in (1) can be zero matrices. Boundary conditions for (1) are given by

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$$x(0,i) = x_0(i), \quad i \in [-\overline{q}_2, 0]$$
$$\text{and} \quad x(t,0) = x_0(t), \quad \dot{x}(t,0) = \dot{x}_0(t), \quad (2)$$
$$t \in [-\overline{q}_1 d, 0].$$

Definition 1. The system (1) is called (internally) positive if for every $x_0(t)$, $\dot{x}_0(t) \in \Re_+^n$, $t \in [-\overline{q}_1 d, 0]$, $x_0(i) \in \Re_+^n$, $i \in [-\overline{q}_2, 0]$ and all inputs $u(t,i) \in \Re_+^m$, $t \geq -\overline{q}_1 d$, $i \geq -\overline{q}_2$ we have $x(t,i) \in \Re_+^n$ and $y(t,i) \in \Re_+^p$ for $t \geq 0 i \in Z_+$.

Theorem 1. The system (1) is positive if and only if

$$A_2 \in M_n, \quad A_0, A_1, A_{j,0}, A_{0,k} \in \Re_+^{n \times n},$$
$$A_0 + A_1 A_2 \in \Re_+^{n \times n}$$
$$B_0, B_{j,0}, B_{0,k} \in \Re_+^{n \times m}, \quad C \in \Re_+^{p \times n}, \quad D \in \Re_+^{p \times m} \quad (3)$$
$$j = 0, 1, ..., \overline{q}_1; \quad k = 0, 1, ..., \overline{q}_2.$$

Proof is the simple combination of the proof for positive 2D continuous-discrete linear systems without delays [6] and standard positive systems with delays [11, 13].

Using the Laplace transform and the Z transform to (1) it is easy to obtain the transfer matrix of the system (1) in the form

$$T(s, z, w, z^{-1}) = C \Bigg[I_n sz - A_0 - A_1 s - A_2 z$$
$$- \sum_{j=1}^{\overline{q}_1} A_{j,0} w^j - \sum_{k=1}^{\overline{q}_2} A_{0,k} z^{-k} \Bigg]^{-1} \quad (4)$$
$$\times [B_0 + \sum_{j=1}^{\overline{q}_1} B_{j,0} w^j + \sum_{k=1}^{\overline{q}_2} B_{0,k} z^{-k}] + D.$$

where $w = e^{-sd}$.

The transfer matrix $T(s, z, w, z^{-1})$ is called proper if

$$\lim_{s,z \to \infty} T(s, z, w, z^{-1}) = K \in \Re^{p \times m} \quad (5)$$

and strictly proper if $K = 0$.

Definition 2. Matrices (3) are called a positive realization of a given transfer matrix $T(s, z, w, z^{-1}) \in \Re^{p \times m}(s, z, w, z^{-1})$ if they satisfy the equality (4).

The positive realization problem under consideration can be stated as follows. Given a proper transfer matrix $T(s, z, w, z^{-1}) \in \Re^{p \times m}(s, z, w, z^{-1})$, find a positive realization with reduced numbers of delays, this is, with numbers of delays in continuous variable less than q_1 and with numbers of delays in discrete variable less than q_2.

In this paper sufficient conditions for the existence of positive realization with reduced numbers of delays will be established and a method for determination of a positive realization with a reduced number of delays of a given transfer matrix $T(s, z, w, z^{-1})$ is proposed.

3. Problem solution

The solution of the positive realization problem is based on the following two lemmas.

Lemma 1. Let $p_k = p_k(s, z, w, z^{-1})$ for $k = 1, 2, ..., 2n-1$ be some polynomials in s, z, w, z^{-1} with nonnegative coefficients and

$$P = P(s, z, w, z^{-1}) = \begin{bmatrix} 0 & 0 & ... & 0 & p_n \\ p_1 & 0 & ... & 0 & p_{n+1} \\ 0 & p_2 & ... & 0 & p_{n+2} \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & p_{n-1} & p_{2n-1} \end{bmatrix}. \quad (6)$$

Then

$$\det[I_n sz - P] = (sz)^n - p_{2n-1}(sz)^{n-1}$$
$$- p_{n-1} p_{2n-2}(sz)^{n-2} - ... - p_2 p_3 ... p_{n-1} p_{n+1}(sz) \quad (7)$$
$$- p_1 p_2 ... p_n.$$

