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Positive stable realizations with system Metzler matrices

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Conditions for the existence of positive stable realizations with system Metzler matrices for linear continuous-time systems are established. A procedure for finding a positive stable realization with system Metzler matrix based on similarity transformation of proper transfer matrices is proposed and demonstrated on numerical examples. It is shown that if the poles of stable transfer matrix are real then the classical Gilbert method can be used to find the positive stable realization.

Key words: positive stable realization, system Metzler matrix, linear continuous-time systems

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [2, 8]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

An overview on the positive realization problem is given in [1, 2, 8]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [3, 4, 12, 13] and the positive minimal realization problem for singular discrete-time systems with delays in [14]. The realization problem for fractional linear systems has been analyzed in [10, 15] and for positive 2D hybrid systems in [11].

In this paper sufficient conditions will be established for the existence of positive stable realizations with system Metzler matrices and procedure for computation of the realizations of proper transfer matrices will be proposed.

The paper is organized as follows. In section 2 some definitions and theorems concerning positive continuous-time linear systems are recalled and the problem formula-

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tion is given. Problem solution is presented in section 3 and 4. In section 3 the method based on the similarity transformation is presented and in section 4 the problem is solved by the use of the Gilbert method. Concluding remarks and open problems are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $\Re^{n\times m}(s)$ – the set of $n\times m$ real matrices in s with real coefficients, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries and the problem formulation

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(i) \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 1 [2, 8] *The system* (1) *is called (internally) positive if* $x(t) \in \Re_+^n$, $y(t) \in \Re_+^n$, $t \geqslant 0$ *for any initial conditions* $x(0) = x_0 \in \Re_+^n$ *and all inputs* $u(t) \in \Re_+^m$, $t \geqslant 0$.

Theorem 1 [2, 8] *The system* (1) *is positive if and only if*

$$A \in M_n,\ \ B \in \Re_+^{n\times m},\ \ C \in \Re_+^{p\times n},\ \ D \in \Re_+^{p\times m}. \tag{2}$$

The transfer matrix of the system (1) is given by

$$T(s) = C[I_n s - A]^{-1}B + D. \tag{3}$$

The transfer matrix is called proper if

$$\lim_{s\to\infty} T(s) = K \in \Re^{p\times m} \tag{4}$$

and it is called strictly proper if $K = 0$.

Definition 2 [2, 8] *Matrices* (2) *are called a positive realization of transfer matrix* $T(s)$ *if they satisfy the equality* (3). *The realization is called (asymptotically) stable if the matrix* A *is a (asymptotically) stable Metzler matrix (Hurwitz Metzler matrix).*

Theorem 2 [8] *The positive realization* (2) *is stable if and only if all coefficients of the polynomial*

$$p_A(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{5}$$

are positive, i.e. $a_i > 0$ *for* $i = 0, 1, \dots, n-1$.

The problem under considerations can be stated as follows.

Given a rational matrix $T(s) \in \Re^{p\times m}$, find a positive stable its realization

$$A \in M_{nS},\ \ B \in \Re_+^{n\times m},\ \ C \in \Re_+^{p\times n},\ \ D \in \Re_+^{p\times m}. \tag{6}$$

where M_{nS} is the set of $n\times n$ (asymptotically) stable Metzler matrices.

3. Problem solution

3.1. SISO systems

First we shall consider the positive single-input single-output (SISO) system (1) with the transfer function

$$T(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}. \tag{7}$$

The positive system with (7) is (asymptotically) stable if and only if $a_i > 0$ for $i = 0, 1, \dots, n-1$ [2, 8]. Knowing the transfer function (7) we can find the matrix D by the use of the formula [2, 8]

$$D = \lim_{s\to\infty} T(s) = b_n \tag{8}$$

and the strictly proper transfer function

$$\begin{aligned} T_{sp}(s) &= T(s) - D = C[I_n s - A]^{-1}B \\ &= \frac{b_n s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} - b_n = \frac{\overline{b}_{n-1}s^{n-1} + \dots + \overline{b}_1 s + \overline{b}_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} \end{aligned} \tag{9}$$

where $\overline{b}_i = b_i - a_i b_n$, $i = 0, 1, \dots, n-1$.

