10.2478/v10170-010-0011-x

Archives of Control Sciences
Volume 20(LVI), 2010
No. 2, *pages 165–186*

Exponential stability of networked control systems with network-induced random delays

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In this paper, the problem of exponential stability for the standard form of the state control, realized in a networked control system structure, is studied. To deal with the problem of stability analysis of the event-time-driven modes in the networked control systems the delayed–dependent exponential stability conditions are reformulated and proven. Based on the delay-time dependent Lyapunov-Krasovskii functional, exponential stability criteria are derived. These criteria are expressed as a set of linear matrix inequalities and their structure can be modified to use the bilinear inequality techniques.

Key words: networked systems, stability analysis, time-delay systems, linear matrix inequality, state feedback

1. Introduction

Recent advances in communication technology lead to an increased use of networked control. Networked control systems (NCS) are control loops closed through a shared communication network, where the network between control system components is used to exchange the information and control signals. The advantage of such structure are most of all simple installation, maintenance and system volume, and increased system agility. However, due to communication channel insertion, induced delay and packet dropout may seriously deteriorate the performance of the system, especially its stability.

During the previous decade, the stability problem of the networked control systems with induced network delays has attracted a lot of attention. The main approach for stability analysis relies on Lyapunov-Krasovskii functional and the linear matrix inequalities (LMI) approach for constructing common Lyapunov function ([7], [20] and the reference therein). For the reason of such delays it is often assumed that the actuator and the controller are sample driven, but once appears a delay overtopping given margin, the

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The work presented in this paper was supported by VEGA, Grant Agency of Ministry of Education and Academy of Science of Slovak Republic under Grant No. 1/0328/08. This support is very gratefully acknowledged.

Received 30.09.2009. Revised 17.4.2010.

system may become unstable. The usual approach ignores in the controller design stage the network delays and in the next design step there is analyzed stability, performance and robustness of the system with respect to the effects and characteristics of network delays and scheduling policy. Some progres review in this research field can be find in [2], [3], [5].

This paper is concerned with the problem of the event-time-driven modes in the networked control systems. Under these in a critical event the switched delay system structure is occasioned, which may include an unstable autonomous system. The paper actualizes, completes and extends the basic idea presented in [16] in coincidence with [10] and [22] to obtain conditions for the exponential stability of such structure. The network-induced time-varying delay upper bound is considered and main attention is focused on LMIs which have to to be analyzed to verify exponentially stability of this control structure. One indirect reason was to regularize the obtained LMIs. The presented LMI approach is computationally efficient as it can be solved numerically (see e.g. [1], [14]), and is based on the redefined Lyapunov-Krasovskii functional and on the Leibniz–Newton formula [13] to eliminate some dead-time dependent terms. Since Lyapunov-Krasovskii functional is used, sufficient conditions for exponential stability are obtained to set the derivative of this functional be negative along all trajectories of the switched system.

2. Problem description

Through this paper the task is concerned with stability analysis of NCS, where the linear dynamic system is given by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in \Re^n$, $\boldsymbol{u}(t) \in \Re^r$, $\boldsymbol{y}(t) \in \Re^m$ are vectors of the state, input and measurable output variables, respectively, and system matrices $\boldsymbol{A} \in \Re^{n\times n}$, $\boldsymbol{B} \in \Re^{n\times r}$ and $\boldsymbol{C} \in \Re^{m\times n}$ are real matrices. It is supposed that the pair $(\boldsymbol{A},\boldsymbol{B})$ is controllable, i.e.

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n \tag{3}$$

and the stable closed-loop system with the linear memoryless state feedback controller of the form

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t) \tag{4}$$

was designed using any standard method, i.e. the controller gain matrix $\boldsymbol{K} \in \Re^{r\times n}$ is known yet.

This control is closed through a shared network to make NCS so the signals from sensors are transmitted in digital form to the controller input and the controller output signals are transmitted in digital form to actuators through this network, where data is sent through the network in packets. It is straightforward to consider that the severity of the network-induced delays is aggravated when data packet dropouts occur during a network transmission. On another side it is supposed that delays in the chain "actuators-system-sensors", localized on the same side of the network, can be neglected. The main problem of the interest is to analyze the exponential stability of such NCS.

2.1. Event-time-driven system

Accepting a network delay-time, the event-time-driven system (1), (2) takes form

$$\dot{\boldsymbol{q}}(t) = \begin{cases} \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{q}(i_k\Delta t), & t \in \langle i_k\Delta t + \tau_k, j_k) \\ \boldsymbol{A}\boldsymbol{q}(t), & t \in \langle j_k, i_{k+1}\Delta t + \tau_{k+1}) \end{cases} \tag{5}$$

where $(i_k : \ k = 1, 2, \ldots)$ are some integers, Δt is the sampling period, and $\tau_k \geqslant 0$ is the time delay, which denotes the time interval from the instant time $i_k\Delta t$ where sensors notes the sample sensor data from the plat to the instant time when actuators transfer the data to the plant.

It is supposed that the next condition is satisfied

$$j_k = \begin{cases} i_{k+1}\Delta t + \tau_{k+1}, & (i_{k+1} - i_k)\Delta t + \tau_{k+1} \leqslant h \\ i_k\Delta t + h, & (i_{k+1} - i_k)\Delta t + \tau_{k+1} > h \end{cases} \tag{6}$$

where h is an upper bound of the intervening time. Event-time-driven mode means, that the controller and the actuator data will be updated once a new packet comes, and this new data can be held during the intervening time less then h. If at the end of this time interval the new data packet has not yet come, the acting data will be set to zero, and will stay zero until the new data package will come. It is naturally that this process starts in the control mode. By this rule obtained the switched delay control system may include an unstable subsystem.

2.2. Basic definitions

Let a nonlinear system, described by the vector differential equation

$$\dot{\boldsymbol{q}}(t) = f(\boldsymbol{q}(t), \boldsymbol{u}(t-\tau), t) \tag{7}$$

is controlled by the memory-free linear controller of the form

$$\boldsymbol{u}(t-\tau) = \boldsymbol{K}\boldsymbol{q}(t-\tau) \tag{8}$$

where $0 < \tau \in \Re$, $\boldsymbol{q}(t) \in \Re^n$, $\boldsymbol{u}(t-\tau) \in \Re^r$, $\boldsymbol{K} \in \Re^{r\times n}$ is the control law gain matrix, $f(\cdot)$ is piecewise smooth and globally Lipschitzian, and the origin is an equilibrium.

If no control input $\boldsymbol{u}(t-\tau)$ is imposed on the system, then the system is said to be in the unforced (autonomous) mode described by

$$\dot{\boldsymbol{q}}(t) = f(\boldsymbol{q}(t),t) \tag{9}$$

and $\boldsymbol{q}(t)$ is said to be a solution (or state trajectory) of (9) over $\langle t_0,t_1)$ if $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$.

Definition 1 (Lyapunov Function [17]) *The equilibrium* $\mathbf{0}$ *of system* (9) *is:*

i. *stable if there exists a continuously differentiable ($\mathcal{C}^1$) function $v(\boldsymbol{q},t) > 0$ such that $\dot{v}(\boldsymbol{q}(t),t) \leqslant 0$;*

ii. *uniformly stable if there exists a $\mathcal{C}^1$ decreasing function $v(\boldsymbol{q},t) > 0$ such that $\dot{v}(\boldsymbol{q}(t),t) \leqslant 0$;*

iii. *asymptotically stable if there exists a $\mathcal{C}^1$ function $v(\boldsymbol{q},t) > 0$ such that $\dot{v}(\boldsymbol{q}(t),t) < 0$;*

iv. *uniformly asymptotically stable if there exists a $\mathcal{C}^1$ decreasing function $v(\boldsymbol{q},t) > 0$ such that $\dot{v}(\boldsymbol{q},t) < 0$;*

v. *exponentially stable if there exist a $\mathcal{C}^1$ function $v(\boldsymbol{q},t) > 0$ and positive real constants α,β,γ, and $p \geqslant 1$ such that for all $\boldsymbol{q}(t)$ and t $\alpha\|\boldsymbol{q}(t)\|^p \leqslant v(\boldsymbol{q}(t),t) \leqslant \beta\|\boldsymbol{q}(t)\|^p$ and $\dot{v}(\boldsymbol{q}(t),t) \leqslant -\gamma\|\boldsymbol{q}(t)\|^p$.*

If (9) is linear and time invariant, then with matrix $\boldsymbol{A} \in \Re^{n\times n}$ equation (9) can be rewritten as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t). \tag{10}$$

It is clear that the origin is always the equilibrium of system (10), and (10) is uniformly asymptotically stable if and only if it is exponentially stable.

