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On generalization of integral control to a class of nonlinear uncertain systems

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The paper concerns a new view on the problem of integral control in the context of nonlinear uncertain systems. It is demonstrated that a standard integral action applied to the linear control system with so-called load disturbances, can be generalized to comprise a class of linearly parameterized nonlinear SISO systems with functional uncertainty. In this case the integral action is turned out to be in fact an adaptation law of unknown parameters. It has been found that the obtained proportional-integral controller's variable gains are the basis functions of the system unknown nonlinearity approximator.

Key words: nonlinear control, uncertain systems, adaptive control, integral action, approximator, functional uncertainty

1. Introduction

It is well known that integral control ensures asymptotic tracking and disturbance rejection if the exogenous signals are constant or asymptotically approach constant limit. This fact is proven for linear systems as well as for wide classes of nonlinear systems (see e.g. [4] for details) in regional or semiglobal sense.

The main goal of this paper is to reinterpret the results of [10] (based on adaptive control concepts) in the framework of integral control, thereby to indicate the connections and analogies between these two seemingly different approaches.

In the paper we consider a general tracking problem of the system influenced by bounded disturbances. In order to obtain a tracking control synthesis we apply an integral action on the system output.

In the first part we consider a standard linear system regulator problem where its basic technique (e.g. pole location) for feedback controller design will be extended to include linear, disturbed tracking systems. In the second part the integral control is proposed for a class of partially known, nonlinear SISO systems. It has been demonstrated that the obtained tracking and disturbance rejection results on the basis of Lyapunov the-

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ory, may be viewed and interpreted in terms of similar form integral action as that used in the former linear case. In this view one may call it a generalized integral action.

2. Tracking via integral control – linear system case

In this section we recall a few facts related to tracking via integral control in the linear systems context and in the layout suitable for further generalization to a class of nonlinear systems. The formulation of an asymptotic tracking problem can be set as follows [6]. Let us consider a dynamical system (object, plant) described by the following equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{1a}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{1b}$$

where $\boldsymbol{y}$, $\boldsymbol{x}$, $\boldsymbol{u}$ are output, state and control vectors respectively and $\boldsymbol{y}_d$ a desired output trajectory. Find a control law for the input $\boldsymbol{u}$ such that starting from any initial state in a region Ω, the tracking error $\boldsymbol{y}(t) - \boldsymbol{y}_d(t)$ tends to zero, while the state $\boldsymbol{x}$ remains bounded.

In this section we assume that the system (1) is of the form (2) i.e. it is linear, time-invariant as well as subject of unknown, constant (or slowly-varying) load disturbances $\boldsymbol{w}$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}_u\boldsymbol{u}(t) + \boldsymbol{B}_w\boldsymbol{w} \tag{2a}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}_y\boldsymbol{x}(t). \tag{2b}$$

Assume also that $\boldsymbol{y}_d$ is constant, desired trajectory.

The problem is to design feedback control law that force the system output $\boldsymbol{y}$ to follow the desired trajectory $\boldsymbol{y}_d$ in the presence of the load disturbances $\boldsymbol{w}$. The task of the controller is therefore twofold: while tracking $\boldsymbol{y}_d$ it is also obliged to mitigate the effect of steady disturbances.

In order to ensure tracking as well as compensating the effect of disturbances we use the integral action as follows:

$$\boldsymbol{e} = \int (\boldsymbol{C}_y\boldsymbol{x} - \boldsymbol{y}_d)dt \tag{3}$$

which represents the integral of the tracking error $\boldsymbol{y} - \boldsymbol{y}_d$. The augmented state model is thus a combination of the plant state equation (2a) and the equation (3) written in differential form:

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C}_y & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u} + \begin{bmatrix} \boldsymbol{B}_w\boldsymbol{w} \\ -\boldsymbol{y}_d \end{bmatrix}. \tag{4}$$

Let us consider now the system (4) without disturbances i.e. the system of the form