A realization of the strictly proper transfer function (9) has the form [2, 7, 8]

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix},\ \overline{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ \overline{C} = \begin{bmatrix} \overline{b}_0 & \overline{b}_1 & \overline{b}_2 & \dots & \overline{b}_{n-1} \end{bmatrix}. \tag{10a}$$

Remark 1. The following realization of the strictly proper transfer function (9) can be also used

$$\bar{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}, \tag{10b}$$

$$\bar{A} = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} \bar{b}_{n-1} & \dots & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 \end{bmatrix}, \tag{10c}$$

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \dots & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{b}_{n-1} \\ \bar{b}_{n-2} \\ \vdots \\ \bar{b}_1 \\ \bar{b}_0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{10d}$$

The realization (10) and (8) of the transfer function is positive if and only if $a_i \leqslant 0$ for $i = 0, 1, \dots, n-2$. This realization by Theorem 2 is unstable.

The solution to the problem under considerations is based on the following lemma.

Lemma 1. *There exists a nonsingular matrix $P \in \Re^{n \times n}$ such that the positive realization*

$$A = P\bar{A}P^{-1} \in M_{nS}, \ \ B = P\bar{B} \in \Re_+^{n \times m}, \ \ C = \bar{C}P^{-1} \in \Re_+^{p \times n}, \ \ D \in \Re_+^{p \times m} \tag{11}$$

is (asymptotically) stable if and only if the matrix $\bar{A}$ is stable.

Proof It is well-known that [7, 9]

$$\det[I_n s - A] = \det[I_n s - \bar{A}] \tag{12}$$

for any nonsingular matrix $P \in \Re^{n \times n}$. Therefore, there exists a nonsingular matrix P such that (11) holds if and only if the matrix $\bar{A}$ is stable. □

In what follows it is assumed that the transfer function (7) is stable.

The following elementary row and column operation on the matrix $\bar{A}$ will be used [7, 9]: $L[i+j\times c]$ $(R[i+j\times c])$ – addition to the i-th row (column) the j-th row (column) multiplied by a scalar c.

To find a positive stable realization (11) of the given transfer function (7) the following procedure will be used.

Procedure 1.

Step 1. Knowing the transfer function (7) and using (8) find the matrix $D\in\Re^{p\times m}$ and the strictly proper transfer function (9).

Step 2. Using $T_{sp}(s)$ and (10) find the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$.

Step 3. Performing suitable elementary row (column) operations $L[i+j\times c]$ $(R[i+j\times c])$ on the matrix $\bar{A}$ find the desired matrix $A=P\bar{A}P^{-1}\in M_{nS}$ and the matrix $P(P^{-1})$ such that $B=P\bar{B}\in\Re_+^{n\times m}$ and $C=\bar{C}P^{-1}\in\Re_+^{p\times n}$.

Remark 2. The matrix P (P^1) can be obtained by performing the elementary row (column) operations on the identity matrix I_n [9].

First we shall demonstrate Procedure 1 on the following simple example.

Example 1. Find the positive stable realization (11) of the transfer function

$$T(s)=\frac{b_2s^2+b_1s+b_0}{s^2+a_1s+a_0}. \tag{13}$$

Using Procedure 1 we obtain the following

Step 1. Using (8) we obtain

$$D=\lim_{s\to\infty}T(s)=b_2 \tag{14}$$

and

$$T_{sp}(s)=T(s)-D=\frac{\bar{b}_1s+\bar{b}_0}{s^2+a_1s+a_0} \tag{15}$$

where $\bar{b}_i=b_i-a_ib_2$, $i=0,1$.

Step 2. The realization (10) of the strictly proper transfer function (15) has the form

$$\bar{A}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix},\quad \bar{B}=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \bar{C}=\begin{bmatrix}\bar{b}_0 & \bar{b}_1\end{bmatrix}. \tag{16}$$

Step 3. To obtain the stable Metzler matrix $A=P\bar{A}P^{-1}$ we perform the following elementary row and column operations on the matrix $\bar{A}$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \xrightarrow{R[1+2\times(-\alpha)]} \begin{bmatrix} -\alpha & 1 \\ \alpha a_1 - a_0 & -a_1 \end{bmatrix} \xrightarrow{L[2+1\times(\alpha)]} \begin{bmatrix} -\alpha & 1 \\ \alpha a_1 - a_0 - \alpha^2 & \alpha - a_1 \end{bmatrix}. \tag{17}$$

If we choose $-\alpha^2 + \alpha a_1 - a_0 = 0$ so that then the matrix (17) is a Metzler matrix of the form

$$A = P\bar{A}P^{-1} = \begin{bmatrix} -\alpha & 1 \\ 0 & \alpha - a_1 \end{bmatrix}. \tag{18}$$

In this case the matrix (18) has two eigenvalues.

