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Computation of positive realization of MIMO hybrid linear systems in the form of second Fornasini-Marchesini model

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The realization problem for positive multi-input and multi-output (MIMO) linear hybrid systems with the form of second Fornasini-Marchesini model is formulated and a method based on the state variable diagram for finding a positive realization of a given proper transfer matrix is proposed. Sufficient conditions for the existence of the positive realization of a given proper transfer matrix are established. A procedure for computation of a positive realization is proposed and illustrated by a numerical example.

Key words: hybrid, 2D system, positive, realization, existence, computation

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of art in positive systems theory is given in the monographs [2, 5]. The realization problem for positive discrete-time and continuous-time systems without and with delays was considered in [1, 2, 5-10]. The reachability, controllability and minimum energy control of positive linear discrete-time systems with delays have been considered in [3, 13]. The relative controllability of stationary hybrid systems has been investigated in [15] and the observability of linear differential-algebraic systems with delays has been

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considered in [16]. A new class of positive 2D hybrid linear system has been introduced in [11], and the realization problem for this class of systems has been considered in [12].

The main purpose of this paper is to present a new method for computation of a positive realization of a given proper transfer matrix (MIMO system) using the state variable diagram method. Sufficient conditions for the existence of a positive realization of a given proper transfer matrix will be established and a procedure for computation of a positive realization will be proposed.

2. Preliminaries and problem formulation

Consider a hybrid system described by the equations [5]

$$\dot{x}(t,i+1) = A_1\dot{x}(t,i) + A_2 x(t,i+1) + B_1\dot{u}(t,i) + B_2 u(t,i+1), \tag{1a}$$

$$y(t,i) = Cx(t,i) + Du(t,i), \quad t \in R_+ = [0,+\infty], \quad i \in Z_+, \tag{1b}$$

where $\dot{x}(t,i) = \dfrac{\partial x(t,i)}{\partial t}$, $x(t,i) \in R^n$, $u(t,i) \in R^m$, $y(t,i) \in R^p$ and A_1, $A_2 \in R^{n\times n}$, B_1, $B_2 \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$ are real matrices.

Boundary conditions for (1a) have the form

$$x(0,i) = x_1(i), \; i \in Z_+ \;\text{ and }\; x(t,0) = x(t), \; \dot{x}(t,0) = \dot{x}(t), \; t \in R_+. \tag{2}$$

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$, M_n be the set of $n\times m$ Metzler matrices (real matrices with nonnegative off-diagonal entries).

Definition 1 [5] *The hybrid system* (1) *is called internally positive if* $x(t,i) \in R_+^n$ *and* $y(t,i) \in R_+^p$, $t \in R_+$, $i \in Z_+$ *for arbitrary boundary conditions* $x(i) \in R_+^n$, $i \in Z_+$, $x(t) \in R_+^n$, $\dot{x}(t) \in R_+^n$, $t \in R_+$ *and inputs* $u(t,i) \in R_+^m$, $\dot{u}(t,i) \in R_+^m$, $t \in R_+$, $i \in Z_+$.

Theorem 1 [5] *The hybrid system* (1) *is internally positive if and only if*

$$\begin{gathered} A_1 \in R_+^{n\times n}, \quad A_2 \in M_n, \quad A_1A_2 \in R_+^{n\times n}, \\ B_1 \in R_+^{n\times m}, \quad B_2 \in R_+^{n\times m}, \quad C \in R_+^{p\times n}, \quad D \in R_+^{p\times m}. \end{gathered} \tag{3}$$

The transfer matrix (4) *of the system* (1) *is given by*

$$T(s,z) = C[I_n sz - A_1 s - A_2 z]^{-1}(B_1 s + B_2 z) + D \quad \in R^{p\times m}(s,z). \tag{4}$$

Definition 2 *The matrices* (3) *are called the positive realization of the transfer matrix* $T(s,z)$ *if they satisfy the equality* (4). *The realization problem can be stated as follow. Given a proper rational matrix* $T(s,z) \in R^{p\times m}(s,z)$, *find its positive realization* (3), *where* $R^{p\times m}(s,z)$ *is the set of* $p\times m$ *rational matrices in* s *and* z.

3. Problem solution for SISO systems

The essence of proposed method for solving of the realization problem for positive 2D hybrid systems will be presented on single-input single-output system (SISO). Consider a hybrid system described by the transfer function

$$T(s,z) = \frac{b_{n_1,n_2}s^{n_1}z^{n_2} + b_{n_1,n_2-1}s^{n_1}z^{n_2-1} + \ldots + b_{11}sz + b_{10}s + b_{01}z + b_{00}}{s^{n_1}z^{n_2} - a_{n_1,n_2-1}s^{n_1}z^{n_2-1} - \ldots - a_{11}sz - a_{10}s - a_{01}z - a_{00}} = \frac{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} b_{i,j}s^i z^j}{s^{n_2}z^{n_2} - \left(\sum_{\substack{i=0\\ i+j\neq n_1+n_2}}^{n_1}\sum_{j=0}^{n_2} a_{i,j}s^i z^j\right)}. \tag{5}$$

Multiplying the nominator and denominator of transfer function (5) by $s^{-n_1}z^{-n_2}$ we obtain

$$T(s,z) = \frac{b_{n_1,n_2} + b_{n_1,n_2-1}z^{-1} + b_{n_1-1,n_2}s^{-1} + \ldots + b_{00}s^{-n_1}z^{-n_2}}{1 - a_{n_1,n_2-1}z^{-1} - a_{n_1-1,n_2}s^{-1} - \ldots - a_{00}s^{-n_1}z^{-n_2}} = \frac{Y}{U}. \tag{6}$$

By defining

$$E = \frac{U}{1 - a_{n_1,n_2-1}z^{-1} - a_{n_1-1,n_2}s^{-1} - \ldots - a_{00}s^{-n_1}z^{-n_2}} \tag{7}$$

we can rewrite (6) in the form

$$\begin{aligned} E &= U + (a_{n_1,n_2-1}z^{-1} + a_{n_1-1,n_2}s^{-1} + \ldots + a_{00}s^{-n_1}z^{-n_2})E, \\ Y &= (b_{n_1,n_2} + b_{n_1,n_2-1}z^{-1} + b_{n_1-1,n_2}s^{-1} + \ldots + b_{00}s^{-n_1}z^{-n_2})E. \end{aligned} \tag{8}$$

Using (8) we may draw the state variable diagram shown in Fig. 1.

