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A Lyapunov functional for a neutral system with a time-varying delay

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Abstract. The paper presents a method of determining of the Lyapunov functional for a linear neutral system with an interval time-varying delay. The Lyapunov functional is constructed for the system with a time-varying delay with a given time derivative, which is calculated on the trajectory of the system with a time-varying delay. The presented method gives analytical formulas for the coefficients of the Lyapunov functional.

Key words: Lyapunov functional, time-varying delay, neutral system.

1. Introduction

The Lyapunov functionals are used to test the stability of the systems. For example Fridman [1] introduced the Lyapunov-Krasovskii functional for examination the stability of the linear retarded and neutral type systems with discrete and distributed delays, which were based on equivalent descriptor form of the original system and obtained delay-dependent and delay-independent conditions in terms of linear matrix inequality (LMI). Ivanescu et al. [2] proceeded with the delay-depended stability analysis for the linear neutral systems, constructed the Lyapunov functional and derived sufficient delay-dependent conditions in terms of linear matrix inequalities (LMIs). Han [3] obtained a delay-dependent stability criterion for the neutral systems with a time-varying discrete delay. This criterion was expressed in the form of LMI and was obtained using the Lyapunov direct method. Han [4] developed the discretized Lyapunov functional approach to investigate the stability of linear neutral systems with mixed neutral and discrete delays. The stability criteria, which are applicable to linear neutral systems with both small and non-small discrete delays, are formulated in the form of LMIs. Han [5] studied the stability problem of linear time delay systems, both retarded and neutral types, using the discrete delay N-decomposition approach to derive some new more general discrete delay dependent stability criteria. Han [6] employed the delay-decomposition approach to derive some improved stability criteria for linear neutral systems and to deduce some sufficient conditions for the existence of a Lyapunov functional for a system with k-non-commensurate neutral time delays of a delayed state feedback controller, which ensure asymptotic stability and a prescribed H1 performance level of the corresponding closed-loop system. Gu and Liu [7] investigated the stability of coupled differential-functional equations using the discretized Lyapunov functional method and delivered the stability condition in the form of LMI, suitable for numerical computation.

The Lyapunov functionals are also used in calculation of the robustness bounds for uncertain time delay systems. For illustration Kharitonov and Zhabko [8] proposed a procedure of construction of the quadratic functionals for the linear retarded type delay systems which could be used for the robust stability analysis of time delay systems. This functional was expressed by means of Lyapunov matrix, which depended on the fundamental matrix of a time delay system. Kharitonov [9] extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and in [10] to the neutral type time delay systems with a discrete and distributed delay. Han [11] investigated the robust stability of uncertain neutral systems with discrete and distributed delays, which has been based on the descriptor model transformation and the decomposition technique, and formulated the stability criteria in the form of LMIs. Han [12] considered the stability for the linear neutral systems with norm-bounded uncertainties in all system matrices and derived a new delay-dependent stability criterion. Neither model transformation nor bounding technique for cross terms is involved through derivation of the stability criterion.

The Lyapunov functionals are also used in computation of the exponential estimates for the solutions of the time delay systems. For instance Kharitonov and Hinrichsen [13] used the Lyapunov matrix to derive exponential estimates for the solutions of exponentially stable time delay systems. Kharitonov and Plischke [14] formulated the necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of a retarded system with one delay. The numerical scheme for construction of the Lyapunov functionals has been proposed by Gu [15]. This method starts with the discretisation of a Lyapunov functional. The scheme is based on LMI techniques.

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The Lyapunov quadratic functionals are also used to calculation of a value of a quadratic performance index of quality in the process of the parametric optimization for the time delay systems. One constructs a Lyapunov functional for the system with a time delay with a given time derivative whose is equal to the negatively defined quadratic form of a system state. The value of that functional at the initial state of the time delay system is equal to the value of a quadratic performance index of quality. For the first time such Lyapunov functional was introduced by Repin [16] for the case of the retarded type time delay linear systems with one delay. Repin [16] delivered also the procedure for determination of the functional coefficients. Duda [17] used the Lyapunov quadratic functional, which was proposed by Repin in the parametric optimization process for systems with a time delay of retarded type and extended the results to the case of a neutral type time delay system in [18]. Duda [19, 20] conducted also the parametric optimization process for the neutral system with two non-commensurate delays and a P-controller, to this end there were used the results presented in [21].

There is another method to achieve a value of a quadratic performance index presented by Górecki and Popek [22] which bases on a characteristic quasipolynomial. Górecki and Białas published two articles [23, 24] whose concern relations between roots of the transcendental equations and their coefficients. These results are helpful in the stability analysis of the time delay systems.

There are papers whose regard the quadratic Lyapunov functionals such that their coefficients are given by the analytical formulas. Duda [25] presented a method of determining of the Lyapunov functional for a linear dynamical system with two lumped retarded type time delays in the general case with non- commensurate delays and presented a special case with commensurate delays in which the Lyapunov functional could be determined by solving of the ordinary differential equations set. Duda [26] presented also a method of determining of the Lyapunov functional for a neutral system with k-non-commensurate delays and in [27] for a linear system with both lumped and distributed delay, and in [28] for a system with a time-varying delay.