Performing the row elementary operations on I_2 we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{L[2+1\times(\alpha)]} \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = P, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R[1+2\times(-\alpha)]} \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} = P^{-1}. \tag{19}$$

The characteristic polynomial of the matrix (17)

$$\begin{vmatrix} s + \alpha & -1 \\ -\alpha a_1 + a_0 + \alpha^2 & s - \alpha + a_1 \end{vmatrix} = s^2 + a_1 s + a_0 \tag{20}$$

is independent of scalar α and has two real zeros.

Remark 3. For the matrix $\bar{A}$ there exists the matrix P such that $A = P\bar{A}P^{-1}$ is the Metzler matrix if and only if the polynomial (20) has two real zeros.

From (16), (11) and (19) we have

$$B = P\bar{B} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = \bar{C}P^{-1} = [\ \bar{b}_0 \quad \bar{b}_1\ ] \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} = [\ \bar{b}_0 - \bar{b}_1\alpha \quad \bar{b}_1\ ]. \tag{21}$$

The desired positive stable realization of the transfer function exists if $\bar{b}_0 - \bar{b}_1\alpha \geqslant 0$ and $\bar{b}_0 - \bar{b}_1\alpha \geqslant 0$ and it is given by (18), (21) and (14).

In particular case if the transfer function (13) has the form

$$T(s) = \frac{2s^2+7s+7}{s^2+3s+2} = 2 + \frac{s+3}{s^2+3s+2} \tag{22}$$

then there exists the desired positive stable realization and it has the form

$$\begin{aligned}
A &= P\bar{A}P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \\
B &= P\bar{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
C &= \bar{C}P^{-1} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \\
D &= [2].
\end{aligned} \tag{23}$$

Example 2. Find a positive stable realization (11) of the transfer function

$$T(s) = \frac{b_3s^3+b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}. \tag{24}$$

Using Procedure 1 we obtain the following

Step 1. Using (8) we obtain

$$D = \lim_{s\to\infty} T(s) = b_3 \tag{25}$$

and

$$T_{sp}(s) = T(s) - D = \frac{\bar{b}_2s^2+\bar{b}_1s+\bar{b}_0}{s^3+a_2s^2+a_1s+a_0} \tag{26}$$

where $\bar{b}_i = b_i - a_ib_3$, $i = 0,1,2$.

Step 2. The realization (10) of the strictly proper transfer function (26) has the form

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \bar{b}_0 & \bar{b}_1 & \bar{b}_2 \end{bmatrix}. \tag{27}$$

Step 3. To obtain the stable Metzler matrix $A = P\bar{A}P^{-1}$ we perform the following elementary row and column operations on the matrix $\bar{A}$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \xrightarrow{R[1+2\times(-\alpha)]}$$

$$\begin{bmatrix} -\alpha & 1 & 0 \\ 0 & 0 & 1 \\ a_1\alpha - a_0 & -a_1 & -a_2 \end{bmatrix} \xrightarrow{L[2+1\times(\alpha)]} \begin{bmatrix} -\alpha & 1 & 0 \\ -\alpha^2 & \alpha & 1 \\ a_1\alpha - a_0 & -a_1 & -a_2 \end{bmatrix} \xrightarrow{R[1+3\times(\alpha^2)]} \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & \alpha & 1 \\ -a_2\alpha^2 + a_1\alpha - a_0 & -a_1 & -a_2 \end{bmatrix} \xrightarrow{L[3+1\times(-\alpha^2)]} \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & \alpha & 1 \\ \alpha^3 - a_2\alpha^2 + a_1\alpha - a_0 & -\alpha^2 - a_1 & -a_2 \end{bmatrix}. \tag{28}$$

If $\alpha^3 - a_2\alpha^2 + a_1\alpha - a_0 \geqslant 0$ then the off-diagonal entries in the first row and in the first column are nonnegative. The submatrix

$$\bar{A}_2 = \begin{bmatrix} \alpha & 1 \\ -\alpha^2 - a_1 & -a_2 \end{bmatrix} \tag{29}$$

has real eigenvalues if $(a_2 - \alpha)^2 - 4(\alpha^2 + a_1 - a_2\alpha) > 0$ (Remark 3). We choose α such that the conditions are satisfied.