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C}_y & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u} \tag{5}$$

and define, following [8], system stabilizability as a feature equivalent to the existence of a feedback control matrix $\hat{\boldsymbol{K}}$ that asymptotically stabilizes the system. Assuming that the extended system (5) is controllable (so *completely* stabilizable – see [8]) there exists a feedback matrix $\hat{\boldsymbol{K}}$ that stabilizes the system (asymptotically). Let apply this feedback law to the system (4). We have then

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C}_y & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \hat{\boldsymbol{K}}\hat{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{B}_w\boldsymbol{w} \\ -\boldsymbol{y}_d \end{bmatrix} \tag{6}$$

where

$$\boldsymbol{u} = -\hat{\boldsymbol{K}}\hat{\boldsymbol{x}} = -[\boldsymbol{K}, \boldsymbol{K}_I] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} = -\boldsymbol{K}\boldsymbol{x} - \boldsymbol{K}_I\boldsymbol{e} = -\boldsymbol{K}\boldsymbol{x} - \boldsymbol{K}_I \int (\boldsymbol{y} - \boldsymbol{y}_d)dt. \tag{7}$$

We can rewrite (6) also in the form

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}\boldsymbol{K}_I \\ \boldsymbol{C}_y & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_w\boldsymbol{w} \\ -\boldsymbol{y}_d \end{bmatrix}. \tag{8}$$

Constant forcing signal $\left[(\boldsymbol{B}_w\boldsymbol{w})^T \ \ \boldsymbol{y}_d^T\right]^T$ and asymptotic stability of unforced system (5) imply the stability of the overall system (8) which leads to the following steady state equation

$$\begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}\boldsymbol{K}_I \\ \boldsymbol{C}_y & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B}_w\boldsymbol{w} \\ \boldsymbol{y}_d \end{bmatrix}. \tag{9}$$

Assuming that the solution of the system (9) exists (well-known matrices rank equality condition) we get the extended steady state vector $\left[\boldsymbol{x}_\infty^T \ \boldsymbol{e}_\infty^T\right]^T$. From the equation (9) we can read out that $\boldsymbol{C}_y\boldsymbol{x} = \boldsymbol{y}_d$, so in the steady state we get $\boldsymbol{y}_\infty = \boldsymbol{y}_d$ i.e. the asymptotic tracking condition. Summing up we can formulate the following theorem.

Theorem 1 *If the system* (5) *is controllable (so completely stabilizable) then the feedback system* (4), (7) *(i.e. system* (8)*) realizes asymptotical tracking of the desired constant trajectory* $\boldsymbol{y}_d$ *in the presence of steady disturbances* $\boldsymbol{w}$.

The important fact from the implemental viewpoint is that no $\boldsymbol{w}$ measurements are required. Note also that in the augmented system (8) the desired trajectory $\boldsymbol{y}_d$ plays the role of additional disturbance and formally there is no difference between $\boldsymbol{w}$ and $\boldsymbol{y}_d$. In this sense there exists a duality i.e. the actual disturbances $\boldsymbol{w}$ (or their part) might be as well treated as the desired trajectory. Note as well that if $\boldsymbol{y} = \boldsymbol{x}$ (or more generally if $\mathrm{rank}(\boldsymbol{C}_y) = \dim(\boldsymbol{x})$) and $\boldsymbol{y}_d \equiv \boldsymbol{0}$, the augmentation of the equation (2a) with (3) is a formal 'trick' which causes a shift of the steady state error ensuing here from coordinates of the original state vector $\boldsymbol{x}$ to the coordinates of vector $\boldsymbol{e}$ which represents a slack (or virtual) variable. In this context one may formulate [9] the following corollary.

Corollary *If the system* (5) *is controllable,* $\mathrm{rank}(\boldsymbol{C}_y) = \dim(\boldsymbol{x})$*) and* $\boldsymbol{y}_d \equiv \boldsymbol{0}$ *then the system's* (8) *equilibrium state has the form* $\left[\boldsymbol{x}_\infty^T \ \boldsymbol{e}_\infty^T\right]^T = \left[\boldsymbol{0} \ \boldsymbol{e}_\infty^T\right]^T$ *i.e.* $\boldsymbol{x}_\infty = \boldsymbol{0}$.