Proof. The expansion of the determinant with respect to the n-th column yields

$$\det[I_n sz - P] = \begin{vmatrix} sz & 0 & ... & 0 & -p_n \\ -p_1 & sz & ... & 0 & -p_{n+1} \\ 0 & -p_2 & ... & 0 & -p_{n+2} \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & -p_{n-1} & sz - p_{2n-1} \end{vmatrix}$$
$$= (sz)^n - p_{2n-1}(sz)^{n-1} - p_{n-1} p_{2n-2}(sz)^{n-2} - ...$$
$$- p_2 p_3 ... p_{n-1} p_{n+1}(sz) - p_1 p_2 ... p_n.$$

Lemma 2. Let $R_n = R_n(s, z, w, z^{-1})$ be the n-th row of the adjoint matrix $[I_n z - P]_{ad}$. Then

$$R_n = [p_1 p_2 ... p_{n-1} \quad p_2 p_3 ... p_{n-1} sz \quad p_3 p_4$$
$$... p_{n-1}(sz)^2 \quad ... \quad p_{n-1}(sz)^{n-2} \quad (sz)^{n-1}]. \quad (8)$$

Proof. Using the well-known equality

$$[I_n sz - P]_{ad}[I_n sz - P] = I_n \det[I_n sz - P]$$

and (7) it is easy to verify that

$$R_n[I_n sz - P] = [0 \quad ... \quad 0 \quad 1] \det[I_n sz - P].$$

In a particular case if the matrix (6) has the form

$$P = \begin{bmatrix} 0 & 0 & ... & 0 & p_2 \\ p_1 & 0 & ... & 0 & p_3 \\ 0 & p_1 & ... & 0 & p_4 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & p_1 & p_{n+1} \end{bmatrix} \quad (9)$$

then

$$\det[I_n sz - P] = (sz)^n - p_{n+1}(sz)^{n-1} - ...$$
$$- p_2 p_1^{n-2}(sz) - p_2 p_1^{n-1} \quad (10)$$

and

$$R_n = [p_1^{n-1} \quad p_1^{n-2} sz \quad ... \quad p_1(sz)^{n-2} \quad (sz)^{n-1}]. \quad (11)$$

The given proper transfer matrix $T = T(s, z, w, z^{-1}) \in \Re^{p \times m}(s, z, w, z^{-1})$ can be always written in the form

$$T = \frac{N}{d} + D, \quad (12)$$

where $N = N(s, z, w, z^{-1}) \in \Re^{p \times m}[s, z, w, z^{-1}]$ and $d = d(s, z, w, z^{-1})$ is a polynomial. From (12) we have

$$D = \lim_{s,z \to \infty} T \qquad (13)$$

since $\lim_{s,z \to \infty} \frac{N}{d} = 0$. The strictly proper transfer matrix is given by

$$T_{sp} = T_{sp}(s, z, s, z^{-1}) = T - D = \frac{N}{d}. \qquad (14)$$

Therefore, the positive realization problem can be reduced to finding the matrices

$$A_2 \in M_n, \quad A_0, A_1, A_{j,0}, A_{0,k} \in \Re_+^{n \times n},$$
$$A_0 + A_1 A_2 \in \Re_+^{n \times n}$$
$$B_0, B_{j,0}, B_{0,k} \in \Re_+^{n \times m}, \quad C \in \Re_+^{p \times n}, \qquad (15)$$
$$j = 0, 1, ..., \overline{q}_1; \quad k = 0, 1, ..., \overline{q}_2.$$

Firstly we consider a single-input single-output ($m = p = 1$, SISO) system with the strictly proper transfer function

$$T_{sp} = T_{sp}(s, z, w, z^{-1}) = \frac{n}{d}, \qquad (16a)$$

where

$$n = n(s, z, w, z^{-1}) = b_{n-1}(sz)^{n-1} + ... + b_1(sz) + b_0, \quad (16b)$$
$$d = d(s, z, w, z^{-1}) = (sz)^n - a_{n-1}(sz)^{n-1} - ... - a_1(sz) - a_0 \qquad (16c)$$

a_k and b_k, $k = 0, 1, ..., n-1$ are polynomials with nonnegative coefficients in s, z, w, z^{-1}.