As a state variable we choose the outputs of integrators ($x_1(t,i)$, $x_2(t,i)$, ..., $x_{n_1}(t,i)$) and of delay elements ($x_{n_1+1}(t,i)$, $x_{n_1+2}(t,i)$, ..., $x_{2n_2}(t,i)$). Using state variable diagram (Fig.1) we can write the following differential and difference equations

$$
\begin{aligned}
&\dot{x}_1(t,i) = x_2(t,i)\\
&\dot{x}_2(t,i) = x_3(t,i)\\
&\vdots\\
&\dot{x}_{n_1-1}(t,i) = x_{n_1}(t,i)\\
&\dot{x}_{n_1}(t,i) = e(t,i)\\
&x_{n_1+1}(t,i+1) = a_{0,n_2-1}x_1(t,i) + a_{1,n_2-1}x_2(t,i) + \ldots\\
&\qquad\qquad + a_{n_1-1,n_2-1}x_{n_1}(t,i) + x_{n_1+2}(t,i) + a_{n_1,n_2-1}e(t,i)\\
&x_{n_1+2}(t,i+1) = a_{0,n_2-2}x_1(t,i) + a_{1,n_2-2}x_2(t,i) + \ldots\\
&\qquad\qquad + a_{n_1-1,n_2-2}x_{n_1}(t,i) + x_{n_1+3}(t,i) + a_{n_1,n_2-2}e(t,i)\\
&\vdots\\
&x_{2,n_2-1}(t,i+1) = a_{0,1}x_1(t,i) + a_{1,1}x_2(t,i) + \ldots\\
&\qquad\qquad + a_{n_1-1,1}x_{n_1}(t,i) + x_{n_2}(t,i) + a_{n_1,1}e(t,i)\\
&x_{2,n_2}(t,i+1) = a_{00}x_1(t,i) + a_{10}x_2(t,i) + \ldots + a_{n_1-1,0}x_{n_1}(t,i) + a_{n_1,0}e(t,i)\\
&x_{2,n_2+1}(t,i+1) = b_{0,n_2-1}x_1(t,i) + b_{1,n_2-1}x_2(t,i) + \ldots\\
&\qquad\qquad + b_{n_1-1,n_2-1}x_{1,n}(t,i) + x_{2,n_2+2}(t,i) + b_{n_1,n_2-1}e(t,i)\\
&x_{2,n_2+2}(t,i+1) = b_{0,n_2-2}x_1(t,i) + b_{1,n_2-2}x_2(t,i) + \ldots\\
&\qquad\qquad + b_{n_1-1,n_2-2}x_{1,n}(t,i) + x_{2,n_2+3}(t,i) + b_{n_1,n_2-2}e(t,i)\\
&\vdots\\
&x_{2n_2-1}(t,i+1) = b_{0,1}x_1(t,i) + b_{1,1}x_2(t,i) + \ldots\\
&\qquad\qquad + b_{n_1-1,1}x_{n_1}(t,i) + x_{2n_2}(t,i) + b_{n_1,1}e(t,i)\\
&x_{2n_2}(t,i+1) = b_{00}x_1(t,i) + b_{10}x_2(t,i) + \ldots + b_{n_1-1,0}x_{n_1}(t,i) + b_{n_1,0}e(t,i)\\
&y(t,i) = b_{0,n_2}x_1(t,i) + b_{1,n_2}x_2(t,i) + \ldots\\
&\qquad\qquad + b_{n_1-1,n_2}x_{n_1}(t,i) + x_{n_2+1}(t,i) + b_{n_1,n_2}e(t,i)
\end{aligned}
\tag{9}
$$

where

$$
e(t,i) = a_{0,n_2}x_1(t,i) + a_{1,n_2}x_2(t,i) + \ldots + a_{n_1-1,n_2}x_{n_1}(t,i) + x_{n_1+1}(t,i) + u(t,i). \tag{10}
$$

By increasing i by one in differential equations of (9) and by applying derivative to differential equations of (9), than substituting (10) into (9) we obtain