Definition 2 *For linear time-invariant autonomous system* (10)*, the following statements are equivalent:*

i. *the autonomous system is asymptotically stable;*

ii. *the autonomous system is exponentially stable.*

Let $\boldsymbol{q}(t,t_0,\boldsymbol{\phi})$ be a solution of (7) at time t with a initial data $\boldsymbol{\phi}$ specified at time t_0, i.e. $\boldsymbol{q}(t_0+\vartheta,t_0,\boldsymbol{\phi}) = \boldsymbol{\phi}(\vartheta)$ for $\vartheta \in \langle -\tau,0\rangle$. Because of the time-invariance of the system, $\boldsymbol{q}(t,t_0,\boldsymbol{\phi}) = \boldsymbol{q}(t-t_0,\boldsymbol{\phi})$ for all $t > t_0$, then the state of the system is $\boldsymbol{q}(t+\vartheta) = \boldsymbol{q}_t(\vartheta)$ for $\vartheta \in \langle -\tau,0\rangle$ and its equilibrium solution is $\boldsymbol{q}_t(\vartheta) \equiv \mathbf{0}$.

Definition 3 *For some* $\tau > 0$ *the equilibrium solution* $\mathbf{0}$ *of system* (7) *is:*

i. *(uniformly) stable if there exists a positive definite continuous functional $v(\boldsymbol{q}_t(\theta))$ whose derivative $\dot{v}(\boldsymbol{q}_t(\theta))$ is negative semi-definite functional;*

ii. *(uniformly) asymptotically stable if there exists a positive definite upper-bounded continuous functional $v(\boldsymbol{q}_t(\theta))$, whose derivative $\dot{v}(\boldsymbol{q}_t(\theta))$ is negative definite functional;*

iii. *(uniformly) exponentially stable if there exist a positive definite continuous functional* $v(\boldsymbol{q}_t(\theta))$ *and positive real constants* $\alpha, \beta, \gamma, \delta$ *such that* $\alpha\|\boldsymbol{q}_t(\theta)\| \leqslant v(\boldsymbol{q}_t(\theta)) \leqslant \beta\|\boldsymbol{q}_t(\theta)\|$, $\dot{v}(\boldsymbol{q}_t(\theta)) \leqslant -\gamma\|\boldsymbol{q}_t(\theta)\|$, *and* $|v\boldsymbol{q}_t(\theta_1)) - v(\boldsymbol{q}_t(\theta_2))| \leqslant \delta\|\boldsymbol{q}_t(\theta_1) - \boldsymbol{q}_t(\theta_2)\|$.

Because of the time-invariance of the system equation, the above definitions in Definition 3. are in fact automatically uniform, but there exist no such equivalencies in the sense of uniform equivalency for linear systems with delays like are given in Definition 2.

If (7) is linear and time invariant, then with matrix $\boldsymbol{A} \in \Re^{n\times n}$, $\boldsymbol{B} \in \Re^{n\times r}$ (7) can be rewritten as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{q}(t-\tau). \tag{11}$$

Since into the control law (4) admit all state variables the are no relative states to some specific state for the remainder of the system.

It can be naturally supposed that the linear controller (8) was designed so that the controlled system (7), (8) free of delay ($\tau \equiv 0$) is asymptotically stable. With this assumption the controlled system be delay-dependent stable, if is unstable for some values of $\tau > 0$. The task of analyze of such controlled system is finding a bound on the delay size which still ensures the stability property, while in the presented NCS formulation, the task is to obtain such as large as possible sub-optimal (maximal allowable) one, introducing the event-time-driven switching (controlled and autonomous) modes. Generally, for the time-varying delay case, delay-dependent stability means that the stability property holds for any continuous (or piece-wise continuous) and bounded time-varying delay function, with any positive and finite bound.

2.3. Basic preliminaries

Lemma 1 (Schur Complement) *Considering matrices* $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{R} = \boldsymbol{R}^T$, $\boldsymbol{S}$ *of appropriate dimensions where* $\det\boldsymbol{R} \neq 0$, *then the following statements are equivalent:*

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} > 0 \Leftrightarrow \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T > 0,\ \boldsymbol{R} > 0. \tag{12}$$

Proof. (e.g. see [1], [9]) Let the linear matrix inequality takes form

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} < 0. \tag{13}$$

Thus, using Gauss elimination it yields

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{S}\boldsymbol{R}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \tag{14}$$

where $\boldsymbol{I}$ is the identity matrix of appropriate dimension. Since

$$\det \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{S}\boldsymbol{R}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = 1 \tag{15}$$

given transform doesn't change the negativity of (13), i.e. yields (12). □

Lemma 2 (Null Complement) *Let* $\boldsymbol{z}(t)$ *is an arbitrary vector,* $\boldsymbol{W}$ *is any nonzero matrix of appropriate dimension and* $\boldsymbol{x}(t)$ *is a differentiable vector function, then*

$$\boldsymbol{z}^T(t)\boldsymbol{W}\left[\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau)-\int\limits_{t-\tau}^{t}\dot{\boldsymbol{x}}(r)\mathrm{d}r\right]=0. \tag{16}$$

Proof. Since Leibniz–Newton formula

$$\int\limits_{t-\tau}^{t}\dot{\boldsymbol{x}}(r)\mathrm{d}r=\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau) \tag{17}$$

implies

$$\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau)-\int\limits_{t-\tau}^{t}\dot{\boldsymbol{x}}(r)\mathrm{d}r=0 \tag{18}$$

then yields (16), too. □

Lemma 3 (Symmetric upper-bounds inequalities) *Let* $f(\boldsymbol{x}(r),s)$, $\boldsymbol{x}(r)\in\Re^n$, $a\in\Re$, $\boldsymbol{X}>0$, $\boldsymbol{X}\in\Re^{n\times n}$ *is a real positive definite and integrable vector function of the form*

$$f(\boldsymbol{x}(r),s)=\boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r) \tag{19}$$

such that there exist well defined integrations as following

$$\int\limits_{-b}^{0}\int\limits_{t+s}^{t}f(\boldsymbol{x}(r),s)drds>0 \tag{20}$$

$$\int\limits_{t-b}^{t}f(\boldsymbol{x}(r),r-t)dr>0 \tag{21}$$

with $b>0$, $b\in\Re$, $t\in\langle 0,\infty)$, *then*

$$\int\limits_{-b}^{0}\int\limits_{t+s}^{t}\boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r)drds\geqslant\int\limits_{-b}^{0}\int\limits_{t+s}^{t}\boldsymbol{x}^T(r)drds\,(c_1^{-1}\boldsymbol{X})\int\limits_{-b}^{0}\int\limits_{t+s}^{t}\boldsymbol{x}(r)drds \tag{22}$$

where

$$\boldsymbol{c}_1=\frac{1}{a^2}(1+abe^{ab}-e^{ab}) \tag{23}$$

and

$$\int_{t-b}^{t} \boldsymbol{x}^T(r)e^{a(r-t)}\boldsymbol{X}\boldsymbol{x}(r)dr \geqslant \int_{t-b}^{t} \boldsymbol{x}^T(r)dr \left(c_2^{-1}e^{ab}\boldsymbol{X}\right) \int_{t-b}^{t} \boldsymbol{x}(r)dr \tag{24}$$

$$\boldsymbol{c}_2 = \frac{1}{a}(-1+e^{ab}). \tag{25}$$

Proof. (e.g. compare [5], [6].) Since for (19) it can be written

$$\boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r) - \boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r) = 0 \tag{26}$$

and according to Schur complement (12) it is true that

$$\begin{bmatrix} \boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r) & \boldsymbol{x}^T(r) \\ \boldsymbol{x}(r) & e^{-as}\boldsymbol{X}^{-1} \end{bmatrix} = 0. \tag{27}$$

Then the double integration of (27) leads to

$$\begin{bmatrix} \int_{-b}^{0}\int_{t+s}^{t} \boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r)\mathrm{d}r\mathrm{d}s & \int_{-b}^{0}\int_{t+s}^{t} \boldsymbol{x}^T(r)\mathrm{d}r\mathrm{d}s \\ \int_{-b}^{0}\int_{t+s}^{t} \boldsymbol{x}(r)\mathrm{d}r\mathrm{d}s & \int_{-b}^{0}\int_{t+s}^{t} e^{-as}\boldsymbol{X}^{-1}\mathrm{d}r\mathrm{d}s \end{bmatrix} \geqslant 0. \tag{28}$$