This paper presents a method of determining of the Lyapunov functional for a linear neutral system with an interval time-varying delay. The Lyapunov functional is constructed for the system with a time-varying delay with a given time derivative which is calculated on the trajectory of the system with a time-varying delay. We assume that a time derivative of the Lyapunov functional is a quadratic form. This assumption enables calculation the value of the integral quadratic performance index for the parametric optimization of a neutral system with an interval time-varying delay. The presented method gives analytical formulas for the coefficients of the Lyapunov functional. The novelty of the result lies in the extension of the Repin method to the neutral system with an interval time-varying delay. To the best of author's knowledge, such extension has not been reported in the literature. There is also presented an example illustrating that method.

2. A mathematical model of a linear neutral system

The linear neutral systems are often used in control theory and in a regulation system. For example if we consider the regulation system with an object with time delay and a PD regulator we obtain a neutral system.

Let us consider a neutral system with a time-varying delay, whose dynamics is described by the functional-differential equation (FDE)

$$\begin{cases} \dfrac{dx(t)}{dt} - C\dfrac{dx(t-\tau(t))}{dt} = Ax(t) + Bx(t-\tau(t)) \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x(t_0+\theta) = \Phi(\theta), \end{cases} \tag{1}$$

where $t \geq t_0$, $\theta \in [-r, 0)$, $\tau(t)$ is a time-varying delay satisfying the condition $0 \leq \tau(t) \leq r$; $\dfrac{d\tau(t)}{dt} \neq 1$ where r is a positive constant A, B, $C \in \mathbb{R}^{n\times n}$ and C is non-singular, $x(t) \in \mathbb{R}^n$, $\Phi \in W^{1,2}([-r,0), \mathbb{R}^n)$.

$W^{1,2}([-r,0), \mathbb{R}^n)$ is a space of all absolutely continuous functions $[-r,0) \to \mathbb{R}^n$ with derivatives in $L^2([-r,0), \mathbb{R}^n)$ a space of Lebesgue square integrable functions on an interval $[-r,0)$ with values in $\mathbb{R}^n$.

The norm in $W^{1,2}([-r,0), \mathbb{R}^n)$ is defined by

$$\| \Phi \|^2_{W^{1,2}} = \int_{-r}^{0} \left(\| \Phi(t) \|^2_{\mathbb{R}^n} + \left\| \frac{d\Phi(t)}{dt} \right\|^2_{\mathbb{R}^n} \right) dt, \tag{2}$$

where $\| \cdot \|_{\mathbb{R}^n}$ is an arbitrary norm in $\mathbb{R}^n$.

The space of initial data is given by the Cartesian product $\mathbb{R}^n \times W^{1,2}([-r,0), \mathbb{R}^n)$.

The theorems of existence, continuous dependence and uniqueness of solutions of Eq. (1) are given in [29].

One can obtain a solution of FDE (1) using a step method [29]. The step method is a basic method for solving FDE with a lumped delay. A solution is found on successive intervals, one after another, by solving an ordinary equation without delay in each interval.

A solution of Eq. (1) is an absolutely continuous function defined for $t \geq t_0 - r$ with values in $\mathbb{R}^n$.

$$x(\cdot) \in W^{1,2}([t_0 - r, \infty), \mathbb{R}^n), \tag{3}$$

where $W^{1,2}([t_0 - r, \infty), \mathbb{R}^n)$ is a space of all absolutely continuous functions with derivatives in a space of Lebesgue square integrable functions on interval $[t_0 - r, \infty)$ with values in $\mathbb{R}^n$.

Definition 1. The zero solution of (1) is **stable** if for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sqrt{\| x(t_0) \|^2_{\mathbb{R}^n} + \| \Phi \|^2_{W^{1,2}}} < \delta$$

implies $\| x(t) \|_{\mathbb{R}^n} \leq \varepsilon$ for $t \geq t_0$.

The zero solution of (1) is **asymptotically stable** if

$$\| x(t) \|_{\mathbb{R}^n} \to 0 \quad as\; t \to \infty$$

The difference equation associated with (1) is given by

$$x(t) = Cx(t - \tau(t)), \quad t \geq t_0. \tag{4}$$

The eigenvalues of the difference Eq. (4) play a fundamental role in the asymptotic behavior of the solutions of the neutral Eq. (1). The difference Eq. (4) is stable when the spectral radius $\gamma(C)$ of the matrix C fulfills the condition

$$\gamma(C) = sup\{|\lambda| : \lambda \in \sigma(C)\} < 1, \tag{5}$$

where the spectrum $\sigma(C)$ is the set of complex numbers λ for which the matrix $\lambda I - C$ is not invertible.

We introduce a new function y, defined by term

$$y(t) = x(t) - Cx(t - \tau(t)) \quad \text{for} \quad t \geq t_0. \tag{6}$$

Thus the Eq. (1) takes a form

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + (AC + B)\,x(t - \tau(t)) \\ y(t) = x(t) - Cx(t - \tau(t)) \\ y(t_0) = x_0 - C\Phi(-\tau(t)) \\ x(t_0 + \theta) = \Phi(\theta). \end{cases} \tag{7}$$

We assume that the matrix C fulfills the condition (5).