Indeed, from the equation (9) we can infer that in the steady state $\dot{\boldsymbol{e}} = \boldsymbol{C}_y\boldsymbol{x} = \boldsymbol{0}$. The last equation yields (under our assumptions) an unique solution $\boldsymbol{x}_\infty = 0$ which implies the thesis.

Remark 1 Since in the steady state is $\boldsymbol{x} = \boldsymbol{0}$ therefore by writing (8) in the form

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}_I\boldsymbol{e} + \boldsymbol{B}_w\boldsymbol{w}$$

it follows that $\boldsymbol{B}_w\boldsymbol{w} - \boldsymbol{B}\boldsymbol{K}_I\boldsymbol{e} = \boldsymbol{0}$. This means that via the extra variable $\boldsymbol{e}$ we can guarantee the cancelation of disturbances.

This corollary is important especially in situations (see section 3.2), where the problem concerns asymptotic stabilization of the state vector. To solve this problem via the output integration the output equation might be introduced entirely artificial with respect to some extra assumptions.

3. Nonlinear tracking control synthesis

In this section a general tracking problem for nonlinear uncertain system (10) is considered. As a first step towards tracking control synthesis the system is transformed to the linear form with not necessarily constant disturbances (27). Next, after introduction of artificial outputs, the previous procedure of integration is applied leading to the asymptotically stabilizing controller synthesis.

Consider an affine-in-control nonlinear SISO system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{\alpha}(\boldsymbol{x}) + \boldsymbol{\beta}(\boldsymbol{x}) \cdot u \tag{10a}$$

$$y = h(\boldsymbol{x}) \tag{10b}$$

where y, $\boldsymbol{x}$, u denote output, state and control variables respectively, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are smooth vector fields on $\Re^n$ and $h : \Re^n \to \Re$ is a smooth function. It is assumed that the functions, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are unknown or may be estimated with a considerable inaccuracy.

By successive differentiation of y with respect to time it is possible to obtain [1], [5] a direct input-output relation between u and y as follows

$$y^{(r)} = f(\boldsymbol{x}) + g(\boldsymbol{x})u \tag{11}$$

where r denotes known system relative degree. The technique can be well systematized and explained using the concept of Lie derivatives [3].

The approach used in this paper assumes that the system (10) is functionally uncertain [1], which means that its unknown nonlinear characteristics are approximated by functional approximators (e.g. polynomials). However, from practical point of view, it seems reasonable to use approximator-like structures where known *model basis functions* contain some portion of the plant specific knowledge [11]. In other words, the unknown functions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the system (10a) are assumed to be the linear combinations of some known model related functions $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$and which represent elementary knowledge of the model.

One may prove the following [10] theorem.

Theorem 2 *If the functions* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *have the form*

$$\boldsymbol{\alpha}(\boldsymbol{x}) = \sum_{i=1}^{m_1} a_i \boldsymbol{\alpha}_i(x), \quad \boldsymbol{\beta}(\boldsymbol{x}) = \sum_{i=1}^{m_2} b_i \boldsymbol{\beta}_i(x) \tag{12}$$

(where a_i, b_i are real unknown parameters) then the scalar functions f, g of the system (11) *may be represented in the similar form:*

$$f(\boldsymbol{x}) = \sum_{i=1}^{n_1} \theta_i^1 f_i(\boldsymbol{x}) + f_0(\boldsymbol{x}), \quad g(\boldsymbol{x}) = \sum_{i=1}^{n_2} \theta_i^2 g_i(\boldsymbol{x}) + g_0(\boldsymbol{x}) \tag{13}$$

where θ_i^1, θ_i^2 are unknown parameters and f_i, g_i (called here the model basis functions*) are known again through the $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$.*

Since a rather complicated formulas for f_i, g_i (as functions of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$) are not principal here, we will omit them.