It is assumed that for the given denominator (18) there exist polynomials

$$p_k(s, z, w, z^{-1}) = p_k + p_k^0 s + p_k^1 z + p_{k,\overline{q}_1}^w w^{\overline{q}_1} + ...$$
$$+ p_{k,1}^w w + p_{k,1}^{z^{-1}} + p_{k,\overline{q}_2}^{z^{-1}} z^{-\overline{q}_2} + ... \qquad (17)$$
$$+ p_{k,1}^{z^{-1}} z^{-1} + p_{k,0}^{z^{-1}}, \quad (\overline{q}_1 \le q_1, \quad \overline{q}_2 \le q_2)$$

with nonnegative coefficients $p_k, p_k^0, ..., p_{k,0}^{z^{-1}}$ such that

$$a_{n-1} = p_{2n-1},$$
$$a_{n-2} = p_{n-1}p_{2n-2}, ..., a_1 = p_2 p_3 ... p_{n-1} p_{n+1}, \qquad (18)$$
$$a_0 = p_1 p_2 ... p_n.$$

In particular case if the matrix P has the form (9) then (18) takes the form

$$a_k = p_1^{n-k-1} p_{k+2}, \quad k = 0, 1, ..., n-1. \qquad (19)$$

Note that if (18) holds then for the given denominator d of (16a) we may find the matrix (6) and next the corresponding matrices A_0, A_1, A_2, $A_{j,0}$, $A_{0,k}$, $j = 0, 1, ..., \overline{q}_1$, $k = 0, 1, ..., \overline{q}_2$ since

$$[I_n sz - P] = I_n sz - A_0 - A_1 s - A_2 z$$
$$- \sum_{j=0}^{\overline{q}_1} A_{j,0} w^j - \sum_{k=0}^{\overline{q}_2} A_{0,k} z^{-k}. \qquad (20)$$

The matrix C is chosen in the form

$$C = [\, 0 \quad ... \quad 0 \quad 1 \,] \in \Re^{1 \times n}. \qquad (21)$$

Taking into account (8), (20), (21) and (4) we obtain

$$C[I_n sz - P]_{ad} \left[B_0 + \sum_{j=0}^{\overline{q}_1} B_{j,0} w^j + \sum_{k=0}^{\overline{q}_2} B_{0,k} z^{-k} \right]$$
$$= R_n \left[B_0 + \sum_{j=0}^{\overline{q}_1} B_{j,0} w^j + \sum_{k=0}^{\overline{q}_2} B_{0,k} z^{-k} \right]$$
$$= [p_1 p_2 ... p_{n-1} \quad p_2 p_3 ... p_{n-1} sz \quad p_3 p_4 ... p_{n-1}(sz)^2 \quad ... \qquad (22)$$
$$p_{n-1}(sz)^{n-2} \quad (sz)^{n-1}]$$
$$\times \left[B_0 + \sum_{j=0}^{\overline{q}_1} B_{j,0} w^j + \sum_{k=0}^{\overline{q}_2} B_{0,k} z^{-k} \right]$$
$$= n(s, z, w, z^{-1}).$$

From (22) it follows that it is possible to find the desired matrices $B_0, B_{j,0}, B_{0,k}$, $j = 0, 1, ..., \overline{q}_1; \quad k = 0, 1, ..., \overline{q}_2$ if there exists the matrix

$$\overline{B} = \overline{B}(w, z^{-1}) = \begin{bmatrix} \overline{b}_0 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix} = \begin{bmatrix} \overline{b}_0(w, z^{-1}) \\ \vdots \\ \overline{b}_{n-1}(w, z^{-1}) \end{bmatrix} \qquad (23)$$
$$= \left[B_0 + \sum_{j=0}^{\overline{q}_1} B_{j,0} w^j + \sum_{k=0}^{\overline{q}_2} B_{0,k} z^{-k} \right]$$

such that

$$R_n \overline{B} = \overline{b}_{n-1}(sz)^{n-1} + p_{n-1} \overline{b}_{n-2}(sz)^{n-2} + ...$$
$$+ p_2 p_3 ... p_{n-1} \overline{b}_1 (sz) + p_1 p_2 ... p_{n-1} \overline{b}_0$$
$$= b_{n-1}(sz)^{n-1} + ... + b_1(sz) + b_0 \qquad (24)$$
$$= n(s, z, w, z^{-1}).$$