$$
\begin{aligned}
&\dot{x}_1(t,i+1) = x_2(t,i+1)\\
&\dot{x}_2(t,i+1) = x_3(t,i+1)\\
&\vdots\\
&\dot{x}_{n_1-1}(t,i+1) = x_{n_1}(t,i+1)\\
&\dot{x}_{n_1}(t,i+1) = a_{0,n_2}x_1(t,i+1) + a_{1,n_2}x_2(t,i+1) + \dots\\
&\qquad\qquad + a_{n_1-1,n_2}x_{n_1}(t,i+1) + x_{n_1+1}(t,i+1) + u(t,i+1)\\
&\dot{x}_{n_1+1}(t,i+1) = \bar{a}_{0,n_2-1}\dot{x}_1(t,i) + \bar{a}_{1,n_2-1}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{a}_{n_1-1,n_2-1}\dot{x}_{n_1}(t,i) + a_{n_1,n_2-1}\dot{x}_{n_1+1}(t,i) + \dot{x}_{n_2+2}(t,i) + a_{n_1,n_2-1}\dot{u}(t,i)\\
&\dot{x}_{n_1+2}(t,i+1) = \bar{a}_{0,n_2-2}\dot{x}_1(t,i) + \bar{a}_{1,n_2-2}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{a}_{n_1-1,n_2-2}\dot{x}_{n_1}(t,i) + a_{n_1,n_2-2}\dot{x}_{n_1+1}(t,i) + \dot{x}_{n_1+3}(t,i) + a_{n_1,n_2-2}\dot{u}(t,i)\\
&\vdots\\
&\dot{x}_{n_2-1}(t,i+1) = \bar{a}_{0,1}\dot{x}_2(t,i) + \bar{a}_2\dot{x}_{1,2}(t,i) + \dots\\
&\qquad\qquad + \bar{a}_{n_1-1,1}\dot{x}_{n_1}(t,i) + a_{n_1,1}\dot{x}_{n_1+1}(t,i) + \dot{x}_{n_2}(t,i) + a_{n_1,1}\dot{u}(t,i)\\
&\dot{x}_{n_2}(t,i+1) = \bar{a}_{00}\dot{x}_1(t,i) + \bar{a}_{10}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{a}_{n_1-1,0}\dot{x}_{n_1}(t,i) + a_{n_1,0}\dot{x}_{n_1+1}(t,i) + a_{n_1,0}\dot{u}(t,i)\\
&\dot{x}_{n_2+1}(t,i+1) = \bar{b}_{0,n_2-1}\dot{x}_1(t,i) + \bar{b}_{1,n_2-1}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{b}_{n_1-1,n_2-1}\dot{x}_{n_1}(t,i) + b_{n_1,n_2-1}\dot{x}_{n_1+1}(t,i) + \dot{x}_{n_2+2}(t,i) + b_{n_1,n_2-1}\dot{u}(t,i)\\
&\dot{x}_{n_2+2}(t,i+1) = \bar{b}_{0,n_2-2}\dot{x}_1(t,i) + \bar{b}_{1,n_2-2}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{b}_{n_2-1,n_2-2}\dot{x}_{n_1}(t,i) + b_{n_1,n_2-2}\dot{x}_{n_1+1}(t,i) + \dot{x}_{n_2+3}(t,i) + b_{n_1,n_2-2}\dot{u}(t,i)\\
&\vdots\\
&\dot{x}_{2n_2-1}(t,i+1) = \bar{b}_{0,1}\dot{x}_1(t,i) + \bar{b}_{1,1}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{b}_{n_1-1,1}\dot{x}_{n_1}(t,i) + b_{n_1,1}\dot{x}_{n_1+1}(t,i) + \dot{x}_{2n_2}(t,i) + b_{n_1,1}\dot{u}(t,i)\\
&\dot{x}_{2n_2}(t,i+1) = \bar{b}_{00}\dot{x}_1(t,i) + \bar{b}_{10}\dot{x}_2(t,i) + \dots\\
&\qquad\qquad + \bar{b}_{n_1-1,0}\dot{x}_{n_1}(t,i) + b_{n_1,0}\dot{x}_{n_1+1}(t,i) + b_{n_1,0}\dot{u}(t,i)\\
&y(t,i) = \bar{b}_{0,n_2}x_1(t,i) + \bar{b}_{1,n_2}x_2(t,i) + \dots\\
&\qquad\qquad + \bar{b}_{n_1-1,n_2}x_{n_1}(t,i) + b_{n_1,n_2}x_{n_1+1}(t,i) + x_{n_2+1}(t,i) + b_{n_1,n_2}u(t,i)
\end{aligned}
\tag{11}
$$

where

$$
\bar{a}_{i,j} = a_{i,j} + a_{i,n_2}a_{n_1,j}, \quad \bar{b}_{i,j} = b_{i,j} + a_{i,n_2}b_{n_1,j}
\qquad \text{for} \quad i = 0,1,\dots,n_1-1, \quad j = 0,1,\dots,n_2-1.
\tag{12}
$$

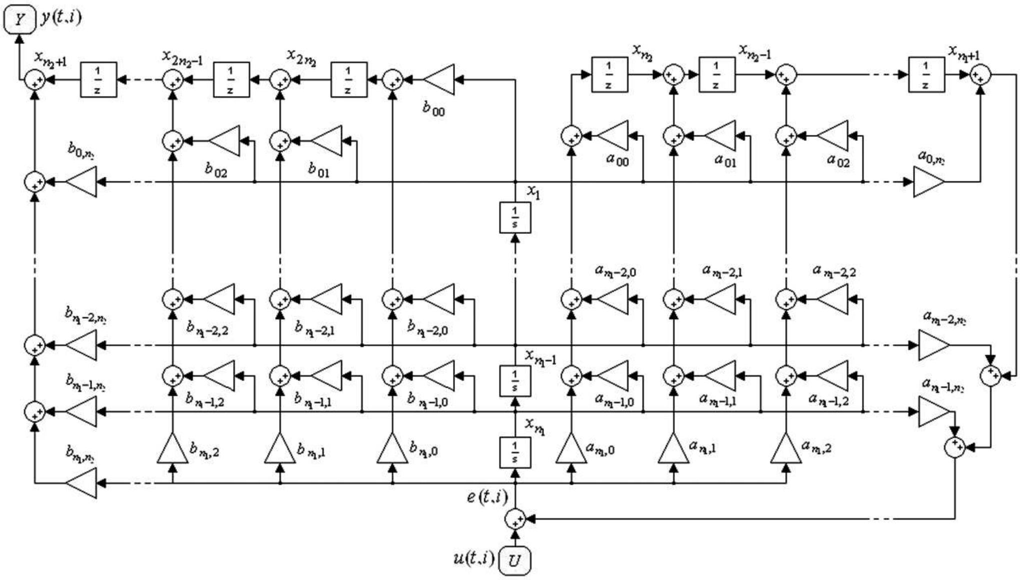

Figure 1. State variable diagram for transfer function (6).