From practical modeling point of view it is important that the vector functions $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ may have components which can be, to some extent, assumed arbitrarily (choice of approximator may depend on our knowledge of the plant). In this respect the system (10) is not only parametrically but also functionally uncertain [1], [10].

The control objective is to force the plant (10) output vector $\boldsymbol{y} = [y, \dot{y}, \cdots, y^{(r-1)}]^T$ to follow a specified desired trajectory $\boldsymbol{y}_d = [y_d, \dot{y}_d, \cdots, y_d^{(r-1)}]^T$ with the state vector $\boldsymbol{x}$ remaining bounded. It is assumed that reference output y_d and its r derivatives are bounded and known, the system zero dynamics is globally exponentially stable (minimum phase condition) as well as the full state and output measurement are accessible. Under the second of these assumptions the model (10), (12) can be transformed, via Theorem 2, to its equivalent form (11), (13).

3.1. The case of exact model

It is assumed in this section that the nonlinear functions f and g of model (11) are known and $g(\boldsymbol{x}) \neq 0, \ \forall \boldsymbol{x} \in \Omega$. A substitution of control law

$$u = \frac{-f(\boldsymbol{x}) + v}{g(\boldsymbol{x})} \tag{14}$$

in the system (11) results in exact cancelation of both nonlinearities ($f(\boldsymbol{x})$ and $g(\boldsymbol{x})$) which yields

$$y^{(r)} = v. \tag{15}$$

To find control $v(t)$ stabilizing this linear system, a standard poles location technique can be used. If v is chosen as

$$v = y_d^{(r)} - \mu_r e^{(r-1)} - \cdots - \mu_1 e \tag{16}$$

where y_d denotes the reference input which y is required to track, $e := y - y_d$ denotes the output tracking error and coefficients μ_i are chosen such that $\Gamma(s) =: s^r + \mu_r s^{r-1} + \cdots + \mu_1 = 0$ is Hurwitz polynomial in the Laplace variable s, then the tracking error and its derivatives converge to zero asymptotically, because the closed-loop dynamics reduce to the equation

$$e^{(r)} + \mu_r e^{(r-1)} + \cdots + \mu_1 e = 0 \tag{17}$$

which, by virtue of the choice of coefficients μ_i is asymptotically stable.

3.2. The case with functional uncertainty

Let us consider now the case when functions f and g are unknown but have the form (13) with $\theta_i^1, i = 1, \cdots n_1$, $\theta_i^2, i = 1, \cdots n_2$ unknown 'true' parameters and the $f_i(\boldsymbol{x})$, $g_i(\boldsymbol{x})$ known model basis functions. At time t the estimates of the functions f and g are respectively given by:

$$\hat{f}(\boldsymbol{x}) = \sum_{i=1}^{n_1} \hat{\theta}_i^1(t) f_i(\boldsymbol{x}) + f_0(\boldsymbol{x}), \quad \hat{g}(\boldsymbol{x}) = \sum_{i=1}^{n_2} \hat{\theta}_i^2(t) g_i(\boldsymbol{x}) + g_0(\boldsymbol{x}) \tag{18}$$

where $\hat{\theta}_i^1$, $\hat{\theta}_i^2$ stands for the estimates of the parameters θ_i^1, θ_i^2, respectively at time t.

Since substitution in the system (11) the control law

$$u = \frac{-\hat{f}(\boldsymbol{x}) + v}{\hat{g}(\boldsymbol{x})} \tag{19}$$

no longer guarantees exact cancelation and thereby a resulting system linearity (like in the former case (14)), then to solve the formulated above tracking problem we derive at first the error equation. Before doing that we introduce the following notations

$$f - \hat{f} = \sum_{i=1}^{n_1} (\theta_i^1 - \hat{\boldsymbol{\theta}}_i^1) f_i(\boldsymbol{x}) = \widetilde{\boldsymbol{\theta}}^{1T} \boldsymbol{w}_1, \quad (g - \hat{g})u = \sum_{i=1}^{n_1} (\theta_i^2 - \hat{\theta}_i^2) g_i(\boldsymbol{x}) u = \widetilde{\boldsymbol{\theta}}^{2T} \boldsymbol{w}_2 \tag{20}$$