The state of the system (7) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \quad \text{for} \quad t \geq t_0, \tag{8}$$

where $y(t) \in \mathbb{R}^n$, $x_t \in W^{1,2}([-r, 0), \mathbb{R}^n)$ and $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0)$.

The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r, 0), \mathbb{R}^n). \tag{9}$$

The norm in the state space X is defined by

$$\| S(t) \|_X^2 = \| y(t) \|_{\mathbb{R}^n}^2 + \| x_t \|_{W^{1,2}}^2 \quad \text{for} \quad t \geq t_0. \tag{10}$$

The controllability of the systems with time delay is presented in [30].

3. A Lyapunov functional

Definition 2. A functional $V : X \to \mathbb{R}$ is **positive definite** if and only if it is continuous and $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.

A functional $V : X \to \mathbb{R}$ is **negative definite** if and only if it is continuous and $V(x) < 0$ for $x \neq 0$ and $V(0) = 0$.

A functional $V : X \times [t_0, \infty) \to \mathbb{R}$ is **positive definite** if it is continuous and there exists a positive definite functional $W : X \to \mathbb{R}$ such that $V(x, t) \geq W(x)$ and $V(0, t) = W(0) = 0$ for $x \in X$ and $t \geq t_0$.

Definition 3. A positive definite functional $V : X \times [t_0, \infty) \to \mathbb{R}$ is **upper bounded** if there exists a positive definite functional $W : X \to \mathbb{R}$ such that $V(x, t) \leq W(x)$ for $x \in X$ and $t \geq t_0$.

Definition 4. We define a time derivative of the functional $V(y(t), x_t, t)$ at $(y(t_0), \Phi, t_0)$ on a trajectory of a system (7) by the formula

$$\begin{aligned} &\frac{dV(y(t_0), \Phi, t_0)}{dt} \\ &= \limsup_{h \to 0} \frac{1}{h} \left[V\left(y\left(t_0 + h\right), x_{t_0+h}, t_0 + h\right) \right. \\ &\qquad \left. - V\left(y(t_0), \Phi, t_0\right)\right]. \end{aligned} \tag{11}$$

Definition 5. We say that $V : X \times [t_0, \infty) \to \mathbb{R}$ is a **Lyapunov functional** if

1. V is a positive definite upper bounded functional
2. V is differentiable
3. A time derivative of V computed according to a formula (11) on the trajectory of the system (7) is negative definite

Existence of the Lyapunov functional for the system (7) is a sufficient condition for asymptotic stability of its zero solution.

From the assumption that the Lyapunov functional is upper bounded results that there exists a functional W such that

$$0 \leq V(y(t), x_t, t) \leq W(y(t), x_t) \quad \text{for} \quad t \geq t_0. \tag{12}$$

When the system (7) is asymptotically stable

$$\lim_{t \to \infty} W(y(t), x_t) = 0 \; implies \; \lim_{t \to \infty} V(y(t), x_t, t) = 0.$$

Hence

$$\begin{aligned} &\int_{t_0}^{\infty} \frac{dV(y(t), x_t, t)}{dt} dt \\ &= \lim_{t \to \infty} V(y(t), x_t, t) - \lim_{t \to t_0} V(y(t), x_t, t) \\ &= -V(\lim_{t \to t_0}(y(t), x_t, t)) = -V(y(t_0), \Phi, t_0). \end{aligned} \tag{13}$$

We assume that the time derivative of the Lyapunov functional V is given as a quadratic form

$$\frac{dV(y(t), x_t, t)}{dt} \equiv -y^T(t) G y(t) \quad \text{for} \quad t \geq t_0, \tag{14}$$

where $G \in \mathbb{R}^{n \times n}$ is a positive definite matrix.

Taking (13) and (14) into account we obtain a relationship

$$J = \int_{t_0}^{\infty} y^T(t) G y(t) dt = V(y_0, \Phi, t_0). \tag{15}$$

Corollary 6. If we construct a Lyapunov functional such that its time derivative computed on the trajectory of the system (7) will be given as a quadratic form (14) we can not only investigate the system (7) stability but also we can calculate a value of a square indicator of quality (15) of the parametric optimization problem.

To calculate the value of the performance index (15), which is equal to the value of the Lyapunov functional at the initial state of the system (7), we need a mathematical formula of the functional.

4. Main result. Determination of the Lyapunov functional

Let us consider a quadratic functional on $X \times [t_0, \infty)$, where X is defined by (9), given by a formula

$$V(y(t), x_t, t) = y^T(t)\alpha(t)y(t) + \int_{-\tau(t)}^{0} y^T(t)\beta(\theta+\tau(t))x_t(\theta)d\theta + \int_{-\tau(t)}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta+\tau(t),\sigma+\tau(t))x_t(\sigma)d\sigma d\theta \tag{16}$$

for $t \geq t_0$ where $\alpha \in C^1([t_0,\infty),\mathbb{R}^{n\times n})$; $\beta \in C^1([0,\tau(t)],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$, $\Omega = \{(\theta,\sigma):\ \theta\in[0,\tau(t)],\ \sigma\in[\theta,0]\}$; $0 \leq \tau(t) \leq r$, where C^1 is a space of all continuous functions with continuous derivative.