Using the equalities

$$\int_{t+s}^{t} e^{-as}\boldsymbol{X}^{-1}\mathrm{d}r = -se^{-as}\boldsymbol{X}^{-1} \tag{29}$$

$$\int_{-b}^{0} -se^{-as}\boldsymbol{X}^{-1}\mathrm{d}s = \frac{s}{a}e^{-as}\boldsymbol{X}^{-1}\Big|_{-b}^{0} - \int_{-b}^{0} \frac{1}{a}e^{-as}\boldsymbol{X}^{-1}\mathrm{d}s = \frac{1}{a^2}(sa+1)e^{-as}\boldsymbol{X}^{-1}\Big|_{-b}^{0} = c\boldsymbol{X}^{-1} \tag{30}$$

with c as given in (23), inequality (28) can be rewritten as

$$\begin{bmatrix} \int_{-b}^{0}\int_{t+s}^{t} \boldsymbol{x}^T(r)e^{as}\boldsymbol{X}\boldsymbol{x}(r)\mathrm{d}r\mathrm{d}s & \int_{-b}^{0}\int_{t+s}^{t} \boldsymbol{x}^T(r)\mathrm{d}r\mathrm{d}s \\ * & c\boldsymbol{X}^{-1} \end{bmatrix} \geqslant 0. \tag{31}$$

It is evident that (28) implies (22).

Analogously using (27) it yields

$$\begin{bmatrix} \int_{t-b}^{t} \boldsymbol{x}^T(r)e^{a(r-t)}\boldsymbol{X}\boldsymbol{x}(r)\mathrm{d}r & \int_{t-b}^{t} \boldsymbol{x}^T(r)\mathrm{d}r \\ * & \int_{t-b}^{t} e^{-a(r-t)}\boldsymbol{X}^{-1}\mathrm{d}r \end{bmatrix} \geqslant 0. \tag{32}$$

Since

$$\int_{t-b}^{t} e^{-a(r-t)}\boldsymbol{X}^{-1}\mathrm{d}r = -\frac{1}{a}e^{-a(r-t)}\boldsymbol{X}^{-1}\Big|_{t-b}^{t} = \frac{1}{a}(-1+e^{ab}) = c_2 \tag{33}$$

the following can be obtained

$$\begin{bmatrix} \int_{t-b}^{t} \boldsymbol{x}^T(r)e^{a(r-t)}\boldsymbol{X}\boldsymbol{x}(r)\mathrm{d}r & \int_{t-b}^{t} \boldsymbol{x}^T(r)\mathrm{d}r \\ * & c_2 e^{-ab}\boldsymbol{X}^{-1} \end{bmatrix} \geqslant 0 \tag{34}$$

which implies (24). This concludes the proof. □

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

3. Exponential stability of the switched system

Defining the delay-dependent Lyapunov–Krasovskii functional candidate as follows

$$\begin{aligned} &v(\boldsymbol{q}(t)) = \\ &= \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \int_{t-h}^{t} \boldsymbol{q}^T(r)e^{\alpha_1(r-t)}\boldsymbol{Q}\boldsymbol{q}(r)\mathrm{d}r + \int_{-h}^{0}\int_{t+s}^{t} \dot{\boldsymbol{q}}^T(r)e^{\alpha_1(s-t)}\boldsymbol{R}\dot{\boldsymbol{q}}(r)\mathrm{d}r\mathrm{d}s > 0 \end{aligned} \tag{35}$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$, $\boldsymbol{R} = \boldsymbol{R}^T > 0$, respectively, and evaluating derivative of $v(\boldsymbol{q}(t))$ with respect to t it can be obtained

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) &= \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \boldsymbol{q}^T(t)\boldsymbol{Q}\boldsymbol{q}(t) + h\dot{\boldsymbol{q}}^T(t)\boldsymbol{R}\dot{\boldsymbol{q}}(t) - \\ &-\alpha_1 \int_{t-h}^{t} \boldsymbol{q}^T(r)e^{\alpha_1(r-t)}\boldsymbol{Q}\boldsymbol{q}(r)\mathrm{d}r - \boldsymbol{q}^T(t-h)e^{-\alpha_1 h}\boldsymbol{Q}\boldsymbol{q}(t-h) - \\ &- \int_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)e^{-\alpha_1 h}\boldsymbol{R}\dot{\boldsymbol{q}}(r)dr - \alpha_1 \int_{-h}^{0}\int_{t+v}^{t} \dot{\boldsymbol{q}}^T(r)e^{\alpha_1(s-t)}\boldsymbol{R}\dot{\boldsymbol{q}}(r)\mathrm{d}r\mathrm{d}s < 0. \end{aligned} \tag{36}$$

3.1. Controlled mode

Theorem 1 *If there exist matrices* $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$, $\boldsymbol{S} \geqslant 0$, $\boldsymbol{V}$ *and scalars* $h > 0$, $\alpha_1 > 0$ *such that*

$$\begin{bmatrix} \boldsymbol{S} & \boldsymbol{V} \\ * & e^{-\alpha_1 h}\boldsymbol{R} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{V}_1 \\ * & \boldsymbol{S}_{22} & \boldsymbol{V}_2 \\ * & * & e^{-\alpha_1 h}\boldsymbol{R} \end{bmatrix} > 0 \tag{37}$$

$$\begin{bmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} & \boldsymbol{U}_{13} \\ * & \boldsymbol{U}_{22} & \boldsymbol{U}_{23} \\ * & * & \boldsymbol{U}_{33} \end{bmatrix} < 0 \tag{38}$$

where

$$\boldsymbol{U}_{11} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} + \boldsymbol{V}_1 + \boldsymbol{V}_1^T + h\boldsymbol{S}_{11} + \alpha_1\boldsymbol{P} \tag{39}$$

$$\boldsymbol{U}_{12} = -\boldsymbol{P}\boldsymbol{B}\boldsymbol{K} - \boldsymbol{V}_1 + \boldsymbol{V}_2^T + h\boldsymbol{S}_{12} \tag{40}$$

$$\boldsymbol{U}_{22} = -\boldsymbol{V}_2 - \boldsymbol{V}_2^T + h\boldsymbol{S}_{22} + e^{-\alpha_1 h}\boldsymbol{Q} \tag{41}$$

$$\boldsymbol{U}_{13} = h\boldsymbol{A}^T\boldsymbol{R} \tag{42}$$

$$\boldsymbol{U}_{23} = -h\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{R} \tag{43}$$

$$\boldsymbol{U}_{33} = -h\boldsymbol{R} \tag{44}$$

then for Lyapunov-Krasovskii functional (35) along the controlled system trajectory it holds

$$v(\boldsymbol{q}(t)) < e^{-\alpha_1(t-t_0)}v(\boldsymbol{q}(t_0)), \quad t_0 = i_k\Delta t. \tag{45}$$

Proof. Since in this case the derivative of Lyapunov–Krasovskii functional takes form (36), then it implies

$$\begin{gathered} \dot{v}(\boldsymbol{q}(t)) + \alpha_1 v(\boldsymbol{q}(t)) = \alpha_1\boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \boldsymbol{q}^T(t)\boldsymbol{Q}\boldsymbol{q}(t) - \\ -\boldsymbol{q}^T(t-h)e^{-\alpha_1 h}\boldsymbol{Q}\boldsymbol{q}(t-h) + h\dot{\boldsymbol{q}}^T(t)\boldsymbol{R}\dot{\boldsymbol{q}}(t) - \int_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)e^{-\alpha_1 h}\boldsymbol{R}\dot{\boldsymbol{q}}(r)dr < 0. \end{gathered} \tag{46}$$