where

$$\boldsymbol{w}_1 = [f_1 \ \ f_2 \ \ \cdots \ \ f_{n_1}]^T, \quad \boldsymbol{w}_2 = [g_1 \ \ g_2 \ \ \cdots \ \ g_{n_1}]^T \tag{21}$$

are *model basis functions* and

$$\widetilde{\boldsymbol{\theta}}^1 = [(\theta_1^1 - \hat{\theta}_1^1) \cdots (\theta_{n_1}^1 - \hat{\theta}_{n_1}^1)]^T, \quad \widetilde{\boldsymbol{\theta}}^2 = [(\theta_1^2 - \hat{\theta}_1^2) \cdots (\theta_{n_2}^2 - \hat{\theta}_{n_2}^2)]^T \tag{22}$$

are vectors of parameters. Moreover $\widetilde{\boldsymbol{\theta}} = \left[\widetilde{\boldsymbol{\theta}}^{1T} \;\; \widetilde{\boldsymbol{\theta}}^{2T}\right]^T$, $\boldsymbol{w} = \left[\boldsymbol{w}_1^T \;\; \boldsymbol{w}_2^T\right]^T$.

To derive the error equations we transform (11) as follows

$$y^{(r)} - v = f + gu - v \tag{23}$$

and then using (19) we get

$$y^{(r)} - v = f + gu - \hat{f} - \hat{g}u = f - \hat{f} + (g - \hat{g})u \tag{24}$$

or finally

$$e^{(r)} + \mu_r e^{(r-1)} + \cdots + \mu_1 e = f - \hat{f} + (g - \hat{g})u. \tag{25}$$

Via definition of a new state vector

$$\boldsymbol{e} = \left[e \;\; \dot{e} \;\cdots\; e^{(r-1)}\right]^T \stackrel{df}{=} [e_1 \;\; e_2 \;\cdots\; e_r]^T \tag{26}$$

we can rewrite (25) in the following matrix form

$$\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{b}\widetilde{\boldsymbol{\theta}}^T \boldsymbol{w} \tag{27}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ -\mu_1 & -\mu_2 & \cdots & -\mu_{r-1} & -\mu_r \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{28}$$

This way the closed loop system (11), (16), (19) has been written in linear-like form (27). Now, treating the second term on the right-hand side of (27) as disturbances one might directly try to apply the technique described in section 2, provided that the disturbances were constant. Despite the fact that it is not the case we will show that after introduction of some (artificial) output the previous idea of integration works here as well. The proof of the system stability will have to be yet delivered separately.

Now we define the output variables keeping in mind the former idea that after integration they should produce slack variables on which the errors due to uncertainties cumulate. One way of defining such output variables (for the reasons to be seen later) is to use the following formula

$$\boldsymbol{z} = -\varepsilon\boldsymbol{\Gamma}\boldsymbol{w} \tag{29}$$

where $\boldsymbol{\Gamma} > 0$ is a diagonal weighting matrix (that determines further an adaptation rate – see(42)) and

$$\varepsilon(t) = \eta_1 e_1 + \cdots + \eta_{r-1} e_{r-1} + e_r := \Psi(s) \tag{30}$$

is defined later (see (35)).

Treating the nonlinear term $\widetilde{\boldsymbol{\theta}}^T\boldsymbol{w}$ (see (27)), representing the error due to uncertainties, as disturbances we may notice that the system (27), (29) is analogous to the system (2) of section 2, so the output integration technique can be applied.

As the counterpart of (3) we will write now the integral of (29) $\widetilde{\boldsymbol{\theta}} = -\int \varepsilon\boldsymbol{\Gamma}\boldsymbol{w}dt$ which in the differential formulation has a form:

$$\dot{\widetilde{\boldsymbol{\theta}}} = -\varepsilon\boldsymbol{\Gamma}\boldsymbol{w}. \tag{31}$$

Observe that, similarly as $\boldsymbol{e}$ in section 2, $\widetilde{\boldsymbol{\theta}}$, interpreted as the augmented system states, plays the role of slack (or virtual) variables on which the errors of uncertainties cumulate. Notice also that alternatively ([10], [7]) formula (31) may be viewed as the parameters adaptation law. Below, via Lyapunov method, we will prove the following theorem.