Defining

$$x(t,i) = \begin{bmatrix} x_1(t,i) \\ \vdots \\ x_{n_1}(t,i) \\ x_{n_1+1}(t,i) \\ \vdots \\ x_{2n_2-1}(t,i) \\ x_{2n_2}(t,i) \end{bmatrix} \tag{13}$$

we can write the equations (11) in the matrix form (1a) and (1b) where

$$A_1 = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \bar{a}_{0,n_2-1} & \dots & \bar{a}_{n_1-1,n_2-1} & a_{n_1,n_2-1} & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \bar{a}_{0,n_2-2} & \dots & \bar{a}_{n_1-1,n_2-2} & a_{n_1,n_2-2} & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \bar{a}_{01} & \dots & \bar{a}_{n_1-1,1} & a_{n_1,1} & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \bar{a}_{00} & \dots & \bar{a}_{n_1-1,0} & a_{n_1,0} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \bar{b}_{0,n_2-1} & \dots & \bar{b}_{n_1-1,n_2-1} & b_{n_1,n_2-1} & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ \bar{b}_{0,n_2-2} & \dots & \bar{b}_{n_1-1,n_2-2} & b_{n_1,n_2-2} & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \bar{b}_{01} & \dots & \bar{b}_{n_1-1,1} & b_{n_1,1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \\ \bar{b}_{00} & \dots & \bar{b}_{n_1-1,0} & b_{n_1,0} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$A_1 \in R^{(n_1+2n_2)\times(n_1+2n_2)},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ a_{0,n_2} & a_{1,n_2} & a_{2,n_2} & \dots & a_{n_1-1,n_2} & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in R^{(n_1+2n_2)\times(n_1+2n_2)}$$

$$B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n_1,n_2-1} \\ \vdots \\ a_{n_1,0} \\ b_{n_1,n_2-1} \\ \vdots \\ b_{n_1,0} \end{bmatrix} \in R^{(n_1+2n_2)\times 1}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \in R^{(n_1+2n_2)\times 1},$$

$$B_{21} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{n_1 \times 1}, \quad B_{22} = [0] \in R^{2n_2 \times 1}, \tag{14}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in R^{1 \times (n_1 + 2n_2)}, \quad C_1 = \begin{bmatrix} \bar{b}_{0,n_2} & \bar{b}_{1,n_2} & \dots & \bar{b}_{n_1-1,n_2} \end{bmatrix} \in R^{1 \times n_1},$$

$$C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix} \in R^{1 \times 2n_2},$$

$$C_{21} = \begin{bmatrix} b_{n_1,n_2} & 0 & \dots & 0 \end{bmatrix} \in R^{1 \times n_2}, \quad C_{22} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in R^{1 \times n_2}$$

$$D = [b_{n_1,n_2}] \in R^{1 \times 1}.$$

Therefore, the following theorem has been proved.

Theorem 2 *There exists a positive realization if all coefficients of the nominator and denominator of $T(s,z)$ are nonnegative.*

If the assumptions of Theorem 2 are satisfied then a positive realization (3) of (5) can be found by the use of the following procedure.

Procedure 1

Step 1. Write the transfer function $T(s,z)$ in the form (6) and the equations (8).

Step 2. Using (8) draw the state variable diagram shown in Fig. 1.

Step 3. Choose the state variables and write equations (9) and (10) in the form (11).

Step 4. Using (11) find the desired realization (14) of transfer function (5).

Example 1

Find a positive realization (3) of the proper transfer function

$$T(s,z) = \frac{6s^2z + 5s^2 + 4sz + 3s + 2z + 1}{s^2z - 0.5s^2 + 0.4sz - 0.3s - 0.2z - 0.1}. \tag{15}$$

In this case $n = 2$ and $m = 1$. Using Procedure 1 we obtain the following.

Step 1. Multiplying the nominator and denominator of transfer function (15) by $s^{-2}z^{-1}$ we obtain

$$T(s,z) = \frac{6+5z^{-1}+4s^{-1}+3s^{-1}z^{-1}+2s^{-2}+s^{-2}z^{-1}}{1-0.5z^{-1}+0.4s^{-1}-0.3s^{-1}z^{-1}-0.2s^{-2}-0.1s^{-2}z^{-1}} = \frac{Y}{U} \tag{16}$$

and

$$\begin{aligned} E &= U + (0.5z^{-1} - 0.4s^{-1} + 0.3s^{-1}z^{-1} + 0.2s^{-2} + 0.1s^{-2}z^{-1})E \\ Y &= (6+5z^{-1}+4s^{-1}+3s^{-1}z^{-1}+2s^{-2}+s^{-2}z^{-1})E \end{aligned} \tag{17}$$

Step 2. State variable diagram has the form shown in Fig. 2

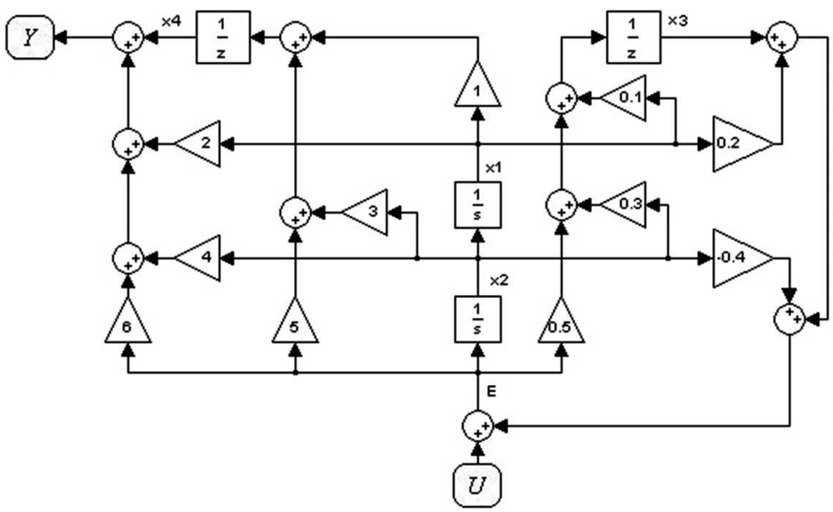

Figure 2. State variable diagram for transfer function (16).