Therefore, the following theorem has been proved.
Theorem 2. There exists the positive realization (15) of the transfer function (14) if it is possible to find the polynomials

$$p_1, p_2, ..., p_{2n-1} \qquad (25)$$

and

$$\overline{b}_{n-1}, ..., \overline{b}_1, \overline{b}_0 \qquad (26)$$

with nonnegative coefficients (except the last coefficient of p_{2n-1}) such that (20) and (24) are satisfied.

Remark 1. There exists a positive realization (3) of the transfer function (12) if the conditions of Theorem 2 are met and additionally

$$\lim_{s,z \to \infty} T(s, z, w, z^{-1}) \in \Re_+^{p \times m}. \qquad (27)$$

If the conditions of Theorem 2 and (27) are met then the positive realization (3) of the transfer function (12) can be computed by the use of the following procedure.

Procedure 1.
Step 1. Using (13) compute the matrix $D \in \Re_+$ and the strictly proper transfer function (14).
Step 2. For given coefficients a_k, $k = 0, 1, ..., n-1$ of the polynomial d using (18) choose polynomials $p_1, p_2, ..., p_{2n-1}$

with nonnegative coefficients and find the matrix P and nonnegative matrices A_0, A_1, $A_{j,0}$, $A_{0,k}$, $j = 0, 1, ..., \overline{q}_1$, $k = 0, 1, ..., \overline{q}_2$, $A_2 \in M_n$ satisfying (20) and $A_0 + A_1 A_2 \in \Re_+^{n \times n}$.

Step 3. Chose the polynomials $\overline{b}_k k = 0, 1, ..., n - 1$ satisfying (24) and find nonnegative matrices B_0, $B_{j,0}$, $B_{0,k}$, $j = 0, 1, ..., \overline{q}_1$, $k = 0, 1, ..., \overline{q}_2$ and the matrix C defined by (21).

Example 1. Compute the positive realization (3) of the transfer function

$$T(s, z, w, z^{-1}) = \frac{X}{Y} \qquad (28)$$

where

$$X = sz(1 + z^{-1}) + s(w + 1) + z(w + 1) + w(w + 1)$$
$$+ z^{-1}(z^{-1} + 1) + z^{-1} + 1,$$

$$Y = (sz)^2 - (s - 2z + w + z^{-1} + 2)sz -$$
$$(s + z + w + z^{-1} + 1)(2s + z + w^2 + z^{-2} + 4).$$

Using Procedure 1 we obtain the following.

Step 1. In this case $D = [0]$ since the transfer function (28) is strictly proper.

Step 2. From denominator of (28) we have

$$a_1 = s - 2z + w + z^{-1} + 2,$$
$$a_0 = (s + z + w + z^{-1} + 1)(2s + z + w^2 + z^{-2} + 4) \qquad (29)$$

and we chose

$$p_1 = s + z + w + z^{-1} + 1,$$
$$p_2 = 2s + z + w^2 + z^{-2} + 4, \qquad (30)$$
$$p_3 = s - 2z + w + z^{-1} + 2.$$

The matrix P has the form

$$P = \begin{bmatrix} 0 & p_2 \\ p_1 & p_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2s + z + w^2 + z^{-2} + 4 \\ s + z + w + z^{-1} + 1 & s - 2z + w + z^{-1} + 2 \end{bmatrix} \qquad (31)$$

and using (20) we obtain

$$P = A_0 + A_1 s + A_2 z + A_{10} w + A_{20} w^2$$
$$+ A_{01} z^{-1} + A_{02} z^{-2}, \qquad (32a)$$

where

$$A_0 = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$
$$A_{20} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad (32b)$$
$$A_{02} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The matrices A_0, A_1, A_{10}, A_{20}, A_{01}, A_{02} have nonnegative entries, the matrix A_2 is a Metzler matrix and they satisfy the condition

$$A_0 + A_1 A_2 = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \quad \in \Re_+^{2 \times 2}. \qquad (33)$$