The details will be shown on the following stable transfer function

$$T(s) = \frac{2s^3 + 15s^2 + 32s + 15}{s^3 + 7s^2 + 14s + 5} = 2 + \frac{s^2 + 4s + 5}{s^3 + 7s^2 + 14s + 5}. \tag{30}$$

In this case

$$D = 2 \tag{31}$$

and

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -14 & -7 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 5 & 4 & 1 \end{bmatrix}. \tag{32}$$

For $\alpha = 1$ we have $\alpha^3 - a_2\alpha^2 + a_1\alpha - a_0 = 3$, $(a_2 - \alpha)^2 - 4(\alpha^2 + a_1 - a_2\alpha) = 4$ and the matrix (29) has the form

$$\bar{A}_2 = \begin{bmatrix} 1 & 1 \\ -15 & -7 \end{bmatrix}. \tag{33}$$

Performing the following elementary operation on the matrix (33) we obtain

$$\bar{A}_2 = \begin{bmatrix} 1 & 1 \\ -15 & -7 \end{bmatrix} \xrightarrow{R[1+2\times(-3)]} \begin{bmatrix} -2 & 1 \\ 6 & -7 \end{bmatrix} \xrightarrow{L[2+1\times(3)]} \begin{bmatrix} -2 & 1 \\ 0 & -4 \end{bmatrix}. \tag{34}$$

Therefore, the matrix A has the form

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 3 & 0 & -4 \end{bmatrix} \tag{35}$$

and performing the elementary row and column operations on the identity matrix I_3 we obtain

$$I_3 \xrightarrow{L[2+1\times(1)]} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{L[3+1\times(-1)]} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{L[3+2\times(3)]} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = P \tag{36}$$

and

$$I_3 \xrightarrow{R[2+1\times(-1)]} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r[1+3\times(1)]} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R[2+3\times(-3)]} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} = P^{-1}. \tag{37}$$

Note that

$$A = P\bar{A}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -14 & -7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} = \tag{38}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 3 & 0 & -4 \end{bmatrix}$$

and it confirms (35). Using (32), (36) and (37) we obtain

$$\begin{aligned} B = P\bar{B} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \\ C = \bar{C}P^{-1} &= \begin{bmatrix} 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}. \end{aligned} \tag{39}$$

The desired positive stable realization of the transfer function (30) is given by (35), (39) and (31).

In general case we perform on the matrix $\bar{A}$ (defined by (10)) the following elementary column and row operations

$$\begin{aligned} \bar{A} = &\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \\ \xrightarrow{R[1+2\times(-\alpha_1)]} &\begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_1\alpha_1 - a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \\ \xrightarrow{L[2+1\times(\alpha_1)]} &\begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ -{\alpha_1}^2 & \alpha_1 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_1\alpha_1 - a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \end{aligned} \tag{40}$$

$$\xrightarrow{R[1+3\times(\alpha_1{}^2)]} \begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ 0 & \alpha_1 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_2\alpha_1{}^2+a_1\alpha_1-a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}$$

$$\xrightarrow{L[3+1\times(-\alpha_1{}^2)]} \begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ 0 & \alpha_1 & 1 & \dots & 0 \\ \alpha^3 & -\alpha_1{}^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_2\alpha_1{}^2+a_1\alpha_1-a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}.$$

Continuing this procedure after $n-1$ steps we obtain the matrix $\bar{A}_n$ with nonnegative off-diagonal entries in the first row and in the first column if there exists such α_1 that $(-1)^{n-1}\alpha_1^n+(-1)^{n-2}a_{n-1}\alpha_1^{n-1}+\ldots+a_1\alpha_1-a_0 \geqslant 0$. If such real α_1 exists then we repeat the procedure for submatrix

$$\bar{A}_{n-1} = \begin{bmatrix} \alpha_1 & 1 & \dots & 0 \\ -\alpha_1{}^2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}. \tag{41}$$

After $n-1$ successful steps we obtain the matrices $\bar{A}_n, \bar{A}_{n-1}, \ldots, \bar{A}_2$ and the desired Metzler matrix A.

Performing the elementary row operations on the matrix I_n we obtain the matrix P and performing the elementary column operations on the matrix I_n we obtain the matrix P^{-1}. Note that the matrices P and P^{-1} are lower triangular with 1 on the diagonals.

Theorem 3. *There exists a positive stable realization* (6) *of the stable transfer function* (7) *if the following conditions are satisfied:*

1) $\lim\limits_{s\to\infty} T(s) = T(\infty) \in \Re_+$,

2) *there exist* $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ *such that the matrices* $\bar{A}_n, \bar{A}_{n-1}, \ldots, \bar{A}_2$ *have nonnegative off-diagonal entries in the first rows and in the first columns,*

3) $\begin{bmatrix} b_0-a_0b_n & b_1-a_1b_n & \dots & b_{n-1}-a_{n-1}b_n \end{bmatrix}P^{-1} \in \Re_+^{1\times n}$