Conjecture 7. We introduce a procedure of determination of the functional (16) coefficients to obtain the Lyapunov functional.

We compute the time derivative of the functional (16) on the trajectory of the system (7) according to the formula (11)

$$\begin{aligned}\frac{dV(y(t),x_t,t)}{dt} &= y^T(t)\left[A^T\alpha(t)+\alpha(t)A+\frac{d\alpha(t)}{dt}\right. \\ &\left.+\beta(\tau(t))\right]y(t) + y^T(t)\left[\left(\alpha(t)+\alpha^T(t)\right)(AC+B)\right. \\ &\left.+\beta(\tau(t))C+\beta(0)\left(\frac{d\tau(t)}{dt}-1\right)\right]x_t(-\tau(t)) \\ &+\int_{-\tau(t)}^{0} y^T(t)\left[A^T\beta(\theta+\tau(t))+\frac{d\beta(\theta+\tau(t))}{dt}\right. \\ &\left.-\frac{d\beta(\theta+\tau(t))}{d\theta}+\delta^T(\theta+\tau(t),\tau(t))\right]x_t(\theta)d\theta \\ &+\int_{-\tau(t)}^{0} x_t^T(-\tau(t))\left[(AC+B)^T\beta(\theta+\tau(t))\right. \\ &+C^T\delta^T(\theta+\tau(t),\tau(t))+\delta(0,\theta+\tau(t)) \\ &\left.\cdot\left(\frac{d\tau(t)}{dt}-1\right)\right]x_t(\theta)d\theta + \int_{-\tau(t)}^{0}\int_{\theta}^{0} x_t^T(\theta) \\ &\cdot\left[\frac{d\delta(\theta+\tau(t),\sigma+\tau(t))}{dt}-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\theta}\right. \\ &\left.-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta\end{aligned} \tag{17}$$

for $t \geq t_0$ where $\alpha \in C^1([t_0,\infty),\mathbb{R}^{n\times n})$; $\beta \in C^1([0,\tau(t)],\mathbb{R}^{n\times n})$; $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$; $\Omega = \{(\theta,\sigma):\ \theta\in[0,\tau(t)],\ \sigma\in[\theta,0]\}$; $0\leq\tau(t)\leq r$.

The time derivative of the Lyapunov functional should be negative definite, therefore we identify the coefficients of the functional (16) assuming that the time derivative (17) satisfies the relationship (14).

From Eqs. (17) and (14) we obtain the set of equations

$$A^T\alpha(t)+\alpha(t)A+\frac{d\alpha(t)}{dt}+\beta(\tau(t)) = -G, \tag{18}$$

$$\begin{aligned}&\left(\alpha(t)+\alpha^T(t)\right)(AC+B)+\beta(\tau(t))C \\ &+\beta(0)\left(\frac{d\tau(t)}{dt}-1\right)=0,\end{aligned} \tag{19}$$

$$\begin{aligned}&A^T\beta(\theta+\tau(t))+\frac{d\beta(\theta+\tau(t))}{dt} \\ &-\frac{d\beta(\theta+\tau(t))}{d\theta}+\delta^T(\theta+\tau(t),\tau(t))=0,\end{aligned} \tag{20}$$

$$\begin{aligned}&(AC+B)^T\beta(\theta+\tau(t))+C^T\delta^T(\theta+\tau(t),\tau(t)) \\ &+\delta(0,\theta+\tau(t))\left(\frac{d\tau(t)}{dt}-1\right)=0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\frac{d\delta(\theta+\tau(t),\sigma+\tau(t))}{dt}-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\theta} \\ &-\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\sigma}=0\end{aligned} \tag{22}$$

for $t\geq t_0$; $\theta\in[-\tau(t),0]$; $\sigma\in[\theta,0]$ where $0\leq\tau(t)\leq r$.

We introduce the new variables

$$\xi = \theta+\tau(t), \tag{23}$$

$$\eta = \sigma+\tau(t). \tag{24}$$

We calculate the derivatives

$$\begin{aligned}\frac{d\delta(\theta+\tau(t),\sigma+\tau(t))}{dt} &= \frac{d\delta(\xi,\eta)}{dt} \\ &= \frac{\partial\delta(\xi,\eta)}{\partial\xi}\frac{d\tau(t)}{dt}+\frac{\partial\delta(\xi,\eta)}{\partial\eta}\frac{d\tau(t)}{dt},\end{aligned} \tag{25}$$

$$\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\theta} = \frac{\partial\delta(\xi,\eta)}{\partial\theta} = \frac{\partial\delta(\xi,\eta)}{\partial\xi}, \tag{26}$$

$$\frac{\partial\delta(\theta+\tau(t),\sigma+\tau(t))}{\partial\sigma} = \frac{\partial\delta(\xi,\eta)}{\partial\sigma} = \frac{\partial\delta(\xi,\eta)}{\partial\eta}, \tag{27}$$

$$\frac{d\beta(\theta+\tau(t))}{dt} = \frac{d\beta(\xi)}{d\xi}\frac{\partial\xi}{\partial t} = \frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt}, \tag{28}$$

$$\frac{d\beta(\theta+\tau(t))}{d\theta} = \frac{d\beta(\xi)}{d\xi}\frac{\partial\xi}{\partial\theta} = \frac{d\beta(\xi)}{d\xi}. \tag{29}$$

The formula (22) takes the form

$$\frac{\partial\delta(\xi,\eta)}{\partial\xi}+\frac{\partial\delta(\xi,\eta)}{\partial\eta}=0, \tag{30}$$

for $t\geq t_0$; $\theta\in[-\tau(t),0]$; $\sigma\in[\theta,0]$; $\xi\in[0,\tau(t)]$, $\eta\in[\xi,\tau(t)]$ where $0\leq\tau(t)\leq r$.