Step 3. From numerator of the transfer function (28) and (24) we have

$$sz(1 + z^{-1}) + s(w + 1) + z(w + 1) + w(w + 1)$$
$$+ z^{-1}(w + 1) + w + 1 = (s + z + w + z^{-1} + 1)(w + 1)$$
$$+ sz(z^{-1} + 1) = p_1 \overline{b}_0 + sz \overline{b}_1 \qquad (34a)$$

and

$$\overline{B} = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \end{bmatrix} = \begin{bmatrix} w + 1 \\ z^{-1} + 1 \end{bmatrix}. \qquad (34b)$$

Hence from (23) we obtain

$$\overline{B} = B_0 + B_{10} w + B_{01} z^{-1} \qquad (35a)$$

and

$$B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad (35b)$$

The matrix c has the form

$$C = [\, 0 \quad 1 \,]. \qquad (36)$$

The desired positive realization of the transfer function (28) is given by (32b), (35b), (36) and $D = [0]$.

4. Extension for multi-input multi-output systems

The proposed method can be extended to multi-input multi-output 2D continuous-discrete linear (MIMO) systems. It is well-known that the proper transfer matrix of the MIMO linear systems with delays can be written in the form

$$T = T(s, z, w, z^{-1}) = \begin{bmatrix} \dfrac{N_{11}}{d_1} & \cdots & \dfrac{N_{1,m}}{d_1} \\ \vdots & \cdots & \vdots \\ \dfrac{N_{p,1}}{d_p} & \cdots & \dfrac{N_{p,m}}{d_p} \end{bmatrix} \qquad (37a)$$
$$+ D \in R^{p \times m}(s, z, w, z^{-1}),$$

where

$$N_{i,j} = N_{i,j}(s, z, w, z^{-1}) = b_{n_{i,j}-1}^{i,j}(sz)^{n_{i,j}-1} + ...$$
$$+ b_1^{i,j}(sz) + b_0^{i,j}, \qquad (37b)$$
$$i = 1, 2, ..., p; \quad j = 1, 2, ..., m,$$
$$d_k = d_k(s, z, w, z^{-1})$$
$$= (sz)^{n_k} - a_{n_k-1}^k(sz)^{n_k-1} - ... - a_1^k(sz) - a_0^k, \qquad (37c)$$
$$k = 1, 2, ..., p$$

and a_j^k, $b_k^{i,j}$ are polynomials in s, z, w, z^{-1}.

Theorem 3. There exists the positive realization

$$A_0 = \text{blockdiag} \begin{bmatrix} A_{0,1} & ... & A_{0,p} \end{bmatrix} \in \Re_+^{n\times n},$$
$$A_1 = \text{blockdiag} \begin{bmatrix} A_{1,1} & ... & A_{1,p} \end{bmatrix} \in \Re_+^{n\times n},$$
$$A_2 = \text{blockdiag} \begin{bmatrix} A_{2,1} & ... & A_{2,p} \end{bmatrix} \in M_n,$$
$$n = n_1 + ... + n_p,$$
$$A_{j,0} = \text{blockdiag} \begin{bmatrix} A_{j,1} & ... & A_{j,p} \end{bmatrix} \in \Re_+^{n\times n},$$
$$j = 1, 2, ..., \overline{q}_1,$$
$$A_{0,k} = \text{blockdiag} \begin{bmatrix} A_{1,k} & ... & A_{p,k} \end{bmatrix} \in \Re_+^{n\times n},$$
$$k = 1, 2, ..., \overline{q}_2,$$

$$B_0 = \begin{bmatrix} B_{11}^0 & ... & B_{1,m}^0 \\ \vdots & ... & \vdots \\ B_{p,1}^0 & ... & B_{p,m}^0 \end{bmatrix} \in \Re_+^{n\times m},$$

$$B_{j,0} = \begin{bmatrix} B_{11}^{j,0} & ... & B_{1,m}^{j,0} \\ \vdots & ... & \vdots \\ B_{p,1}^{j,0} & ... & B_{p,m}^{j,0} \end{bmatrix} \in \Re_+^{n\times m}, \quad j = 1, 2, ..., \overline{q}_1,$$