With known matrix $\boldsymbol{K}$ of the control law (5) it can be written

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{q}(t-h). \tag{47}$$

Defining

$$\boldsymbol{w}^T(t) = \begin{bmatrix} \boldsymbol{q}(t)^T & \boldsymbol{q}^T(t-h) \end{bmatrix} \tag{48}$$

$$\boldsymbol{s}^T(t,r) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{q}^T(t-h) & \dot{\boldsymbol{q}}^T(r) \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}^T(t) & \dot{\boldsymbol{q}}^T(r) \end{bmatrix} \tag{49}$$

with (16) it holds

$$\boldsymbol{w}^T(t)\boldsymbol{V}\left[\boldsymbol{q}(t) - \boldsymbol{q}(t-h) - \int_{t-h}^{t} \dot{\boldsymbol{q}}(r)dr\right] + \left[\boldsymbol{q}(t) - \boldsymbol{q}(t-h) - \int_{t-h}^{t} \dot{\boldsymbol{q}}(r)dr\right]^T \boldsymbol{V}^T\boldsymbol{w}(t) = 0 \tag{50}$$

$$\begin{gathered} \boldsymbol{w}^T(t)\boldsymbol{V}\begin{bmatrix} 1 & -1 \end{bmatrix}\boldsymbol{w}(t) + \boldsymbol{w}^T(t)\begin{bmatrix} 1 & -1 \end{bmatrix}^T\boldsymbol{V}^T\boldsymbol{w}(t) - \\ -\boldsymbol{w}^T(t)\boldsymbol{V}\int_{t-h}^{t} \dot{\boldsymbol{q}}(r)\mathrm{d}r - \int_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)\mathrm{d}r\,\boldsymbol{V}^T\boldsymbol{w}(t) = 0 \end{gathered} \tag{51}$$

respectively, where

$$\boldsymbol{V}^T = \begin{bmatrix} \boldsymbol{V}_1^T & \boldsymbol{V}_2^T \end{bmatrix} \tag{52}$$

$$\boldsymbol{w}^T(t)\boldsymbol{V}\begin{bmatrix} 1 & -1 \end{bmatrix}\boldsymbol{w}(t) + \boldsymbol{w}^T(t)\begin{bmatrix} 1 & -1 \end{bmatrix}^T\boldsymbol{V}^T\boldsymbol{w}(t) = \boldsymbol{w}^T(t)\boldsymbol{U}_1\boldsymbol{w}(t) \tag{53}$$

$$\boldsymbol{U}_1 = \boldsymbol{V}\begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\boldsymbol{V}^T = \begin{bmatrix} \boldsymbol{V}_1 + \boldsymbol{V}_1^T & -\boldsymbol{V}_1 + \boldsymbol{V}_2^T \\ * & -\boldsymbol{V}_2 - \boldsymbol{V}_2^T \end{bmatrix}. \tag{54}$$

Using (48) it implies, too

$$\boldsymbol{q}^T(t-h)e^{-\alpha_1 h}\boldsymbol{Q}\boldsymbol{q}(t-h) = \boldsymbol{w}^T(t)\boldsymbol{U}_2\boldsymbol{w}(t) \tag{55}$$

where

$$\boldsymbol{U}_2 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ * & e^{-\alpha_1 h}\boldsymbol{Q} \end{bmatrix}. \tag{56}$$

Also it is possible to write

$$h\dot{\boldsymbol{q}}^T(t)\boldsymbol{R}\dot{\boldsymbol{q}}(t) = \boldsymbol{w}^T(t)\begin{bmatrix} \boldsymbol{A}^T \\ -\boldsymbol{K}^T\boldsymbol{B}^T \end{bmatrix} h\boldsymbol{R}\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{K}\boldsymbol{B} \end{bmatrix}\boldsymbol{w}(t) = \boldsymbol{w}^T(t)\boldsymbol{U}_3\boldsymbol{w}(t) \tag{57}$$

$$\boldsymbol{U}_3 = \begin{bmatrix} h\boldsymbol{A}^T\boldsymbol{R} \\ -h\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{R} \end{bmatrix}(h\boldsymbol{R})^{-1}\begin{bmatrix} h\boldsymbol{R}\boldsymbol{A} & -h\boldsymbol{R}\boldsymbol{K}\boldsymbol{B} \end{bmatrix} \tag{58}$$

and

$$\begin{aligned} &\alpha_1\boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{Q}\boldsymbol{q}(t) = \boldsymbol{w}^T(t)\boldsymbol{U}_4\boldsymbol{w}(t) = \\ &= \boldsymbol{w}^T(t)\left[\begin{bmatrix} \boldsymbol{A}^T \\ -\boldsymbol{K}^T\boldsymbol{B}^T \end{bmatrix}\boldsymbol{P}\begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\boldsymbol{P}\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{K} \end{bmatrix}\right]\boldsymbol{w}(t) + \\ &\quad + \boldsymbol{w}^T(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}(\alpha_1\boldsymbol{P} + \boldsymbol{Q})\begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{w}(t) \end{aligned} \tag{59}$$

$$\boldsymbol{U}_4 = \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \alpha_1\boldsymbol{P} + \boldsymbol{Q} & -\boldsymbol{P}\boldsymbol{B}\boldsymbol{K} \\ * & \boldsymbol{0} \end{bmatrix}. \tag{60}$$

On the other hand, for $h > 0$ and any semi-positive definite matrix $\boldsymbol{S} \geqslant 0$, it is true

$$h\boldsymbol{w}^T(t)\boldsymbol{S}\boldsymbol{w}(t) - h\boldsymbol{w}^T(t)\boldsymbol{S}\boldsymbol{w}(t) = h\boldsymbol{w}^T(t)\boldsymbol{S}\boldsymbol{w}(t) - \int_{t-h}^{t}\boldsymbol{w}^T(t)\boldsymbol{S}\boldsymbol{w}(t)\mathrm{d}r = 0 \tag{61}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ * & \boldsymbol{S}_{22} \end{bmatrix} \geqslant 0 \tag{62}$$

which offer the possibility to combine elements in the integrals in (46), (51), and (62) as follows

$$\boldsymbol{w}^T(t)\boldsymbol{S}\boldsymbol{w}(t) + \dot{\boldsymbol{q}}^T(r)e^{-\alpha_1 h}\boldsymbol{R}\dot{\boldsymbol{q}}(r) + \dot{\boldsymbol{q}}^T(r)\boldsymbol{V}^T\boldsymbol{w}(t) + \boldsymbol{w}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(r) = \boldsymbol{s}^T(t,r)\boldsymbol{S}^\bullet\boldsymbol{s}(t,r) \tag{63}$$

$$\boldsymbol{S}^\bullet = \begin{bmatrix} \boldsymbol{S} & \boldsymbol{V} \\ * & e^{-\alpha_1 h}\boldsymbol{R} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{V}_1 \\ * & \boldsymbol{S}_{22} & \boldsymbol{V}_2 \\ * & * & e^{-\alpha_1 h}\boldsymbol{R} \end{bmatrix}. \tag{64}$$

Inequality (46), together with (50) and (61) can be written now in the form

$$\dot{v}(\boldsymbol{q}(t)) + \alpha_1 v(\boldsymbol{q}(t)) \leqslant \boldsymbol{w}^T(t)\boldsymbol{U}^\bullet\boldsymbol{w}(t) - \int\limits_{t-h}^{t} \boldsymbol{s}^T(t,r)\boldsymbol{S}^\bullet\boldsymbol{s}(t,r)dr < 0 \tag{65}$$

where

$$\boldsymbol{U}^\bullet = h\boldsymbol{S} + \boldsymbol{U}_1 + \boldsymbol{U}_2 + \boldsymbol{U}_3 + \boldsymbol{U}_4. \tag{66}$$

It is evident, that (65) is negative if

$$\boldsymbol{U}^\bullet < 0, \qquad \boldsymbol{S}^\bullet \geqslant 0. \tag{67}$$