Proof If the condition 1) is met then $D = T(\infty) \in \Re_+$. From Step 3 Procedure 1 it follows that if the condition 2) is satisfied then

$$A = P\bar{A}P^{-1} \in M_{nS} \quad \text{and} \quad B = P\bar{B} \in \Re_+^{n\times m}$$

since the n-th column of the matrix P is equal to B. Taking into account that

$$\bar{C} = [\ \bar{b}_0 \quad \bar{b}_1 \quad \dots \quad \bar{b}_{n-1}\ ] = [\ b_0 - a_0 b_n \quad b_1 - a_1 b_n \quad \dots \quad b_{n-1} - a_{n-1} b_n\ ]$$

we have $C \in \Re_+^{1\times n}$ if the condition 3) is met. □

Remark 4. For different $\alpha_1, \alpha_2, \dots, \alpha_n$ we obtain different forms of the matrices A and P and therefore, different positive stable realizations (11). All those realizations are related by similarity transformations.

3.2. MIMO systems

Consider a stable positive continuous-time linear system (1) with a given proper transfer matrix of the form

$$T(s) = \begin{bmatrix} T_{11}(s) & \dots & T_{1,m}(s) \\ \vdots & \dots & \vdots \\ T_{p,1}(s) & \dots & T_{p,m}(s) \end{bmatrix} \in \Re^{p\times m}(s),\ \ T_{i,j}(s) = \frac{n_{i,j}(s)}{d_{i,j}(s)},\ \ i = 1,\dots,p;\ \ j = 1,\dots,m \tag{42}$$

where $\Re^{p\times m}(s)$ is the set of proper rational real matrices in s.

With slight modifications Procedure 1 can be also used to find positive stable realizations with system Metzler matrices of the transfer matrix (42).

Step 1. The matrix D can be found by the use of the formula

$$D = \lim_{s\to\infty} T(s) \tag{43}$$

and the strictly proper transfer matrix

$$T_{sp}(s) = T(s) - D \tag{44}$$

which can be written in the form

$$T_{sp}(s) = \begin{bmatrix} \dfrac{N_{11}(s)}{d_1(s)} & \dots & \dfrac{N_{1,m}(s)}{d_m(s)} \\ \vdots & \dots & \vdots \\ \dfrac{N_{p,1}(s)}{d_1(s)} & \dots & \dfrac{N_{p,m}(s)}{d_m(s)} \end{bmatrix} = N(s)D^{-1}(s) \tag{45a}$$

where

$$N(s)=\begin{bmatrix} N_{11}(s) & \dots & N_{1,m}(s) \\ \vdots & \dots & \vdots \\ N_{p,1}(s) & \dots & N_{p,m}(s) \end{bmatrix}, \quad D(s)=\operatorname{diag}[\, d_1(s) \quad \dots \quad d_m(s) \,],$$

$$d_j(s)=s^{d_j}+a_{j,d_j-1}s^{d_j-1}+\dots+a_{j,1}s+a_{j,0}, \tag{45b}$$

$$N_{i,j}(s)=c_{i,j}^{d_j-1}s^{d_j-1}+\dots+c_{i,j}^{1}s+c_{i,j}^{0}, \quad i=1,\dots,p; \quad j=1,\dots,m.$$

Step 2. A realization of (45) has the form [7]

$$\bar{A}=\operatorname{blockdiag}[\, \bar{A}_1 \quad \dots \quad \bar{A}_m \,], \quad \bar{A}_j=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{j,0} & -a_{j,1} & -a_{j,2} & \dots & -a_{j,n-1} \end{bmatrix},$$

$$\bar{B}=\operatorname{blockdiag}[\, \bar{b}_1 \quad \dots \quad \bar{b}_m \,], \quad \bar{b}_j=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^{d_j}, \quad j=1,\dots,m, \tag{46}$$

$$\bar{C}=\begin{bmatrix} c_{11}^0 & c_{11}^1 & \dots & c_{11}^{d_1-1} & \dots & c_{1,m}^0 & c_{1,m}^1 & \dots & c_{1,m}^{d_m-1} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{p,1}^0 & c_{p,1}^1 & \dots & c_{p,1}^{d_1-1} & \dots & c_{p,m}^0 & c_{p,m}^1 & \dots & c_{p.m}^{d_m-1} \end{bmatrix}.$$

Step 3. Performing suitable elementary row and column operations on the matrix $\bar{A}$ find the desired matrix $A=P\bar{A}P^{-1}\in M_{nS}$ and the matrix P (P^{-1}) such that $B=P\bar{B}\in\Re_+^{n\times m}$ and $C=\bar{C}P^{-1}\in\Re_+^{p\times n}$.