$$B_{0,k} = \begin{bmatrix} B_{11}^{0,k} & ... & B_{1,m}^{0,k} \\ \vdots & ... & \vdots \\ B_{p,1}^{0,k} & ... & B_{p,m}^{0,k} \end{bmatrix} \in \Re_+^{n\times m}, \quad k = 1, 2, ..., \overline{q}_2,$$

$$C = \text{blockdiag} \begin{bmatrix} C_1 & ... & C_p \end{bmatrix} \in \Re_+^{p\times n},$$
$$C_i = \begin{bmatrix} 0 & ... & 0 & 1 \end{bmatrix} \in \Re_+^{1\times n},$$
$$i = 1, 2, ..., p$$

(38)

of the strictly proper transfer matrix (37) if it is possible to find the polynomials in (s, z, w, z^{-1})

$$p_1^i, p_2^i, ..., p_{2n_i-1}^i, \quad i = 1, 2, ..., p$$ (39a)

and

$$b_{n_{i,j}-1}^{i,j}, ..., b_1^{i,j}, b_0^{i,j},$$
$$i = 1, 2, ..., p; \qquad j = 1, 2, ..., m$$ (39b)

with nonnegative coefficients (expect the last coefficients of $p_{2n_i-1}^i$) such that the conditions

$$\det[I_n sz - P_i]$$

$$= \begin{vmatrix} sz & 0 & ... & 0 & -p_{n_i}^i \\ -p_1^i & sz & ... & 0 & -p_{n_i+1}^i \\ 0 & -p_2^i & ... & 0 & -p_{n_i+2}^i \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & -p_{n_i-1}^i & sz - p_{2n_i-1}^i \end{vmatrix} = d_i, \quad (40)$$

$$i = 1, 2, ..., p$$

$$\overline{b}_{n_i-1}^{i,j}(sz)^{n_i-1} + p_{n_i-1}^i \overline{b}_{n_i-2}^{i,j}(sz)^{n_i-2} + ...$$
$$+ p_2^i p_3^i ... p_{n_i-1}^i \overline{b}_1^{i,j}(sz) + p_1^i p_2^i ... p_{n_i-1}^i \overline{b}_0^{i,j}$$
$$= b_{n_i-1}^{i,j}(sz)^{n_i-1} + ... + b_1^{i,j}(sz) + b_0^{i,j} = N_{i,j},$$
$$i = 1, 2, ..., p; \qquad j = 1, 2, ..., m$$ (41)

are satisfied.

Proof. If the polynomials (39) have nonnegative coefficients (expect the last coefficients of $p_{n_i-1}^i$) then

$$[I_n sz - P] = \text{blockdiag}_{i=1,2,...,p}$$

$$\cdot \left[I_{n_i} sz - A_{0,i} - A_{1,i} s - A_{2,i} z \right.$$
$$\left. - \sum_{j=0}^{\overline{q}_1} A_{i,1} w^j - \sum_{k=0}^{\overline{q}_2} A_{i,k} z^{-k} \right]$$ (42)

and (39a) and (39b) hold. If the coefficients of the polynomials (39) are nonnegative and (41) holds then the matrices (38) satisfy the equality

$$\begin{bmatrix} N_{11} & ... & N_{1,m} \\ \vdots & ... & \vdots \\ N_{p,1} & ... & N_{p,m} \end{bmatrix}$$

$$= C[I_n sz - P]_{ad} \left[B_0 + \sum_{j=0}^{\overline{q}_1} B_{j,0} w^j + \sum_{k=0}^{\overline{q}_2} B_{0,k} z^{-k} \right].$$ (43)

If the conditions of Theorem 3 are satisfied then the positive realization of (37) can be found by the use of procedure similar to the Procedure 1.

Remark 2. The state variable diagram method presented in [23] for continuous-time systems with delays can be also extended to 2D continuous-discrete linear systems.

5. Concluding remarks

A new method for determination of positive realizations with reduced numbers of delays of 2D continuous-discrete linear systems has been proposed. Using the proposed method it is possible to find a positive realization with reduced numbers of delays in state and input variable. Sufficient conditions for the existence of positive realizations have been established and the procedure for finding the positive realizations has been proposed. The procedure has been illustrated by the numerical example. The proposed method can be extended to fractional continuous and discrete linear systems.

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