Using Schur complement property (66) can be partitioned as

$$\begin{bmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} & \boldsymbol{U}_{13} \\ * & \boldsymbol{U}_{22} & \boldsymbol{U}_{23} \\ * & * & \boldsymbol{U}_{33} \end{bmatrix} \tag{68}$$

where

$$\boldsymbol{U}_{11} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{V}_1 + \boldsymbol{V}_1^T + h\boldsymbol{S}_{11} + \alpha_1\boldsymbol{P} + \boldsymbol{Q} \tag{69}$$

$$\boldsymbol{U}_{12} = -\boldsymbol{P}\boldsymbol{B}\boldsymbol{K} - \boldsymbol{V}_1 + \boldsymbol{V}_2^T + h\boldsymbol{S}_{12} \tag{70}$$

$$\boldsymbol{U}_{22} = -\boldsymbol{V}_2 - \boldsymbol{V}_2^T + h\boldsymbol{S}_{22} + e^{-\alpha_1 h}\boldsymbol{Q} \tag{71}$$

$$\boldsymbol{U}_{13} = h\boldsymbol{A}^T\boldsymbol{R} \tag{72}$$

$$\boldsymbol{U}_{23} = -h\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{R} \tag{73}$$

$$\boldsymbol{U}_{33} = -h\boldsymbol{R} \tag{74}$$

from which implies (37)–(44). Therefore, (65) gives

$$e^{\alpha_1 t}\dot{v}(\boldsymbol{q}(t)) + e^{\alpha_1 t}\alpha_1 v(\boldsymbol{q}(t)) < 0 \tag{75}$$

and integration of (75) from t_0 to t results in

$$\int_{t_0}^{t} (e^{\alpha_1 r}\dot{v}(\boldsymbol{q}(r)) + e^{\alpha_1 r}\alpha_1 v(\boldsymbol{q}(r)))\mathrm{d}r = e^{\alpha_1 r}v(\boldsymbol{q}(r))|_{t_0}^{t} = e^{\alpha_1 t}v(\boldsymbol{q}(t)) - e^{\alpha_1 t_0}v(\boldsymbol{q}(t_0)) < 0 \tag{76}$$

$$v(\boldsymbol{q}(t)) < e^{-\alpha_1(t-t_0)}v(\boldsymbol{q}(t_0)) \tag{77}$$

respectively, which implies (45). This concludes the proof. □

3.2. Autonomous mode

Theorem 2 *Given matrices* $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$, *and scalar* $\alpha_1 > 0$, *if there exist scalar* $\alpha_2 > 0$ *and matrices* $\boldsymbol{W}_l$, $l = 1,2,3$ *such that*

$$\begin{bmatrix} \boldsymbol{E} & -\boldsymbol{W}_1 + \boldsymbol{W}_2^T & hd_1\boldsymbol{R} + \boldsymbol{W}_3^T & h\boldsymbol{W}_1 \\ * & -e^{-\alpha_1 h}\boldsymbol{Q} - \boldsymbol{W}_2 - \boldsymbol{W}_2^T & -\boldsymbol{W}_3^T & h\boldsymbol{W}_2 \\ * & * & -d_1\boldsymbol{R} - d_2\boldsymbol{Q} & h\boldsymbol{W}_3 \\ * & * & * & -he^{-\alpha_1 h}\boldsymbol{R} \end{bmatrix} < 0 \tag{78}$$

where

$$\boldsymbol{E} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} + \boldsymbol{Q} - \alpha_2\boldsymbol{P} - h^2 d_1\boldsymbol{R} + \boldsymbol{W}_1 + \boldsymbol{W}_1^T \tag{79}$$

$$d_i = (\alpha_1 + \alpha_2)c_i^{-1},\ i = 1,2, \quad \boldsymbol{c}_1 = \frac{1}{\alpha_1^2}(1 + \alpha_1 h e^{\alpha_1 h} - e^{\alpha_1 h}),\ \boldsymbol{c}_2 = -\frac{1}{\alpha_1}(1 - e^{\alpha_1 h}) \tag{80}$$

then for Lyapunov-Krasovskii functional (35) along the autonomous system trajectory it yields

$$v(\boldsymbol{q}(t)) < e^{\alpha_2(t-t_0)}v(\boldsymbol{q}(t_0)), \quad t_0 = j_k. \tag{81}$$

Proof. Using the autonomous system equation implying from (1) as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) \tag{82}$$

this follows after substitution into (36)

$$\begin{aligned} &\dot{v}(\boldsymbol{q}(t)) - \alpha_2 v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} + \boldsymbol{Q} - \alpha_2\boldsymbol{P}\right)\boldsymbol{q}(t) - \\ &-\boldsymbol{q}^T(t-h)e^{-\alpha_1 h}\boldsymbol{Q}\boldsymbol{q}^T(t-h)) - \int_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)e^{-\alpha_1 h}\boldsymbol{R}\dot{\boldsymbol{q}}(r)dr - (\alpha_1 + \alpha_2)\dot{v}^{\bullet}(\boldsymbol{q}(t)) < 0 \end{aligned} \tag{83}$$

where

$$\dot{v}^{\bullet}(\boldsymbol{q}(t)) = \int_{-h}^{0}\int_{t+s}^{t} \dot{\boldsymbol{q}}^T(r)e^{\alpha_1(s-t)}\boldsymbol{R}\dot{\boldsymbol{q}}(r)\mathrm{d}r\mathrm{d}s + \int_{t-h}^{t} \boldsymbol{q}^T(r)e^{\alpha_1(r-t)}\boldsymbol{Q}\boldsymbol{q}(r)\mathrm{d}r. \tag{84}$$

Using (22), (23) and (24) it yields

$$\begin{aligned}\dot v^{\bullet}(\boldsymbol{q}(t)) \geqslant \dot v^{\circ}(\boldsymbol{q}(t)) = \dot v^{\circ}_R(\boldsymbol{q}(t)) + \dot v^{\circ}_Q(\boldsymbol{q}(t)) = \\ = \int_{-h}^{0}\int_{t+s}^{t} \dot{\boldsymbol{q}}^T(r)\mathrm{d}r\mathrm{d}s\,(c_1^{-1}\boldsymbol{R}) \int_{-h}^{0}\int_{t+s}^{t} \dot{\boldsymbol{q}}(r)\mathrm{d}r\mathrm{d}s + \int_{t-h}^{t} \boldsymbol{q}^T(r)\mathrm{d}r\,(c_2^{-1}\boldsymbol{Q}) \int_{t-h}^{t} \boldsymbol{q}(r)\mathrm{d}r\end{aligned} \tag{85}$$

where notations (80) was introduced. Since it can be written

$$\int_{-h}^{0}\int_{t+v}^{t} \dot{\boldsymbol{q}}(r)\mathrm{d}r\mathrm{d}v = \int_{-h}^{0} \big(\boldsymbol{q}(t) - \boldsymbol{q}(t+v)\big)\mathrm{d}v = h\boldsymbol{q}(t) - \int_{-h}^{0} \boldsymbol{q}(t+v)\mathrm{d}v = h\boldsymbol{q}(t) - \int_{t-h}^{t} \boldsymbol{q}(r)\mathrm{d}r \tag{86}$$

then

$$\dot v^{\circ}_R(\boldsymbol{q}(t)) = \left[h\boldsymbol{q}(t) - \int_{t-h}^{t} \boldsymbol{q}(r)\mathrm{d}r\right]^T \boldsymbol{c}_1^{-1}\boldsymbol{R} \left[h\boldsymbol{q}(t) - \int_{t-h}^{t} \boldsymbol{q}(r)\mathrm{d}r\right] \tag{87}$$

and with notation

$$\boldsymbol{p}^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{q}^T(t-h) & \int_{t-h}^{t} \boldsymbol{q}^T(r)\mathrm{d}r \end{bmatrix} \tag{88}$$

(83) takes the form

$$\begin{aligned}-(\alpha_1+\alpha_2)\dot v^{\bullet}(\boldsymbol{q}(t)) = \boldsymbol{p}^T(t)\boldsymbol{T}_1\boldsymbol{p}(t) = \\ = -(\alpha_1+\alpha_2)\boldsymbol{p}^T(t)\left[\begin{bmatrix} h \\ 0 \\ -1 \end{bmatrix} \boldsymbol{c}_1^{-1}\boldsymbol{R} \begin{bmatrix} h & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{c}_2^{-1}\boldsymbol{Q} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right]\boldsymbol{p}(t)\end{aligned} \tag{89}$$

where

$$\boldsymbol{T}_1 = \begin{bmatrix} -h^2 d_1\boldsymbol{R} & \boldsymbol{0} & hd_1\boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ hd_1\boldsymbol{R} & \boldsymbol{0} & -d_1\boldsymbol{R} - d_2\boldsymbol{Q} \end{bmatrix}, \quad d_i = (\alpha_1+\alpha_2)c_i^{-1}, \quad i = 1,2. \tag{90}$$

With (88) it can be written

$$\begin{aligned}\boldsymbol{q}^T(t)\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} + \boldsymbol{Q} - \alpha_2\boldsymbol{P}\right)\boldsymbol{q}(t) - \boldsymbol{q}^T(t-h))e^{-\alpha_1 h}\boldsymbol{Q}\boldsymbol{q}^T(t-h)) = \\ = \boldsymbol{p}^T(t)\boldsymbol{T}_2\boldsymbol{p}(t)\end{aligned} \tag{91}$$

where

$$\boldsymbol{T}_2 = \begin{bmatrix} \boldsymbol{E}^{\circ} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -e^{-\alpha_1 h}\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{E}^{\circ} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} - \alpha_2\boldsymbol{P} + \boldsymbol{Q}. \tag{92}$$