Theorem 3 *The coupled system* (27)*,* (31) *is stable whereas the subsystem* (27) *is asymptotically stable.*

Proof Since $\boldsymbol{A}$ is a stability matrix, there exists a positive definite matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{I} \tag{32}$$

where $\boldsymbol{I}$ is the identity matrix. We choose the following Lyapunov function

$$V(\boldsymbol{e},\widetilde{\boldsymbol{\theta}}) = \boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e} + \widetilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{\theta}} \tag{33}$$

whose time derivative along the trajectories of (27), (31) is given by

$$\dot{V} = \boldsymbol{e}^T\boldsymbol{P}(\boldsymbol{A}\boldsymbol{e} + \boldsymbol{b}\widetilde{\boldsymbol{\theta}}^T\boldsymbol{w}) + (\boldsymbol{e}^T\boldsymbol{A}^T + \boldsymbol{w}^T\widetilde{\boldsymbol{\theta}}\boldsymbol{b}^T)\boldsymbol{P}\boldsymbol{e} + 2\widetilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\widetilde{\boldsymbol{\theta}}}.$$

Using the fact that $\boldsymbol{P} = \boldsymbol{P}^T$ as well as that $\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{b}$ is a scalar yields

$$\dot{V} = -\boldsymbol{e}^T\boldsymbol{e} + 2\widetilde{\boldsymbol{\theta}}^T(\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{b}\boldsymbol{w} + \boldsymbol{\Gamma}^{-1}\dot{\widetilde{\boldsymbol{\theta}}}) \leqslant 0. \tag{34}$$

To make it decreasing along these trajectories and thereby establish bounded $\boldsymbol{e}$ and $\widetilde{\boldsymbol{\theta}}$, one should put $\dot{\widetilde{\boldsymbol{\theta}}} = -\boldsymbol{\gamma}\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{b}\boldsymbol{w}$, which by denoting

$$\varepsilon(t) = \boldsymbol{e}^T\boldsymbol{P}\boldsymbol{b} \tag{35}$$

leads to (31). However, to verify that $\boldsymbol{e} \to 0$ as $t \to \infty$ we use Barbalat's lemma [6]. To check the uniform continuity of $\dot{V}$ it is enough to prove that the second derivative of V i.e.

$$\ddot{V} = -2\boldsymbol{e}^T\dot{\boldsymbol{e}} = -2\boldsymbol{e}^T(\boldsymbol{A}\boldsymbol{e} + \boldsymbol{b}\widetilde{\boldsymbol{\theta}}^T\boldsymbol{w}) \tag{36}$$

is bounded. This in turn needs $\boldsymbol{w}$, a continuous function of $\boldsymbol{x}$ to be bounded. Note that if $\boldsymbol{e}$ and $\boldsymbol{y}_d$ are bounded, it is implied that $\boldsymbol{y}$ is bounded. These facts and assumed stable zero dynamics imply that the state $\boldsymbol{x}$ is bounded. Now (if we assume that $g(\boldsymbol{x})$ is bounded away from zero) it follows that $\boldsymbol{w}$ is bounded. □

Remark 2 Note that, although $\boldsymbol{e}$ converges to zero the system (27), (31) is not asymptotically stable because $\widetilde{\boldsymbol{\theta}}$ is only guaranteed to be bounded.

The proven stability of the coupled system (27), (31) (and asymptotical stability of (27)) guarantees the tracking property as well as compensation of disturbances (here nonlinearities) with reference to the system (11) or (which is the same) to the original control synthesis problem (10). Observe that the second term on the right hand side of (27) i.e. $\boldsymbol{b}\widetilde{\boldsymbol{\theta}}^T\boldsymbol{w}$ also tends to zero (compare Remark 1) so the introduction of extra variables $\widetilde{\boldsymbol{\theta}}$ leads, similarly to the linear case, to the cancelation of disturbances.