Step 3. Using state variable diagram we can write the following equations

$$\begin{aligned} &\dot{x}_1(t,i) = x_2(t,i) \\ &\dot{x}_2(t,i) = 0.2x_1(t,i) - 0.4x_2(t,i) + x_3(t,i) + u(t,i) \\ &x_3(t,i+1) = 0.2x_1(t,i) + 0,1x_2(t,i) + 0,5x_3(t,i) + 0.5u(t,i) \\ &x_4(t,i+1) = 2x_1(t,i) + x_2(t,i) + 5x_3(t,i) + 5u(t,i) \\ &y(t,i) = 3,2x_1(t,i) + 1.6x_2(t,i) + 6x_3(t,i) + x_4(t,i) + 6u(t,i) \end{aligned} \tag{18}$$

Step 4. The desired realization of (15) has the form

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0.1 & 0.5 & 0 \\ 2 & 1 & 5 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.2 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

$$C = \begin{bmatrix} 3.2 & 1.6 & 6 & 1 \end{bmatrix}, \quad D = [6]$$

Obtained realization is positive because the conditions of Theorem 1 are satisfied. The following example shows that the conditions of Theorem 2 are not necessary for the existence of a positive realization.

4. Generalization for MIMO systems

Consider the m-inputs and p-outputs 2D hybrid linear system (1) with the proper transfer matrix

$$T(s,z) = \begin{bmatrix} T_{11}(s,z) & \dots & T_{1m}(s,z) \\ \vdots & \vdots & \vdots \\ T_{p1}(s,z) & \dots & T_{pm}(s,z) \end{bmatrix} \in R^{p\times m}(s,z) \tag{20}$$

where

$$T_{kl}(s,z) = \frac{\sum_{i=0}^{n_{1kl}} \sum_{j=0}^{n_{2kl}} b_{i,j}^{kl} s^i z^j}{s^{n_{1kl}} z^{n_{2kl}} - \left(\sum_{i=0}^{n_{1kl}} \sum_{\substack{j=0 \\ i+j \neq n_{1kl}+n_{2kl}}}^{n_{2kl}} a_{i,j}^{kl} s^i z^j \right)} \quad \text{for} \quad k = 1,2,\dots,p, \quad l = 1,2,\dots,m. \tag{21}$$

It is well-known [5] that the 2D transfer matrix (20) can be always written in the

$$T(s,z)=\begin{bmatrix} \frac{n_{11}(s,z)}{d_1(s,z)} & \dots & \frac{n_{1m}(s,z)}{d_m(s,z)} \\ \vdots & \vdots & \vdots \\ \frac{n_{p1}(s,z)}{d_1(s,z)} & \dots & \frac{n_{pm}(s,z)}{d_m(s,z)} \end{bmatrix} = \begin{bmatrix} \frac{N_1(s,z)}{d_1(s,z)} & \dots & \frac{N_m(s,z)}{d_m(s,z)} \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} N_l(s,z) &= [\ n_{1l}(s,z) \quad \dots \quad n_{pl}(s,z)\ ]^T \\ n_{kl}(s,z) &= b^{kl}_{n_{1kl},n_{2kl}} s^{n_{1kl}} z^{n_{2kl}} + b^{kl}_{n_{1kl},n_{2kl}-1} s^{n_{1kl}} z^{n_{2kl}-1} + \dots \\ &\qquad + b^{kl}_{11} sz + b^{kl}_{10} s + b^{kl}_{01} z + b^{kl}_{00} \\ d_l(s,z) &= s^{n_{1l}} z^{n_{2l}} - a^l_{n_{1l},n_{2l}-1} s^{n_{1l}} z^{n_{2l}-1} - \dots \\ &\qquad - a^l_{11} sz - a^l_{10} s - a^l_{01} z - a^l_{00} \\ n_{1l} &= n_{1kl}, \quad n_{2l} = n_{2kl}, \\ k &= 1,2,\dots,p; \quad l = 1,2,\dots,m \end{aligned} \tag{23}$$

and T denotes the transpose.

In a similar way as for SISO systems, multiplying the nominator and denominator of each element of transfer matrix (22) by $s^{-n_{1l}} z^{-n_{2l}}$ we obtain

$$E_l = U_l + \bar{d}_l(s,z) E_l$$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} \bar{n}_{11}(s,z) & \dots & \bar{n}_{1m}(s,z) \\ \vdots & \vdots & \vdots \\ \bar{n}_{p1}(s,z) & \dots & \bar{n}_{pm}(s,z) \end{bmatrix} \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \tag{24}$$

where

$$\begin{aligned} \bar{d}_l(s,z) &= a^l_{n_{1l},n_{2l}-1} z^{-1} + a^l_{n_{1l}-1,n_{2l}} s^{-1} + \dots + a^l_{00} s^{-n_{1l}} z^{-n_{2l}} \\ \bar{n}_{kl}(s,z) &= b^{kl}_{n_{1kl},n_{2kl}} + b^{kl}_{n_{1kl},n_{2kl}-1} z^{-1} + b^{kl}_{n_{1kl}-1,n_{2kl}} s^{-1} + \dots + b^{kl}_{00} s^{-n_{1kl}} z^{-n_{2kl}} \\ k &= 1,2,\dots,p; \quad l = 1,2,\dots,m \end{aligned} \tag{25}$$

Similarly as for SISO systems using (24) we may draw a suitable state variable diagram for the MIMO system with the proper transfer matrix (22). Using the state variable diagram we may write the set of differential and difference equations in the form (11) (case for SISO systems). Defining vectors

$$x(t,i)=\begin{bmatrix} x_1(t,i) \\ \vdots \\ x_m(t,i) \end{bmatrix} \quad \text{where} \quad x_k(t,i)=\begin{bmatrix} x_{k,1}(t,i) \\ \vdots \\ x_{k,n_{1l}}(t,i) \\ x_{k,n_{1_l}+1}(t,i) \\ \vdots \\ x_{k,(p+1)n_{2_l}-1}(t,i) \\ x_{k,(p+1)n_{2_l}}(t,i) \end{bmatrix} \quad \text{for} \quad l=1,2,\ldots,m$$

$$u(t,i)=\begin{bmatrix} u_1(t,i) \\ \vdots \\ u_m(t,i) \end{bmatrix} \quad \text{and} \quad y(t,i)=\begin{bmatrix} y_1(t,i) \\ \vdots \\ y_p(t,i) \end{bmatrix} \tag{26}$$