Null constraint (16) can be adapted into this solution in the next forms

$$\boldsymbol{p}^T(t)\boldsymbol{W}\left[\boldsymbol{q}(t)-\boldsymbol{q}(t-h)-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}(r)\mathrm{d}r\right]+\left[\boldsymbol{q}^T(t)-\boldsymbol{q}^T(t-h)-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)\mathrm{d}r\right]^T\boldsymbol{W}^T\boldsymbol{p}(t)=0 \tag{93}$$

$$\begin{gathered}\boldsymbol{p}^T(t)\boldsymbol{W}\left[\begin{array}{ccc}1 & -1 & 0\end{array}\right]\boldsymbol{p}(t)+\boldsymbol{p}^T(t)\left[\begin{array}{ccc}1 & -1 & 0\end{array}\right]^T\boldsymbol{W}^T\boldsymbol{p}(t)-\\ -\boldsymbol{p}^T(t)\boldsymbol{W}\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}(r)\mathrm{d}r-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)\mathrm{d}r\boldsymbol{W}^T\boldsymbol{p}(t)=\\ =\boldsymbol{p}^T(t)\boldsymbol{T}_3\boldsymbol{p}(t)-\boldsymbol{p}^T(t)\boldsymbol{W}\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}(r)\mathrm{d}r-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)\mathrm{d}r\boldsymbol{W}^T\boldsymbol{p}(t)=0\end{gathered} \tag{94}$$

respectively, where

$$\boldsymbol{W}^T=\left[\begin{array}{ccc}\boldsymbol{W}_1^T & \boldsymbol{W}_2^T & \boldsymbol{W}_3^T\end{array}\right] \tag{95}$$

$$\boldsymbol{T}_3=\boldsymbol{W}\left[\begin{array}{ccc}1 & -1 & 0\end{array}\right]+\left[\begin{array}{c}1\\ -1\\ 0\end{array}\right]\boldsymbol{W}^T=\left[\begin{array}{ccc}\boldsymbol{W}_1+\boldsymbol{W}_1^T & -\boldsymbol{W}_1+\boldsymbol{W}_2^T & \boldsymbol{W}_3^T\\ \boldsymbol{W}_2-\boldsymbol{W}_1^T & -\boldsymbol{W}_2-\boldsymbol{W}_2^T & -\boldsymbol{W}_3^T\\ \boldsymbol{W}_3 & -\boldsymbol{W}_3 & \boldsymbol{0}\end{array}\right]. \tag{96}$$

Therefore, inequality (83) together with (93) takes form

$$\begin{gathered}\dot{v}(\boldsymbol{q}(t))-\alpha_2 v(\boldsymbol{q}(t))\leqslant\boldsymbol{p}^T(t)\boldsymbol{T}_{123}\boldsymbol{p}(t)-\\ -\boldsymbol{p}^T(t)\boldsymbol{W}\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}(r)\mathrm{d}r-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)\mathrm{d}r\boldsymbol{W}^T\boldsymbol{p}(t)-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)e^{-\alpha_1 h}\boldsymbol{R}\dot{\boldsymbol{q}}(r)\mathrm{d}r<0\end{gathered} \tag{97}$$

where

$$\boldsymbol{T}_{123}=\boldsymbol{T}_1+\boldsymbol{T}_2+\boldsymbol{T}_3. \tag{98}$$

Then it possible to denote

$$\begin{gathered}-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)e^{-\alpha_1 h}\boldsymbol{R}\dot{\boldsymbol{q}}(r)\mathrm{d}r-\boldsymbol{p}^T(t)\boldsymbol{W}\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}(r)\mathrm{d}r-\int\limits_{t-h}^{t}\dot{\boldsymbol{q}}^T(r)\mathrm{d}r\boldsymbol{W}^T\boldsymbol{p}(t)\leqslant\\ \leqslant\dot{\boldsymbol{p}}^T(t)\left[\begin{array}{c}0\\ 0\\ -1\end{array}\right](-c_3^{-1}\boldsymbol{R})\left[\begin{array}{ccc}0 & 0 & -1\end{array}\right]\dot{\boldsymbol{p}}(t)+\\ +\boldsymbol{p}^T(t)\boldsymbol{W}\left[\begin{array}{ccc}0 & 0 & -1\end{array}\right]\dot{\boldsymbol{p}}(t)+\dot{\boldsymbol{p}}^T(t)\left[\begin{array}{c}0\\ 0\\ -1\end{array}\right]\boldsymbol{W}^T\boldsymbol{p}(t)=\dot{v}^{\diamond}(t)\end{gathered} \tag{99}$$

where

$$c_3=he^{\alpha_1 h}. \tag{100}$$

Completing the right side of (99) to square with notation

$$\boldsymbol{Z} = -c_3^{-1}\boldsymbol{R} \tag{101}$$

gives

$$\begin{aligned} \dot{v}_1(t) &= -\boldsymbol{p}^T(t)\boldsymbol{W}\boldsymbol{Z}^{-1}\boldsymbol{W}^T\boldsymbol{p}(t)+ \\ &+\left[\dot{\boldsymbol{p}}^T(t)\begin{bmatrix}0\\0\\-1\end{bmatrix}\boldsymbol{Z}+\boldsymbol{p}^T(t)\boldsymbol{W}\right]\boldsymbol{Z}^{-1}\left[\boldsymbol{W}^T\boldsymbol{p}(t)+\boldsymbol{Z}\begin{bmatrix}0 & 0 & -1\end{bmatrix}\dot{\boldsymbol{p}}(t)\right] = \\ &= -\boldsymbol{p}^T(t)\boldsymbol{W}\boldsymbol{Z}^{-1}\boldsymbol{W}^T\boldsymbol{p}(t)+\dot{\boldsymbol{\theta}}(t) \end{aligned} \tag{102}$$

where

$$\dot{\boldsymbol{\theta}}(t) = \left[\dot{\boldsymbol{p}}^T(t)\begin{bmatrix}0\\0\\-1\end{bmatrix}\boldsymbol{Z}+\boldsymbol{p}^T(t)\boldsymbol{W}\right]\boldsymbol{Z}^{-1}\left[\boldsymbol{W}^T\boldsymbol{p}(t)+\boldsymbol{Z}\begin{bmatrix}0 & 0 & -1\end{bmatrix}\dot{\boldsymbol{p}}(t)\right]. \tag{103}$$

Since for $\boldsymbol{Z}<0$ is $\dot{\boldsymbol{\theta}}(t)<0$, it is obvious that

$$\dot{v}(\boldsymbol{q}(t))-\alpha_2 v(\boldsymbol{q}(t)) \leqslant \dot{\boldsymbol{\theta}}(t)+\boldsymbol{p}^T(t)\boldsymbol{T}^{\bullet}\boldsymbol{p}(t)<0 \tag{104}$$

if

$$\boldsymbol{T}^{\bullet} = \boldsymbol{T}_1+\boldsymbol{T}_2+\boldsymbol{T}_3-\boldsymbol{W}\boldsymbol{Z}^{-1}\boldsymbol{W}^T<0. \tag{105}$$

Using Schur complement property for $\boldsymbol{W}\boldsymbol{Z}^{-1}\boldsymbol{W}^T$, i.e.