Remark 5. The strictly proper transfer matrix (44) can be also written in the form

$$T_{sp}(s)=\begin{bmatrix} \frac{\bar{N}_{11}(s)}{d_1(s)} & \dots & \frac{\bar{N}_{1,m}(s)}{d_1(s)} \\ \vdots & \dots & \vdots \\ \frac{\bar{N}_{p,1}(s)}{d_p(s)} & \dots & \frac{\bar{N}_{p,m}(s)}{d_p(s)} \end{bmatrix}=\bar{D}^{-1}(s)\bar{N}(s) \tag{47a}$$

where

$$N(s)=\begin{bmatrix}\bar{N}_{11}(s) & \dots & \bar{N}_{1,m}(s)\\ \vdots & \dots & \vdots\\ \bar{N}_{p,1}(s) & \dots & \bar{N}_{p,m}(s)\end{bmatrix},\quad D(s)=\text{diag}[\ \bar{d}_1(s)\ \ \dots\ \ \bar{d}_p(s)\ ],$$

$$\bar{d}_i(s)=s^{d_i}+\bar{a}_{i,d_i-1}s^{d_i-1}+\dots+\bar{a}_{i,1}s+\bar{a}_{i,0}, \tag{47b}$$

$$\bar{N}_{i,j}(s)=\bar{b}_{i,j}^{\bar{d}_i-1}s^{\bar{d}_i-1}+\dots+\bar{b}_{i,j}^{1}s+\bar{b}_{i,j}^{0},\quad i=1,\dots,p;\quad j=1,\dots,m.$$

A realization of (47) has the form [7]

$$\bar{A}=\text{blockdiag}[\ \bar{A}_1\ \ \dots\ \ \bar{A}_p\ ],\quad \bar{A}_i=\begin{bmatrix}0 & 0 & \dots & 0 & -\bar{a}_{i,0}\\ 1 & 0 & \dots & 0 & -\bar{a}_{i,1}\\ 0 & 1 & \dots & 0 & -\bar{a}_{i,2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & -\bar{a}_{i,n-1}\end{bmatrix},$$

$$\bar{B}=\begin{bmatrix}\bar{b}_{11}^{0} & \bar{b}_{12}^{0} & \dots & \bar{b}_{1,m}^{0}\\ \bar{b}_{11}^{1} & \bar{b}_{12}^{1} & \dots & \bar{b}_{1,m}^{1}\\ \vdots & \vdots & \dots & \vdots\\ \bar{b}_{11}^{\bar{d}_1-1} & \bar{b}_{12}^{\bar{d}_1-1} & \dots & \bar{b}_{1,m}^{\bar{d}_1-1}\\ \bar{b}_{21}^{0} & \bar{b}_{22}^{0} & \dots & \bar{b}_{2,m}^{0}\\ \vdots & \vdots & \dots & \vdots\\ \bar{b}_{p,1}^{\bar{d}_p-1} & \bar{b}_{p,2}^{\bar{d}_p-1} & \dots & \bar{b}_{p,m}^{\bar{d}_p-1}\end{bmatrix}, \tag{48}$$

$$\bar{C}=\text{blockdiag}[\ C_1\ \ \dots\ \ C_p\ ],\quad C_i=[\ 0\ \ \dots\ \ 0\ \ 1\ ]\in\Re^{1\times d_i},\quad i=1,\dots,p.$$

Theorem 4. *There exists a positive stable realization* (6) *of the stable transfer matrix* (42) *if the following conditions are satisfied:*

1 $\lim_{s\to\infty}T(s)=T(\infty)\in\Re_+^{p\times m}$,

2 *the same as in Theorem 3,*

3 $\bar{C}P^{-1}\in\Re_+^{p\times n}$.

Proof is similar to the proof of Theorem 3.

Remark 6. For a chosen $\bar{A}$ there exists many Metzler matrices A and matrices P (P^{-1}). All those positive, stable realizations are related by similarity transformations.