The formula (20) takes the form

$$\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta(\xi)}{d\xi}+A^T\beta(\xi)+\delta^T(\xi,\tau(t))=0. \tag{31}$$

The formula (21) takes the form

$$\begin{aligned}&(AC+B)^T\beta(\xi)+C^T\delta^T(\xi,\tau(t)) \\ &+\delta(0,\xi)\left(\frac{d\tau(t)}{dt}-1\right)=0.\end{aligned} \tag{32}$$

The solution of Eq. (22) is given by a formula

$$\begin{aligned}\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)&=\delta\left(\xi,\eta\right)\\&=f(\xi-\eta)=f\left(\theta-\sigma\right)\end{aligned}\tag{33}$$

for $t\geq t_0$; $\theta\in[-\tau(t),0]$; $\sigma\in[\theta,0]$; $0\leq\tau(t)\leq r$; where $f\in C^1\left([-r,r],\mathbb{R}^{n\times n}\right)$.

From formula (31) we get

$$\begin{aligned}&\delta^T\left(\xi,\tau(t)\right)=f^T\left(\xi-\tau(t)\right)\\&=-\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta(\xi)}{d\xi}-A^T\beta(\xi)\end{aligned}\tag{34}$$

We put the term (34) into (32). After calculations we get

$$C^T\frac{d\beta(\xi)}{d\xi}=\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\xi)+\delta\left(0,\xi\right).\tag{35}$$

From the relation (34) we can determine the term $\delta(0,\xi)=f(-\xi)$

$$\begin{aligned}f(-\xi)=&\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta^T(-\xi+\tau(t))}{d\xi}\\&-\beta^T\left(-\xi+\tau(t)\right)A\end{aligned}\tag{36}$$

and put it into (35). In this way we get the formula

$$\begin{aligned}&C^T\frac{d\beta(\xi)}{d\xi}-\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta^T(-\xi+\tau(t))}{d\xi}\\&=\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\xi)-\beta^T\left(-\xi+\tau(t)\right)A\end{aligned}\tag{37}$$

for $\xi\in[0,\tau(t)]$ where $0\leq\tau(t)\leq r$.

We determine the formula (37) for the new variable $-\xi+\tau(t)$. After calculations we obtain the formula

$$\begin{aligned}&\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta(\xi)}{d\xi}-\frac{d\beta^T(-\xi+\tau(t))}{d\xi}C\\&=\left(\frac{d\tau(t)}{dt}-1\right)^{-1}\beta^T(-\xi+\tau(t))B-A^T\beta(\xi).\end{aligned}\tag{38}$$

In this way we obtained the set of differential equations

$$\begin{cases}C^T\frac{d\beta(\xi)}{d\xi}-\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta^T(-\xi+\tau(t))}{d\xi}\\=\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\xi)-\beta^T\left(-\xi+\tau(t)\right)A\\\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta(\xi)}{d\xi}-\frac{d\beta^T(-\xi+\tau(t))}{d\xi}C\\=\left(\frac{d\tau(t)}{dt}-1\right)^{-1}\beta^T(-\xi+\tau(t))B-A^T\beta(\xi)\end{cases}\tag{39}$$

for $t\geq t_0$, $\xi\in[0,\tau(t)]$ where $0\leq\tau(t)\leq r$ with the initial conditions $\beta\left(0\right)$ and $\beta\left(\tau\left(t\right)\right)$.

We can reshape the set of Eqs. (39) to the form

$$\begin{cases}C^T\frac{d\beta(\xi)}{d\xi}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(\xi)}{d\xi}\\=\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(\xi)+\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\xi)C\\-\beta^T\left(-\xi+\tau(t)\right)(AC+B)\\C^T\frac{d\beta(-\xi+\tau(t))}{d\xi}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(-\xi+\tau(t))}{d\xi}\\=\beta^T\left(\xi\right)(AC+B)-\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(-\xi+\tau(t))\\-\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(-\xi+\tau(t))C\end{cases}\tag{40}$$

for $t\geq t_0$, $\xi\in[0,\tau(t)]$ where $0\leq\tau(t)\leq r$ with the initial conditions $\beta\left(0\right)$ and $\beta\left(\tau\left(t\right)\right)$.