$$-\boldsymbol{W}\boldsymbol{Z}^{-1}\boldsymbol{W}^T = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{W}\\ * & \boldsymbol{Z}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{W}\\ * & -h^{-1}e^{-\alpha_1 h}\boldsymbol{R}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0} & h\boldsymbol{W}\\ * & -he^{-\alpha_1 h}\boldsymbol{R}\end{bmatrix} \tag{106}$$

and combining it with (90), (92), and (96), inequality (105) can now be rewritten as follows

$$\boldsymbol{T}^{\bullet} = \begin{bmatrix}\boldsymbol{E} & -\boldsymbol{W}_1+\boldsymbol{W}_2^T & hd_1\boldsymbol{R}+\boldsymbol{W}_3^T & h\boldsymbol{W}_1\\ * & -e^{-\alpha_1 h}\boldsymbol{Q}-\boldsymbol{W}_2-\boldsymbol{W}_2^T & -\boldsymbol{W}_3^T & h\boldsymbol{W}_2\\ * & * & -d_1\boldsymbol{R}-d_2\boldsymbol{Q} & h\boldsymbol{W}_3\\ * & * & * & -he^{-\alpha_1 h}\boldsymbol{R}\end{bmatrix}<0 \tag{107}$$

$$\boldsymbol{E} = \boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}+h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A}+\boldsymbol{Q}-\alpha_2\boldsymbol{P}-h^2d_1\boldsymbol{R}+\boldsymbol{W}_1+\boldsymbol{W}_1^T \tag{108}$$

from whose imply (78), (79). Therefore it also holds from (104)

$$e^{-\alpha_2 t}\dot{v}(\boldsymbol{q}(t))-e^{-\alpha_2 t}\alpha_2 v(\boldsymbol{q}(t))<0. \tag{109}$$

Integrating (109) from t_0 to t results in the formula

$$\begin{gathered}\int_{t_0}^{t} e^{-\alpha_2 r}\dot{v}(\boldsymbol{q}(r))\mathrm{d}r - \int_{t_0}^{t} e^{-\alpha_2 r}\alpha_2 v(\boldsymbol{q}(r))\mathrm{d}r = e^{-\alpha_2 r}v(\boldsymbol{q}(r))|_{t_0}^{t} = \\ = e^{-\alpha_2 t}v(\boldsymbol{q}(t)) - e^{-\alpha_2 t_0}v(\boldsymbol{q}(t_0)) < 0\end{gathered} \tag{110}$$

from which implies (81). This concludes the proof. □

3.3. Derivatives

Using l'Hopital (Bernoulli) rule, it is evident that

$$\lim_{\alpha_1 \to 0} d_1 = \lim_{\alpha_1 \to 0} \frac{f_1(\alpha_1)}{g_1(\alpha_1)} = \lim_{\alpha_1 \to 0} \frac{f_1''(\alpha_1)}{g_1''(\alpha_1)} = 2\frac{\alpha_2}{h^2} \tag{111}$$

$$\lim_{\alpha_1 \to 0} d_2 = \lim_{\alpha_1 \to 0} \frac{f_2(\alpha_1)}{g_2(\alpha_1)} = \lim_{\alpha_1 \to 0} \frac{f_1'(\alpha_1)}{g_1'(\alpha_1)} = \frac{\alpha_2}{h} \tag{112}$$

where

$$f_1(\alpha_1) = (\alpha_1 + \alpha_2)\alpha_1^2, \qquad g_1(\alpha_1) = 1 + \alpha_1 h e^{\alpha_1 h} - e^{\alpha_1 h} \tag{113}$$

$$f_2(\alpha_1) = -(\alpha_1 + \alpha_2)\alpha_1, \qquad g_2(\alpha_1) = 1 - e^{\alpha_1 h} \tag{114}$$

respectively. These imply the next corollary

Corollary 1

- *If for* $\alpha_1 = 0$ *there exist matrices* $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$, $\boldsymbol{S} \geqslant 0$, $\boldsymbol{V}$ *and scalar* $h > 0$ *such that*

$$\begin{bmatrix} \boldsymbol{S} & \boldsymbol{V} \\ * & \boldsymbol{R} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{V}_1 \\ * & \boldsymbol{S}_{22} & \boldsymbol{V}_2 \\ * & * & \boldsymbol{R} \end{bmatrix} > 0 \tag{115}$$

$$\begin{bmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} & \boldsymbol{U}_{13} \\ * & \boldsymbol{U}_{22} & \boldsymbol{U}_{23} \\ * & * & \boldsymbol{U}_{33} \end{bmatrix} < 0 \tag{116}$$

where

$$\boldsymbol{U}_{11} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} + \boldsymbol{V}_1 + \boldsymbol{V}_1^T + h\boldsymbol{S}_{11} \tag{117}$$

$$\boldsymbol{U}_{12} = -\boldsymbol{P}\boldsymbol{B}\boldsymbol{K} - \boldsymbol{V}_1 + \boldsymbol{V}_2^T + h\boldsymbol{S}_{12} \tag{118}$$

$$\boldsymbol{U}_{22} = -\boldsymbol{V}_2 - \boldsymbol{V}_2^T + h\boldsymbol{S}_{22} + \boldsymbol{Q} \tag{119}$$

$$\boldsymbol{U}_{13} = h\boldsymbol{A}^T\boldsymbol{R} \tag{120}$$

$$\boldsymbol{U}_{23} = -h\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{R} \tag{121}$$

$$\boldsymbol{U}_{33} = -h\boldsymbol{R} \tag{122}$$

then for Lyapunov-Krasovskii functional (35) *along the controlled system trajectory it yields*

$$v(\boldsymbol{q}(t)) < v(\boldsymbol{q}(t_0)), \quad t_0 = i_k\Delta t. \tag{123}$$

- *Given matrices* $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$ *and scalar* $h > 0$, *if there exist scalar* $\alpha_2 > 0$ *and matrices* $\boldsymbol{W}_l$, $l = 1,2,3$ *such that*

$$\begin{bmatrix} \boldsymbol{E}_2 & -\boldsymbol{W}_1 + \boldsymbol{W}_2^T & 2\alpha_2\boldsymbol{R} + h\boldsymbol{W}_3^T & h\boldsymbol{W}_1 \\ * & -\boldsymbol{Q} - \boldsymbol{W}_2 - \boldsymbol{W}_2^T & -h\boldsymbol{W}_3^T & h\boldsymbol{W}_2 \\ * & * & -2\alpha_2\boldsymbol{R} - \alpha_2 h\boldsymbol{Q} & h\boldsymbol{W}_3 \\ * & * & * & -h\boldsymbol{R} \end{bmatrix} < 0 \tag{124}$$

where

$$\boldsymbol{E}_2 = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} + \boldsymbol{Q} - \alpha_2\boldsymbol{P} - 2\alpha_2\boldsymbol{R} + \boldsymbol{W}_1 + \boldsymbol{W}_1^T \tag{125}$$

then for Lyapunov-Krasovskii functional (35) *and* $\alpha_1 = 0$ *along the autonomous system trajectory it yields*

$$v(\boldsymbol{q}(t)) < e^{\alpha_2(t-t_0)}v(\boldsymbol{q}(t_0)), \quad t_0 = j_k. \tag{126}$$

Remark 1 It is evident that with $\alpha_1 = \alpha_2 = 0$ matrix (98) is a singular matrix and the strategy starting with (99) cannot be applied. In this case it is possible to define the vector

$$\boldsymbol{r}^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{q}^T(t-h) & \int\limits_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)\mathrm{d}r \end{bmatrix} \tag{127}$$

and to adapt null constraint (16) into solution as follows

$$\boldsymbol{r}^T(t)\boldsymbol{H}\left[\boldsymbol{q}(t) - \boldsymbol{q}(t-h) - \int\limits_{t-h}^{t} \dot{\boldsymbol{q}}(r)\mathrm{d}r\right] + \left[\boldsymbol{q}^T(t) - \boldsymbol{q}^T(t-h) - \int\limits_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)\mathrm{d}r\right]^T \boldsymbol{H}^T\boldsymbol{r}(t) = 0 \tag{128}$$

$$\boldsymbol{r}^T(t)\boldsymbol{H}\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}\boldsymbol{r}(t) + \boldsymbol{r}^T(t)\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T \boldsymbol{H}^T\boldsymbol{r}(t) = \boldsymbol{r}^T(t)\boldsymbol{T}_3^{\circ}\boldsymbol{r}(t) = 0 \tag{129}$$

respectively, where

$$\boldsymbol{H}^T = \begin{bmatrix} \boldsymbol{H}_1^T & \boldsymbol{H}_2^T & \boldsymbol{H}_3^T \end{bmatrix} \tag{130}$$

$$\boldsymbol{T}_3^{\circ} = \boldsymbol{H}\begin{bmatrix} 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\boldsymbol{H}^T = \begin{bmatrix} \boldsymbol{H}_1 + \boldsymbol{H}_1^T & -\boldsymbol{H}_1 + \boldsymbol{H}_2^T & -\boldsymbol{H}_1 + \boldsymbol{W}_3^T \\ * & -\boldsymbol{H}_2 - \boldsymbol{H}_2^T & -\boldsymbol{H}_2 - \boldsymbol{H}_3^T \\ * & * & -\boldsymbol{H}_3 - \boldsymbol{H}_3^T \end{bmatrix}. \tag{131}$$

Thus, writing the next

$$\int_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)\boldsymbol{R}\dot{\boldsymbol{q}}(r)\mathrm{d}r \geqslant \int_{t-h}^{t} \dot{\boldsymbol{q}}^T(r)\mathrm{d}r(h^{-1}\boldsymbol{R}) \int_{t-h}^{t} \dot{\boldsymbol{q}}(r)\mathrm{d}r = \boldsymbol{r}^T(t)\boldsymbol{T}_4\boldsymbol{r}(t) \tag{132}$$

where

$$\boldsymbol{T}_4 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & h^{-1}\boldsymbol{R} \end{bmatrix} \tag{133}$$

it is evident, that

$$\dot{v}(\boldsymbol{q}(t)) \leqslant \boldsymbol{r}^T(t)\boldsymbol{T}^\circ\boldsymbol{r}(t) < 0 \tag{134}$$

where

$$\boldsymbol{T}^\circ = \boldsymbol{T}_2 + \boldsymbol{T}_3^\circ - \boldsymbol{T}_4 < 0. \tag{135}$$

This remark implies the next corollary.