we may write the set of equations in the form

$$\begin{aligned} &\dot{x}(t,i+1)=A_1\dot{x}(t,i)+A_2x(t,i+1)+B_1\dot{u}(t,i)+B_2u(t,i+1) \\ &y(t,i)=Cx(t,i)+Du(t,i) \end{aligned} \tag{27}$$

where

$$A_1=blockdiag\,[\,A_1^1 \quad \ldots \quad A_1^m\,],\quad A_2=blockdiag\,[\,A_2^1 \quad \ldots \quad A_2^m\,],$$

$$B_1=blockdiag\,[\,B_1^1 \quad \ldots \quad B_1^m\,],\quad B_2=blockdiag\,[\,B_2^1 \quad \ldots \quad B_2^m\,],$$

$$C=\begin{bmatrix} C_1^1 & C_2^1 \\ \vdots & \vdots \\ C_1^p & C_2^p \end{bmatrix},\quad D=\begin{bmatrix} b_{n_{111},n_{211}}^{11} & \ldots & b_{n_{11m},n_{21m}}^{1m} \\ \vdots & \vdots & \vdots \\ b_{n_{1p1},n_{2p1}}^{p1} & \ldots & b_{n_{1pm},n_{2pm}}^{pm} \end{bmatrix} \tag{28}$$

and

$$
A_1^l = \begin{bmatrix}
0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\bar{a}^l_{0,n_{2_l}-1} & \dots & \bar{a}^l_{n_{1_l}-1,n_{2l}-1} & a^l_{n_{1l},n_{2_l}-1} & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\bar{a}^l_{0,n_{2_l}-2} & \dots & \bar{a}^l_{n_{1l}-1,n_{2l}-2} & a^l_{n_{1_l},n_{2l}-2} & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
\bar{a}^l_{01} & \dots & \bar{a}^l_{n_{1_l}-1,1} & a^l_{n_{1l},1} & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\
\bar{a}^l_{00} & \dots & \bar{a}^l_{n_{1_l}-1,0} & a^l_{n_{1l},0} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\bar{b}^{kl}_{0,n_{2_l}-1} & \dots & \bar{b}^{kl}_{n_{1_l}-1,n_{2l}-1} & b^{kl}_{n_{1_l},n_{2l}-1} & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\
\bar{b}^{kl}_{0,n_{2l}-2} & \dots & \bar{b}^{kl}_{n_{1l}-1,n_{2l}-2} & b^{kl}_{n_{1l},n_{2l}-2} & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{b}^{kl}_{01} & \dots & \bar{b}^{kl}_{n_{1_l}-1,1} & b^{kl}_{n_{1_l},1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \\
\bar{b}^{kl}_{00} & \dots & \bar{b}^{kl}_{n_{1_l}-1,0} & b^{kl}_{n_{1l},0} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0
\end{bmatrix}
$$

$$A_1^l \in R^{(n_{1l}+(p+1)n_{2_l})\times(n_{1_l}+(p+1)n_{2_l})},$$

$$
A_2^l = \begin{bmatrix}
0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\
a^l_{0,n_{2_l}} & a^l_{1,n_{2_l}} & a^l_{2,n_{2l}} & \dots & a^l_{n_{1_l}-1,n_{2_l}} & 1 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0
\end{bmatrix}
$$

$$A_2^l \in R^{(n_{1l}+(p+1)n_{2_l})\times(n_{1_l}+(p+1)n_{2_l})}$$

$$B_1^l = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a^l_{n_{1_l},n_{2_l}-1} \\ \vdots \\ a^l_{n_{1_l},0} \\ b^{kl}_{n_{1l},n_{2l}-1} \\ \vdots \\ b^{kl}_{n_{1_l},0} \end{bmatrix} \in R^{(n_{1l}+(p+1)n_{2_l})\times 1}, \quad \begin{array}{l} B_2^l = \begin{bmatrix} B^l_{21} \\ B^l_{22} \end{bmatrix} \in R^{(n_{1l}+(p+1)n_{2_l})\times 1} \\ B^l_{21} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{n_{1l}\times 1}, \\ B^l_{22} = [0] \in R^{(p+1)n_{2l}\times 1} \end{array} \tag{29}$$

$$C_1^k = \begin{bmatrix} C_1^{k1} & \dots & C_1^{km} \end{bmatrix}, \quad C_1^{kl} = \begin{bmatrix} \bar{b}^{kl}_{0,n_{2l}} & \bar{b}^{kl}_{1,n_{2l}} & \dots & \bar{b}^{kl}_{n_{1l}-1,n_{2l}} \end{bmatrix} \in R^{1\times n_{1l}},$$

$$C_2^k = \begin{bmatrix} C_2^{k1} & \dots & C_2^{km} \end{bmatrix}, \quad C_2^{kl} = \begin{bmatrix} C^{kl}_{21} & \dots & C^{kl}_{2,p+1} \end{bmatrix} \in R^{1\times(p+1)n_{2l}},$$

$$C^{kl}_{21} = \begin{bmatrix} b^{kl}_{n_{1l},n_{2l}} & 0 & \dots & 0 \end{bmatrix} \in R^{1\times n_{2l}}, \quad C^{kl}_{2,k+1} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in R^{1\times n_{2l}}$$

$$D = \left[b^{kl}_{n_{1_l},n_{l2}} \right] \in R^{p\times m}$$

and

$$\bar{a}^l_{i_l j_l} = a^l_{i_l j_l} + a^l_{i_l n_{2l}} a^l_{n_{1l} j_l}, \quad \bar{b}^{kl}_{i_l j_l} = b^{kl}_{i_l j_l} + a^l_{i_l n_{2l}} b^{kl}_{n_{1l} j_l}, \\ \text{for} \quad i_l = 0,1,\dots,n_{1l}-1, \quad j_l = 0,1,\dots,n_{2l}-1. \tag{30}$$

Summing up the considerations we obtain for the MIMO hybrid linear system the following theorem.

Theorem 3 *There exists a positive realization if all coefficients of the numerators and denominators of the transfer matrix* (20) *are nonnegative.*

The procedure given for SISO systems with slight modifications can be also used for finding a positive realization of the transfer matrix (20).