Example 3. Find a positive stable realization (11) of the transfer matrix

$$T(s) = \begin{bmatrix} \frac{s+3}{s+1} & \frac{2s+5}{s+2} \\ \frac{1}{s+2} & \frac{s+4}{s+3} \end{bmatrix}. \tag{49}$$

Using Procedure 1 we obtain the following

Step 1. Using (43) for (49) we obtain

$$D = \lim_{s\to\infty} T(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \tag{50}$$

and the strictly proper transfer matrix

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{1}{s+3} \end{bmatrix} \tag{51}$$

which can be written in the form

$$T_{sp}(s) = \begin{bmatrix} \frac{2s+4}{s^2+3s+2} & \frac{s+3}{s^2+5s+6} \\ \frac{s+1}{s^2+3s+2} & \frac{s+2}{s^2+5s+6} \end{bmatrix} = N(s)D^{-1}(s) \tag{52a}$$

where

$$N(s) = \begin{bmatrix} 2s+4 & s+3 \\ s+1 & s+2 \end{bmatrix}, \quad D(s) = \begin{bmatrix} s^2+3s+2 & 0 \\ 0 & s^2+5s+6 \end{bmatrix}. \tag{52b}$$

Step 2. The realization (46) of (52) has the form

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 4 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}. \tag{53}$$

Step 3. To obtain the stable Metzler matrix $A = P\bar{A}P^{-1}$ we perform the following elementary row and column operations on the matrix $\bar{A}$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix} \xrightarrow[R[3+4\times(-2)]]{R[1+2\times(-1)]}$$
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 4 & -5 \end{bmatrix} \xrightarrow[L[4+3\times(2)]]{L[2+1\times(1)]} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} = A. \tag{54}$$

Performing the elementary row operations on the matrix I_4 we obtain the P matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[L[4+3\times(2)]]{L[2+1\times(1)]} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = P. \tag{55}$$

Using (11) and (53) we obtain

$$A = P\bar{A}P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$
$$B = P\bar{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{56}$$
$$C = \bar{C}P^{-1} = \begin{bmatrix} 4 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The desired positive stable realization is given by (56).

4. System with real poles

In this section using Gilbert method [7] a procedure for finding positive stable realizations with system Metzler matrices will be presented for transfer matrices with real negative poles.

Consider a linear continuous-time system with m-inputs, p-outputs and the strictly proper transfer matrix

$$T_{sp}(s) = \frac{N(s)}{d(s)} \in \Re^{p \times m}(s) \tag{57}$$

where $N(s) \in \Re^{p \times m}[s]$ (the set of $p \times m$ polynomial matrices) and

$$d(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0. \tag{58}$$

It is assumed that the equation $d(s) = 0$ has only distinct real negative roots $s_1, s_2, \ldots, s_n$ ($s_i \neq s_j$ for $i \neq j$), i.e. $d(s) = (s - s_1)(s - s_2)\ldots(s - s_n)$. In this case the transfer matrix (57) can be written in the form

$$T_{sp}(s) = \sum_{i=1}^{n} \frac{T_i}{s - s_i} \tag{59}$$

where

$$T_i = \lim_{s \to s_i} (s - s_i) T_{sp}(s) = \frac{N(s_i)}{\prod\limits_{j=1, j \neq i}^{n} (s_i - s_j)}, \quad i = 1, \ldots, n. \tag{60}$$

Let

$$\operatorname{rank} T_i = r_i \leqslant \min(p, m). \tag{61}$$

It is easy to show [7] that

$$T_i = C_i B_i, \quad \operatorname{rank} C_i = \operatorname{rank} B_i = r_i, \quad i = 1, \ldots, n \tag{62a}$$

where

$$C_i = \begin{bmatrix} C_{i,1} & C_{i,2} & \ldots & C_{i,r_i} \end{bmatrix} \in \Re^{p \times r_i}, \quad B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r_i} \end{bmatrix} \in \Re^{r_i \times m}. \tag{62b}$$

We shall show that the matrices are the desired positive stable realization with system Metzler matrix

$$A = \text{blockdiag}[\ I_{r_1}s_1 \quad \dots \quad I_{r_n}s_n\ ], \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \quad C = [\ C_1 \quad \dots \quad C_n\ ]. \tag{63}$$

Using (63), (62) and (59) we obtain

$$\begin{aligned} T(s) &= C[Is - A]^{-1}B = \\ &[\ C_1 \quad \dots \quad C_n\ ]\left(\text{blockdiag}[\ I_{r_1}(s-s_1)^{-1} \quad \dots \quad I_{r_n}(s-s_n)^{-1}\ ]\right)\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \\ &= \sum_{i=1}^{n} \frac{C_iB_i}{s-s_i} = \sum_{i=1}^{n} \frac{T_i}{s-s_i}. \end{aligned} \tag{64}$$

From (63) it follows that:

1) if $s_1, s_2, \dots, s_n$ are real negative then the matrix A is stable and is a Metzler matrix,

2) if

$$T_i \in \Re_+^{p\times m} \quad \text{for} \quad i = 1,\dots,n \tag{65}$$

then

$$C_i \in \Re_+^{p\times r_i} \quad \text{and} \quad B_i \in \Re_+^{r_i\times m} \quad \text{for} \quad i = 1,\dots,n \tag{66}$$

and $B \in \Re_+^{\bar{n}\times m}, C \in \Re_+^{p\times \bar{n}}, \bar{n} = \sum_{i=1}^{n} r_i$.