There holds the relationship between $\beta(\xi)$ and $\beta(-\xi+\tau(t))$

$$\beta\left(\xi\right)|_{\xi=\frac{\tau(t)}{2}}=\beta\left(-\xi+\tau\left(t\right)\right)|_{\xi=\frac{\tau(t)}{2}}.\tag{41}$$

We calculate the derivative of Eq. (19) with respect to t

$$\begin{aligned}&\left(\frac{d\alpha(t)}{dt}+\frac{d\alpha^T(t)}{dt}\right)(AC+B)+\frac{d\beta(\tau(t))}{dt}C\\&+\frac{d\beta(0)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)+\frac{d^2\tau(t)}{dt^2}\beta(0)=0,\end{aligned}\tag{42}$$

where

$$\frac{d\beta(0)}{dt}=\left.\frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt}\right|_{\xi=0},\tag{43}$$

$$\frac{d\beta(\tau(t))}{dt}=\left.\frac{d\beta(\xi)}{d\xi}\frac{d\tau(t)}{dt}\right|_{\xi=\tau(t)}.\tag{44}$$

From Eq. (40) it results that

$$\begin{aligned}&C^T\frac{d\beta(0)}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(0)}{dt}\\&=\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(0)+\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}\\&\cdot B^T\beta(0)C-\frac{d\tau(t)}{dt}\beta^T\left(\tau(t)\right)(AC+B),\end{aligned}\tag{45}$$

$$\begin{aligned}&C^T\frac{d\beta(\tau(t))}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(\tau(t))}{dt}\\&=\frac{d\tau(t)}{dt}\beta^T\left(0\right)(AC+B)-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(\tau(t))\\&-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\tau(t))C.\end{aligned}\tag{46}$$

From Eq. (18) we obtain

$$\frac{d\alpha(t)}{dt}=-A^T\alpha(t)-\alpha(t)A-\beta(\tau(t))-G.\tag{47}$$

We put the term (47) into Eq. (42). After calculations we get

$$\begin{aligned}&\left[A^T\left(\alpha(t)+\alpha^T(t)\right)+\left(\alpha(t)+\alpha^T(t)\right)A\right](AC+B)\\&+\left(\beta(\tau(t))+\beta^T(\tau(t))\right)(AC+B)-\frac{d^2\tau(t)}{dt^2}\beta(0)\\&-\frac{d\beta(\tau(t))}{dt}C-\frac{d\beta(0)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)\\&=-\left(G+G^T\right)(AC+B).\end{aligned}\tag{48}$$

The matrix $\alpha(t)$, the initial conditions of the system (40) and $\frac{d\beta(0)}{dt}$, $\frac{d\beta(\tau(t))}{dt}$ we obtain by solving the set of algebraic Eqs. (48), (19), (45), (46) and (41). We write that set of the equations below

$$\begin{aligned}&\left[A^T\left(\alpha(t)+\alpha^T(t)\right)+\left(\alpha(t)+\alpha^T(t)\right)A\right](AC+B)\\&+\left(\beta(\tau(t))+\beta^T(\tau(t))\right)(AC+B)-\frac{d^2\tau(t)}{dt^2}\beta(0)\\&-\frac{d\beta(\tau(t))}{dt}C-\frac{d\beta(0)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)\\&=-\left(G+G^T\right)(AC+B),\end{aligned}\tag{49}$$

$$\begin{aligned}&\left(\alpha(t)+\alpha^T(t)\right)(AC+B)+\beta\left(\tau(t)\right)C\\&+\beta(0)\left(\frac{d\tau(t)}{dt}-1\right)=0,\end{aligned}\tag{50}$$

$$\begin{aligned}&C^T\frac{d\beta(0)}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(0)}{dt}\\&=\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)\\&\cdot A^T\beta(0)+\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(0)C\\&-\frac{d\tau(t)}{dt}\beta^T\left(\tau(t)\right)(AC+B),\end{aligned}\tag{51}$$

$$\begin{aligned}&C^T\frac{d\beta(\tau(t))}{dt}C-\left(\frac{d\tau(t)}{dt}-1\right)^2\frac{d\beta(\tau(t))}{dt}\\&=\frac{d\tau(t)}{dt}\beta^T(0)(AC+B)\\&-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)A^T\beta(\tau(t))\\&-\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt}-1\right)^{-1}B^T\beta(\tau(t))C,\end{aligned}\tag{52}$$

$$\beta(\xi)\mid_{\xi=\frac{\tau(t)}{2}}=\beta\left(-\xi+\tau(t)\right)\mid_{\xi=\frac{\tau(t)}{2}}.\tag{53}$$

Having the solution of the set of differential equations (40) and taking into account the formulas (23), (33) and (36) we can get the matrices

$$\beta\left(\theta+\tau(t)\right)=\beta(\xi)\mid_{\xi=\theta+\tau(t)},\tag{54}$$

$$\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)=f(\sigma-\theta),\tag{55}$$

where

$$\begin{aligned}f(\rho)=&-\left(\frac{d\tau(t)}{dt}-1\right)\frac{d\beta^T(\rho+\tau(t))}{d\rho}\\&-\beta^T\left(\rho+\tau(t)\right)A\end{aligned}\tag{56}$$

for $t\ge t_0$; $\theta\in[-\tau(t),0]$; $\sigma\in[\theta,0]$ where $0\le\tau(t)\le r$.