It is easy to notice that the combined system (27), (31) has in fact the form

$$\begin{cases} \dot{\boldsymbol{e}} = \hat{\boldsymbol{A}}\boldsymbol{e} + \boldsymbol{b}\hat{u} \\ \dot{\widetilde{\boldsymbol{\theta}}} = -\varepsilon\boldsymbol{\Gamma}\boldsymbol{w} \end{cases} \tag{37}$$

where the proportional-like control is

$$\hat{u} = -\boldsymbol{K}\boldsymbol{e} - \boldsymbol{K}_I\widetilde{\boldsymbol{\theta}}. \tag{38}$$

In consequence we have

$$\begin{cases} \dot{\boldsymbol{e}} = (\hat{\boldsymbol{A}} - \boldsymbol{b}\boldsymbol{K})\boldsymbol{e} - \boldsymbol{b}\boldsymbol{K}_I\widetilde{\boldsymbol{\theta}} \\ \dot{\widetilde{\boldsymbol{\theta}}} = -\boldsymbol{\Gamma}\boldsymbol{w}\boldsymbol{b}^T\boldsymbol{P}\boldsymbol{e} \end{cases} \tag{39}$$

or

$$\begin{bmatrix} \dot{\boldsymbol{e}}(t) \\ \dot{\widetilde{\boldsymbol{\theta}}}(t) \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{A}} - \boldsymbol{b}K & -\boldsymbol{b}\boldsymbol{K}_I \\ -\boldsymbol{\Gamma}\boldsymbol{w}\boldsymbol{b}^T\boldsymbol{P} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}(t) \\ \widetilde{\boldsymbol{\theta}}(t) \end{bmatrix} \tag{40}$$

where

$$\boldsymbol{K} = [\mu_1, \cdots, \mu_r], \quad \boldsymbol{K}_I = -\boldsymbol{w}^T, \quad \text{and} \quad \hat{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Notice that the term $\boldsymbol{\Gamma}\boldsymbol{w}\boldsymbol{b}^T\boldsymbol{P}$ in (40) makes sense only after being multiplied by $\boldsymbol{e}(t)$. As we can observe that the structure of (40) matches to the former, linear case (8). The only difference consists in another mechanism of the disturbances rejection. In the first case (8) the constant disturbances $\boldsymbol{B}_w\boldsymbol{w}$ are just cancelled by an extra term $\boldsymbol{B}\boldsymbol{K}_I\boldsymbol{e}$ while here the variable 'disturbances' $\boldsymbol{b}\widetilde{\boldsymbol{\theta}}^T\boldsymbol{w}$ (see (27)) tends to zero by virtue of the proposed system structure. In the same way we can see that the defined above extended-state proportional-like controller (38) fully corresponds to the formula (7) (when restricted to the scalar control case).

Let us rewrite the controller (38) in the form

$$\hat{u} = u_{lin} + u_{apr} = -\boldsymbol{K}\boldsymbol{e} - \boldsymbol{K}_I\widetilde{\boldsymbol{\theta}}. \tag{41}$$

The constant gains μ_i, in the original state, can be optimized (e.g. via LQR technique) while the augmented state, variable gains $\boldsymbol{K}_I = -\boldsymbol{w}^T$ depend on the choice of unknown nonlinearity approximator structure. In this way the choice of approximator may be interpreted as appropriate gains selection and is thereby an immanent part of the controller synthesis process. Note that the term $-\boldsymbol{K}_I\widetilde{\boldsymbol{\theta}} = \boldsymbol{w}^T\widetilde{\boldsymbol{\theta}}$ in (41) has a dual meaning, on one hand it is an extra control u_{apr}, on the other hand it is an error of nonlinearity approximation that while being zero makes also the control u_{apr} equal to zero. In such a case a full compensation of nonlinearity takes place so the system is completely linear.