Corollary 2

- *Given matrices* $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$ *and scalar* $h > 0$ *satisfying* (115) *through* (122)*, if there exist matrices* $\boldsymbol{H}_l$, $l = 1,2,3$ *such that*

$$\begin{bmatrix} \boldsymbol{E}_{00} & -\boldsymbol{H}_1 + \boldsymbol{H}_2^T & -h(\boldsymbol{H}_1 - \boldsymbol{H}_3^T) \\ * & -\boldsymbol{Q} - \boldsymbol{H}_2 - \boldsymbol{H}_2^T & -h(\boldsymbol{H}_2 + \boldsymbol{H}_3^T) \\ * & * & -h\boldsymbol{R} - h^2(\boldsymbol{H}_3 + \boldsymbol{H}_3^T) \end{bmatrix} < 0 \tag{136}$$

where

$$\boldsymbol{E}_{00} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + h\boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} + \boldsymbol{Q} + \boldsymbol{H}_1 + \boldsymbol{H}_1^T \tag{137}$$

then for Lyapunov-Krasovskii functional (35) and $\alpha_1 = 0$, $\alpha_2 = 0$ *along the autonomous system trajectory it holds*

$$v(\boldsymbol{q}(t)) < v(\boldsymbol{q}(t_0)), \qquad t_0 = j_k. \tag{138}$$

Remark 2 Solving all matrix inequalities together, i.e. (37), (38), as well as (78), it can be obtained the average decay degree $\alpha_2^\star$ for which the switched networked control system is exponentially stable.

It is evident, that other modifications can be obtained setting $\boldsymbol{Q} = \mathbf{0}$, $\boldsymbol{R} = \mathbf{0}$, respectively.

4. Illustrative example

The numerical example is provided below to illustrate the main results. It is assumed that the parameters of the system (1), (2) are

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^T.$$

The system is unstable and defining the desired closed-loop system matrix eigenvalue spectrum

$$\rho(\boldsymbol{A} - \boldsymbol{g}\boldsymbol{k}^T) = \{-0.5,\ -5.0\ -10.0\}$$

the feedback gain vector $\boldsymbol{k}^T$ which stabilize this system was designed as follows

$$\boldsymbol{k}^T = \begin{bmatrix} 30.0000 & 56.5000 & 12.5000 \end{bmatrix}.$$

Solving (37), (38), (78) for LMI matrix variables $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$, $\boldsymbol{S} > 0$, $\boldsymbol{V}$, $\boldsymbol{W}$ by Self–Dual–Minimization (SeDuMi) package for Matlab [14], with the average design parameters of the Lyapunov–Krasovskii functional (35) initialization

$$\alpha_1 = 0.45, \quad \alpha_2 = 2.0, \quad h = 0.038$$

the problem was solved as feasible with matrices

$$\boldsymbol{P} = \begin{bmatrix} 0.0083 & 0.0037 & 0.0009 \\ 0.0037 & 0.0054 & 0.0012 \\ 0.0009 & 0.0012 & 0.0004 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} 0.0011 & 0.0017 & 0.0003 \\ 0.0017 & 0.0027 & 0.0005 \\ 0.0003 & 0.0005 & 0.0001 \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} 0.0021 & 0.0017 & 0.0005 \\ 0.0017 & 0.0051 & 0.0008 \\ 0.0005 & 0.0008 & 0.0002 \end{bmatrix}$$

$$\boldsymbol{S} = \begin{bmatrix} 1.8548 & 2.5422 & 0.8341 & -0.6326 & -0.5222 & -0.2227 \\ 2.5422 & 3.5916 & 1.1673 & -0.8149 & -0.7193 & -0.3032 \\ 0.8341 & 1.1673 & 0.3822 & -0.2737 & -0.2393 & -0.1014 \\ -0.6326 & -0.8149 & -0.2737 & 0.2659 & 0.2277 & 0.0873 \\ -0.5222 & -0.7193 & -0.2393 & 0.2277 & 0.2709 & 0.0874 \\ -0.2227 & -0.3032 & -0.1014 & 0.0873 & 0.0874 & 0.0326 \end{bmatrix}$$

$$\boldsymbol{V}=\begin{bmatrix} -0.0445 & -0.0597 & -0.0172 \\ -0.0599 & -0.0790 & -0.0235 \\ -0.0199 & -0.0271 & -0.0078 \\ 0.0090 & 0.0146 & 0.0033 \\ 0.0075 & 0.0178 & 0.0032 \\ 0.0030 & 0.0056 & 0.0012 \end{bmatrix}, \quad \boldsymbol{W}=\begin{bmatrix} -0.0735 & -0.0360 & -0.0132 \\ -0.0353 & -0.1375 & -0.0201 \\ -0.0132 & -0.0202 & -0.0057 \\ 0.0745 & 0.0367 & 0.0134 \\ 0.0355 & 0.1406 & 0.0206 \\ 0.0131 & 0.0204 & 0.0058 \\ -0.0335 & -0.0624 & -0.0129 \\ -0.0248 & -0.0987 & -0.0166 \\ -0.0031 & -0.0120 & -0.0022 \end{bmatrix}.$$

It is evident that $\boldsymbol{Q}\neq\boldsymbol{0}$ regularizes the solution.

One can apply this in the structure, where (37), (38) are used for initialization and obtained matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}>0$, $\boldsymbol{R}>0$, $\boldsymbol{S}>0$, $\boldsymbol{V}$ are included in the optimization with respect to $\alpha_2\rightarrow\alpha_{2min}$ based on (78).

5. Concluding remarks

The significance of NCS problems is tied to the recent ample interest in designing control strategy for networked systems. This paper motivations imply the next facts:

- For applications within the scope of large-scale distributed and networked systems the network induced time-delay is by nature time-varying and sometimes randomized. Generally, if a the delay τ is unknown but constant, then the energy of $\boldsymbol{q}(t-\tau)$ is the same as the energy of $\boldsymbol{q}(t)$, and a simple but conservative delay-independent stability criterion can be used for stability analyze. When the delay parameter is time-varying, stability analysis is more involved since systems with time-varying delays are comparatively less stable than those with constant time delays, and it is obvious that stability criteria for system with constant time delays cannot be easily generalized for time-varying delay systems.

- It is known that delay-independent Lyapunov-Krasovskii functional to be very conservative since it considers the delay as a norm bounded uncertainty and implies in a fact the delay part does not help for the stabilization. Therefore it was interesting to develop criterion based on the more sophisticated Lyapunov-Krasovskii functional.

We observe that the used criterion gives better stability margins without an improvement of the computational complexity of the optimization problem to be solved. Although deriving these conditions may not be a trivial matter, once they are available, using them is as simple as changing LMIs structure in program routines. Note, a simple quadratic function can be used to analyze the autonomous mode, but without direct connection to the controlled mode.

Therefore, the main objective of this paper was to present a method of determining the delay-dependent stability criteria for event–time–driven modes in the networked control system. In particular, with used Lyapunov–Krasovskii functional, there were introduced the additional design parameters to regularize LMI feasibility results and to obtain the size of the available margins under which the system can stay stable. The presented procedure was developed in such a way that simpler LMI structures can be derived directly from.

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