In this way we obtained all coefficients of the functional (16). This coefficients depend on the matrices A, B and C of the system (7). The time derivative of the functional (16) is negative definite. When the matrices $\alpha(t)$, $\beta(\theta+\tau(t))$ and $\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)$ for $t\ge t_0;\theta\in[-\tau(t),0];\sigma\in[\theta,0]$ are positive definite the functional (16) becomes the Lyapunov functional.

The Lyapunov functional for a neutral system with an interval time-varying delay given by formula (16) is more general than the functional proposed by Repin [16].

Example 8. Let us consider a system described by equation

$$\begin{cases}\dfrac{dx(t)}{dt}-c\dfrac{dx(t-\tau(t))}{dt}=ax(t)+bx(t-\tau(t))\\x(t_0)=x_0\\x(t_0+\theta)=\Phi(\theta)\in\mathbb{R}\end{cases}\tag{57}$$

$t\ge t_0$; $\Phi\in W^{1,2}([-r,0),\mathbb{R})$; $x(t)\in\mathbb{R}$; a, b, $c\in\mathbb{R}$;$\theta\in[-r,0)$;$\mid c\mid<1$; $\tau(t)$ is a time-varying delay satisfying the condition $0\le\tau(t)\le r$; $\frac{d\tau(t)}{dt}\neq1$; where r is positive constant.

We can reshape the Eq. (57) to the form

$$\begin{cases}\dfrac{dy(t)}{dt}=ay(t)+(ac+b)x(t-\tau(t))\\y(t)=x(t)-cx(t-\tau(t))\\y(t_0)=x_0-c\Phi(-\tau(t))\\x(t_0+\theta)=\Phi(\theta)\end{cases}\tag{58}$$

$t\ge t_0$; $\Phi\in W^{1,2}([-r,0),\mathbb{R})$; $x(t)\in\mathbb{R}$; a, b, $c\in\mathbb{R}$;$\mid c\mid<1$; $\theta\in[-r,0)$;$\tau(t)$ is a time-varying delay satisfying the condition $0\le\tau(t)\le r$; $\frac{d\tau(t)}{dt}\neq1$; where r is positive constant.

The Lyapunov functional is given by a formula

$$\begin{aligned}V\left(y(t),x_t,t\right)&=\alpha(t)y^2(t)\\&+\int_{-\tau(t)}^{0}\beta\left(\theta+\tau(t)\right)y(t)\,x_t(\theta)\,d\theta\\&+\int_{-\tau(t)}^{0}\int_{\theta}^{0}\delta\left(\theta+\tau(t),\sigma+\tau(t)\right)x_t(\theta)\,x_t(\sigma)\,d\sigma d\theta,\end{aligned}\tag{59}$$

where

$$x_t\in W^{1,2}([-r,0),\mathbb{R}),$$

$$x_t(\theta)=x(t+\theta)\ for\ \theta\in[-r,0).$$

We obtain the coefficients of the functional as below.

The Eq. (40) takes the form

$$\begin{bmatrix} \dfrac{d\beta(\xi)}{d\xi} \\ \dfrac{d\beta\left(-\xi+\tau(t)\right)}{d\xi} \end{bmatrix} = \begin{bmatrix} p_1 & -p_2 \\ p_2 & -p_1 \end{bmatrix} \begin{bmatrix} \beta(\xi) \\ \beta\left(-\xi+\tau(t)\right) \end{bmatrix} \tag{60}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$ $0 \leq \tau(t) \leq r$, where

$$p_1 = \frac{\left(\dfrac{d\tau(t)}{dt} - 1\right) a + \dfrac{bc}{\frac{d\tau(t)}{dt} - 1}}{c^2 - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2}, \tag{61}$$

$$p_2 = \frac{ac+b}{c^2 - \left(\dfrac{d\tau(t)}{dt} - 1\right)^2}. \tag{62}$$

The fundamental matrix of the differential Eq. (60) is given by formula

$$Q = \begin{bmatrix} ch\lambda\xi + \dfrac{p_1}{\lambda} sh\lambda\xi & -\dfrac{p_2}{\lambda} sh\lambda\xi \\ \dfrac{p_2}{\lambda} sh\lambda\xi & ch\lambda\xi - \dfrac{p_1}{\lambda} sh\lambda\xi \end{bmatrix}, \tag{63}$$

where

$$\lambda = \frac{\sqrt{\dfrac{b^2 - a^2\left(\frac{d\tau(t)}{dt} - 1\right)^2}{c^2 - \left(\frac{d\tau(t)}{dt} - 1\right)^2}}}{\left(\dfrac{d\tau(t)}{dt} - 1\right)}. \tag{64}$$

Hence

$$\begin{bmatrix} \beta(\xi) \\ \beta(-\xi+\tau(t)) \end{bmatrix} = Q(\xi) \begin{bmatrix} \beta(0) \\ \beta(\tau(t)) \end{bmatrix} \tag{65}$$

for $t \geq t_0$, $\xi \in [0, \tau(t)]$ where $0 \leq \tau(t) \leq r$.