The above considerations allow for an alternative view on the original control system synthesis problem (10). The dynamic nonlinear controller (38), (or (41)) should be conceived not as a nonlinear controller but rather as a parameter adaptation scheme. This approach, from the adaptive control standpoint, can be summarized in the following theorem [11].

Theorem 4 *The closed-loop system* (11)*,* (19) *and* (16)*, after introduction of the parameter update law,*

$$\dot{\widetilde{\boldsymbol{\theta}}} = -\varepsilon\boldsymbol{\Gamma}\boldsymbol{w} \quad \text{or} \quad \dot{\hat{\boldsymbol{\theta}}} = \varepsilon\boldsymbol{\Gamma}\boldsymbol{w} \tag{42}$$

yields bounded $\mathbf{y}(t)$ *asymptotically converging to* $\mathbf{y}_d(t)$*.*

Note that the integrators (31) are interpreted here as the parameters adaptation laws (42).

Although a practical realization of the considered above adaptive control system ((11), (19), (16) and (42)) might be illustrated as in Fig. 1. this scheme in fact implicitly comprises also the proportional-like control (38) – an analogue of the classical linear system feedback control of section 2.

Presented here point of view, in contrast to the adaptive control standpoint, reveals an important feature of the parameters (the differences $\widetilde{\boldsymbol{\theta}}$). They are variables whose basic role is accumulation of the errors due to disturbances (cancelation of nonlinearities). In the adaptive control context this feature i.e. the fact that the variables $\widetilde{\boldsymbol{\theta}}$ do not tend to zero, has been considered as a drawback [11]; in the presented here 'integral action standpoint' it is a desired value.

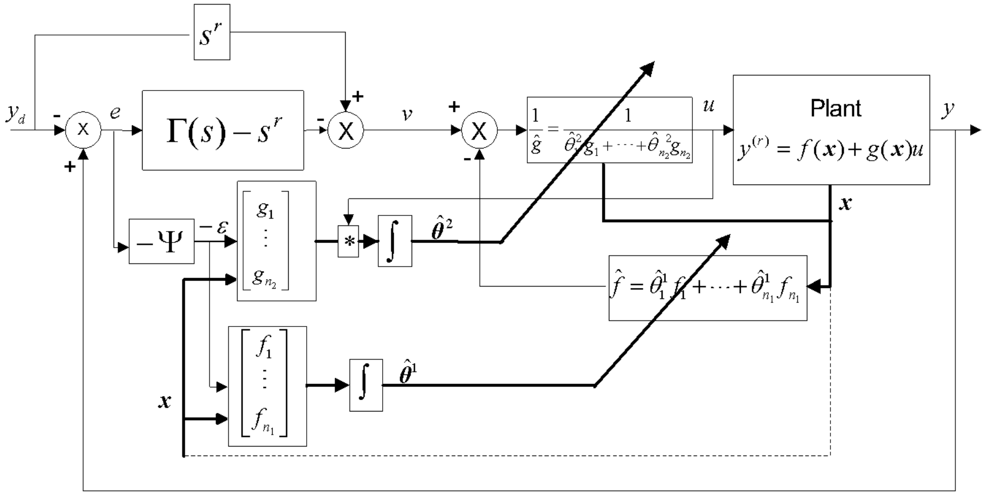

Figure 1. Model basis functions adaptive control scheme.

4. Conclusions

The problem of integral control in the context of nonlinear uncertain systems was considered. In the first part it has been proven that the integral control in the feedback loop of linear time-invariant plant, plus a standard control synthesis technique guarantee asymptotical tracking of the desired trajectory in the presence of constant disturbances. Duality between the disturbances and the desired trajectory input was observed. In the sequel the integral control technique was generalized to include a class of nonlinear, linearly parameterized SISO systems with uncertain parameters. It has been found out that the controller variable gains are in fact the basis functions of the system unknown nonlinearity approximator. A part of the controller with these gains has also a dual meaning: on one hand it is an extra control signal component, on the other hand it is an error of nonlinearity approximation.

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