Example 2

Find a positive realization (3) of the proper transfer matrix

$$T(s,z)=\begin{bmatrix} \dfrac{b_{11}^{11}sz+b_{10}^{11}s+b_{01}^{11}z+b_{00}^{11}}{sz-a_{10}^{1}s-a_{01}^{1}z-a_{00}^{1}} & \dfrac{b_{21}^{12}s^2z+b_{20}^{12}s^2+b_{11}^{12}sz+b_{10}^{12}s+b_{01}^{12}z+b_{00}^{12}}{s^2z-a_{20}^{2}s^2-a_{11}^{2}sz-a_{10}^{2}s-a_{01}^{2}z-a_{00}^{2}} \\[2ex] \dfrac{b_{11}^{21}sz+b_{10}^{21}s+b_{01}^{21}z+b_{00}^{21}}{sz-a_{10}^{1}s-a_{01}^{1}z-a_{00}^{1}} & \dfrac{b_{21}^{22}s^2z+b_{20}^{22}s^2+b_{11}^{22}sz+b_{10}^{22}s+b_{01}^{22}z+b_{00}^{22}}{s^2z-a_{20}^{2}s^2-a_{11}^{2}sz-a_{10}^{2}s-a_{01}^{2}z-a_{00}^{2}} \end{bmatrix} \tag{31}$$

In this case there are $p=2$ outputs and $m=2$ inputs. Using Procedure we obtain the following.

Step 1. Multiplying numerators and denominator of the first column by $s^{-1}z^{-1}$ and multiplying numerators and denominator of the second column by $s^{-2}z^{-1}$ we obtain

$$T(s,z)=\begin{bmatrix} \dfrac{b_{11}^{11}+b_{10}^{11}z^{-1}+b_{01}^{11}s^{-1}+b_{00}^{11}s^{-1}z^{-1}}{1-a_{10}^{1}z^{-1}-a_{01}^{1}s^{-1}-a_{00}^{1}s^{-1}z^{-1}} & \dfrac{b_{21}^{12}+b_{20}^{12}z^{-1}+b_{11}^{12}s^{-1}+b_{10}^{12}s^{-1}z^{-1}+b_{01}^{12}s^{-2}+b_{00}^{12}s^{-2}z^{-1}}{1-a_{20}^{2}z^{-1}-a_{11}^{2}s^{-1}-a_{10}^{2}s^{-1}z^{-1}-a_{01}^{2}s^{-2}-a_{00}^{2}s^{-2}z^{-1}} \\[2ex] \dfrac{b_{11}^{21}+b_{10}^{21}z^{-1}+b_{01}^{21}s^{-1}+b_{00}^{21}s^{-1}z^{-1}}{1-a_{10}^{1}z^{-1}-a_{01}^{1}s^{-1}-a_{00}^{1}s^{-1}z^{-1}} & \dfrac{b_{21}^{22}+b_{20}^{22}z^{-1}+b_{11}^{22}s^{-1}+b_{10}^{22}s^{-1}z^{-1}+b_{01}^{22}s^{-2}+b_{00}^{22}s^{-2}z^{-1}}{1-a_{20}^{2}z^{-1}-a_{11}^{2}s^{-1}-a_{10}^{2}s^{-1}z^{-1}-a_{01}^{2}s^{-2}-a_{00}^{2}s^{-2}z^{-1}} \end{bmatrix} \tag{32}$$

and

$$\begin{aligned} E_1 &= U_1+(a_{10}^{1}z^{-1}+a_{01}^{1}s^{-1}+a_{00}^{1}s^{-1}z^{-1})E_1 \\ E_2 &= U_2+(a_{20}^{2}z^{-1}+a_{11}^{2}s^{-1}+a_{10}^{2}s^{-1}z^{-1}+a_{01}^{2}s^{-2}+a_{00}^{2}s^{-2}z^{-1})E_2 \\ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &= \begin{bmatrix} b_{11}^{11}+b_{10}^{11}z^{-1}+b_{01}^{11}s^{-1}+b_{00}^{11}s^{-1}z^{-1} & b_{21}^{12}+b_{20}^{12}z^{-1}+b_{11}^{12}s^{-1}+b_{10}^{12}s^{-1}z^{-1}+b_{01}^{12}s^{-2}+b_{00}^{12}s^{-2}z^{-1} \\ b_{11}^{21}+b_{10}^{21}z^{-1}+b_{01}^{21}s^{-1}+b_{00}^{21}s^{-1}z^{-1} & b_{21}^{22}+b_{20}^{22}z^{-1}+b_{11}^{22}s^{-1}+b_{10}^{22}s^{-1}z^{-1}+b_{01}^{22}s^{-2}+b_{00}^{22}s^{-2}z^{-1} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \end{aligned} \tag{33}$$

Step 2 State variable diagram for (33) has the form shown in Fig. 3.

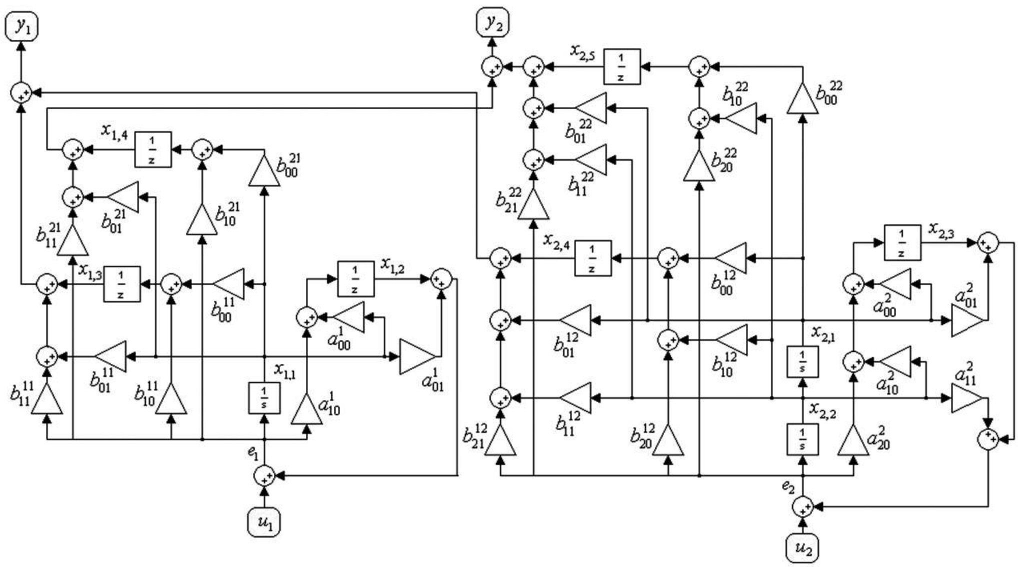

Figure 3. State variable diagram for transfer function (32).