We need the initial conditions of the set of differential Eqs. (60) to obtain

$$\beta(\theta + \tau(t)) = \beta\left(\xi\right)|_{\xi=\theta+\tau(t)}, \tag{66}$$

$$\delta\left(\theta+\tau(t), \sigma+\tau(t)\right) = f\left(\sigma - \theta\right), \tag{67}$$

$$f(\rho) = -\left(\frac{d\tau(t)}{dt} - 1\right) \frac{d\beta(\rho+\tau(t))}{d\rho} - a\beta\left(\rho+\tau(t)\right) \tag{68}$$

for $t \geq t_0$; $\theta \in [-\tau(t), 0]$; $\sigma \in [\theta, 0]$ where $0 \leq \tau(t) \leq r$.

The initial conditions of the differential Eq. (60) and the coefficient $\alpha(t)$ we get by solving of a set of Eqs. (49) to (53) which take the form as below

$$\begin{aligned} &4a\left(ac+b\right)\alpha(t) \\ &+ \left(-cp_2\frac{d\tau(t)}{dt} - \frac{d^2\tau(t)}{dt^2} - p_1\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)\right)\beta(0) \\ &+ \left(2\left(ac+b\right) + cp_1\frac{d\tau(t)}{dt} + p_2\frac{d\tau(t)}{dt}\left(\frac{d\tau(t)}{dt} - 1\right)\right)\beta(\tau(t)) \\ &= -2w\left(ac+b\right), \end{aligned} \tag{69}$$

$$2\left(ac+b\right)\alpha(t) + \left(\frac{d\tau(t)}{dt} - 1\right)\beta(0) + c\beta(\tau(t)) = 0, \tag{70}$$

$$\begin{aligned} &\left(ch\frac{\lambda\tau(t)}{2} + \frac{p_1 - p_2}{\lambda} sh\frac{\lambda\tau(t)}{2}\right)\beta(0) \\ &+ \left(\frac{p_1 - p_2}{\lambda} sh\frac{\lambda\tau(t)}{2} - ch\frac{\lambda\tau(t)}{2}\right)\beta(\tau(t)) = 0. \end{aligned} \tag{71}$$

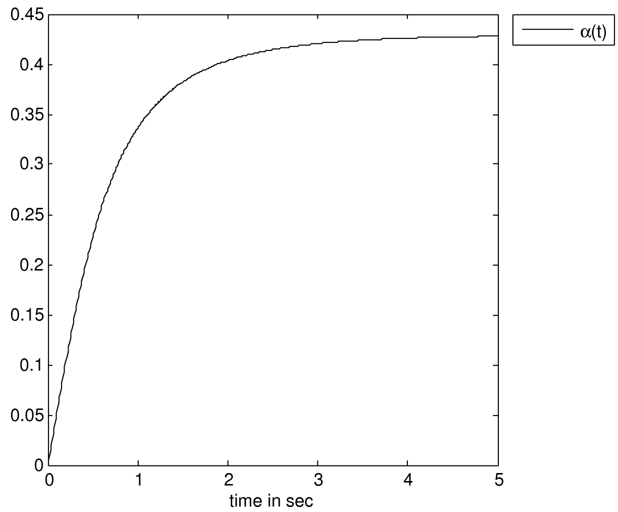

Fig. 1. Parameter $\alpha(t)$

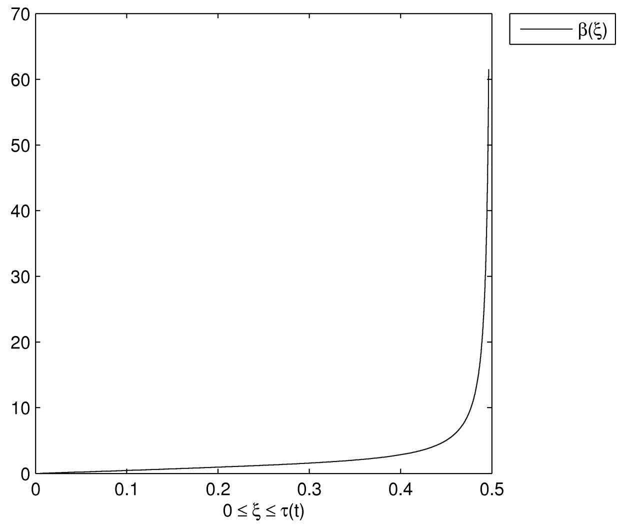

Fig. 2. Parameter $\beta(\xi)$

The figures show graphs of functions $\alpha(t)$ and $\beta(\xi)$, obtained with the Matlab code, for given values of parameters $a = -1$, $b = -0.5$, $c = 0.5$, $w = 1$, $\tau(t) = r\left(1 - \exp\left(-\frac{t}{T}\right)\right)$, $r = 0.5$, $T = 1$ of the system (57).

From figures implies that the system (58) is stable for given values of parameters a, b, c because $\alpha(t)$ and $\beta(\xi)$ are positive.

5. Conclusions

The paper presents the procedure of determining of the coefficients of the Lyapunov functional given by formula (16) for a linear system with an interval time-varying delay, described by Eq. (7). This article extends the method presented by Repin to the neutral system with an interval time-varying delay. The presented method allows achieving the analytical formulas on the coefficients of the Lyapunov functional, which can be used to examine the stability of the time delay systems with an interval time-varying delay and in the process of the parametric optimization for calculation of the square index of the quality given by formula (15).

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