Step 3. Using state variable diagram we can write the following equations

$$
\begin{aligned}
\dot{x}_{1,1}(t,i) &= e_1(t,i)\\
x_{1,2}(t,i+1) &= a^1_{00}x_{1,1}(t,i)+a^1_{10}e_1(t,i)\\
x_{1,3}(t,i+1) &= b^{11}_{00}x_{1,1}(t,i)+b^{11}_{10}e_1(t,i)\\
x_{1,4}(t,i+1) &= b^{21}_{00}x_{1,1}(t,i)+b^{21}_{10}e_1(t,i)\\
\dot{x}_{2,1}(t,i) &= x_{2,2}(t,i)\\
\dot{x}_{2,2}(t,i) &= e_2(t,i)\\
x_{2,3}(t,i+1) &= a^2_{00}x_{2,1}(t,i)+a^2_{10}x_{2,2}(t,i)+a^2_{20}e_2(t,i)\\
x_{2,4}(t,i+1) &= b^{12}_{00}x_{2,1}(t,i)+b^{12}_{10}x_{2,2}(t,i)+b^{12}_{20}e_2(t,i)\\
x_{2,5}(t,i+1) &= b^{22}_{00}x_{2,1}(t,i)+b^{22}_{10}x_{2,2}(t,i)+b^{22}_{20}e_2(t,i)\\
y_1(t,i) &= x_{1,3}(t,i)+b^{11}_{01}x_{1,1}(t,i)+b^{11}_{11}e_1(t,i)+x_{2,4}(t,i)+b^{12}_{01}x_{2,1}(t,i)\\
&\quad +b^{12}_{11}x_{2,2}(t,i)+b^{12}_{21}e_2(t,i)\\
y_2(t,i) &= x_{1,4}(t,i)+b^{21}_{01}x_{1,1}(t,i)+b^{21}_{11}e_1(t,i)+x_{2,5}(t,i)+b^{22}_{01}x_{2,1}(t,i)\\
&\quad +b^{22}_{11}x_{2,2}(t,i)+b^{22}_{21}e_2(t,i)
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
e_1(t,i) &= a^1_{01}x_{1,1}(t,i)+x_{1,2}(t,i)+u_1(t,i)\\
e_2(t,i) &= a^2_{01}x_{2,1}(t,i)+a^2_{11}x_{2,2}(t,i)+x_{2,3}(t,i)+u_2(t,i).
\end{aligned}
\tag{35}
$$

Step 4. Substituting (35) into (34) and taking into account (26)–(28), the desired realization of (31) has the form

$$A_1 = \begin{bmatrix} A_1^1 & 0 \\ 0 & A_1^2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_2^1 & 0 \\ 0 & A_2^2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_1^1 & 0 \\ 0 & B_1^2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} B_2^1 & 0 \\ 0 & B_2^2 \end{bmatrix}, \quad C = [\, C_1 \quad C_2 \,] = \begin{bmatrix} C_1^1 & C_2^1 \\ C_1^2 & C_2^2 \end{bmatrix}, \tag{36}$$

$$D = \begin{bmatrix} b_{11}^{11} & b_{21}^{12} \\ b_{11}^{21} & b_{21}^{22} \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{a}_{00}^1 & a_{10}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{b}_{00}^{11} & b_{10}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{b}_{00}^{21} & b_{10}^{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{a}_{00}^2 & \bar{a}_{10}^2 & a_{10}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{b}_{00}^{12} & \bar{b}_{10}^{12} & b_{10}^{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{b}_{00}^{22} & \bar{b}_{10}^{22} & b_{10}^{22} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a_{01}^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{01}^2 & a_{11}^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ a_{10}^1 & 0 \\ b_{10}^{11} & 0 \\ b_{10}^{21} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a_{20}^2 \\ 0 & b_{20}^{12} \\ 0 & b_{20}^{22} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{matrix} C_1 = \begin{bmatrix} \bar{b}_{01}^{11} & \bar{b}_{01}^{12} & \bar{b}_{11}^{12} \\ \bar{b}_{01}^{21} & \bar{b}_{01}^{22} & \bar{b}_{11}^{22} \end{bmatrix}, \\ C_2 = \begin{bmatrix} b_{11}^{11} & 1 & 0 & b_{21}^{12} & 1 & 0 \\ b_{11}^{21} & 0 & 1 & b_{21}^{22} & 0 & 1 \end{bmatrix}, \\ D = \begin{bmatrix} b_{11}^{11} & b_{21}^{12} \\ b_{11}^{21} & b_{21}^{22} \end{bmatrix} \end{matrix}$$

with (30).

5. Concluding remarks

A method for computation of a positive realization of a given proper transfer matrix of 2D hybrid linear systems has been proposed. Sufficient conditions for the existence of a positive realization of a given proper transfer matrix have been established. A procedure for computation of a positive realization has been proposed. The effectiveness of the procedure has been illustrated by a numerical example. In general case the proposed procedure does not provide a minimal realization of a given transfer matrix. An open problem is formulation of the necessary and sufficient conditions for the existence of solution of the positive realization problem for 2D hybrid systems in the general case. Extension of those considerations for 2D hybrid systems described by models with structures similar to the 2D general model [14] or the 2D first Fornasini-Marchesini model [18] are also open problems.

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