If $T(\infty) \in \Re_+^{p\times m}$ then from (43) we have $D \in \Re_+^{p\times m}$. Therefore, the following theorem has been proved.

Theorem 5. *There exists a positive stable realization* (63), (43) *of the proper transfer matrix* (42) *if the following conditions are satisfied:*

1) *The poles of $T(s)$ are distinct real and negative $s_i \neq s_j$ for $i \neq j$, $s_i < 0$, $i = 1,\dots,n$.*

2) $T_i \in \Re_+^{p\times m}$ *for $i = 1,\dots,n$.*

3) $T(\infty) \in \Re_+^{p\times m}$.

If the conditions of Theorem 5 are satisfied the following procedure can be used to find the desired positive stable realization with system Metzler matrix.

Procedure 2

Step 1. Using (43) find the matrix D and the strictly proper transfer matrix (44) and write it in the form (57).

Step 2. Find the real zeros $s_1, s_2, \ldots, s_n$ of the polynomial (58).

Step 3. Using (60) find the matrices $T_1, \ldots, T_n$ and their decomposition (62).

Step 4. Using (63) find the matrices A, B, C.

Example 4. Using Procedure 2 find a positive stable realization with system Metzler matrix of the transfer matrix (49).

Step 1. The matrix D with nonnegative entries has the form (50) and the strictly proper transfer matrix given by (51).

Step 2. The transfer matrix (51) can be written in the form

$$T_{sp}(s) = \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{bmatrix} = \frac{N(s)}{d(s)}. \tag{67}$$

In this case $d(s) = (s+1)(s+2)(s+3)$, $s_1 = -1$, $s_2 = -2$, $s_3 = -3$ and the condition 1) of Theorem 5 is met.

Step 3. Using (60) and (62) we obtain

$$T_1 = \frac{1}{(s+2)(s+3)} \begin{bmatrix} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{bmatrix} \Bigg|_{s=-1} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$r_1 = \operatorname{rank} T_1 = 1, \quad T_1 = C_1 B_1, \tag{68a}$$

$$B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

$$T_2 = \frac{1}{(s+1)(s+3)} \begin{bmatrix} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{bmatrix} \Bigg|_{s=-2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$r_2 = \operatorname{rank} T_2 = 2, \quad T_2 = C_2 B_2, \tag{68b}$$

$$B_2 = \begin{bmatrix} B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$T_3 = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{bmatrix} \Bigg|_{s=-3} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$r_3 = \text{rank}T_3 = 1, \quad T_3 = C_3B_3, \tag{68c}$$

$$B_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From (68) it follows that the conditions 2) of Theorem 5 are satisfied.

Step 4. Using (63) and (68) we obtain

$$A = \begin{bmatrix} I_{r_1}s_1 & 0 & 0 \\ 0 & I_{r_2}s_2 & 0 \\ 0 & 0 & I_{r_1}s_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{69}$$

$$C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The desired positive stable realization of (49) is given by (69) and (50). This approach can be extended for transfer matrices with multiple real negative poles [16].

Remark 7. If the polynomial (58) has real and complex zeros then a combination of the Procedure 1 and Procedure 2 is recommended to find the positive stable realizations with system Metzler matrices of the transfer matrix (57).

5. Concluding remarks

Conditions for the existence of positive stable realizations with system Metzler matrices of proper transfer matrices have been established (Theorem 3). The procedure based on similarity transformation for finding of the positive stable realization (Procedure 1) has been proposed and its efficiency has been demonstrated on numerical examples. It has been shown that if the poles of stable transfer matrix are real then the classical Gilbert method can be used to find the positive stable realizations. Conditions for the existence of positive stable realizations of proper transfer matrices by the use of the Gilbert method have been established (Theorem 5) and Procedure 2 for its computation has been proposed. The procedure has been illustrated by numerical example. If the transfer matrix has real and complex poles then a combination of the procedures for finding its positive stable realization has been recommended (Remark 7).

The following are open problems:

1) find necessary and sufficient conditions for the existence of positive stable realizations with system Metzler matrices of proper transfer matrices,

2) give a method for finding positive stable realizations with system Metzler matrices which is not based on the similarity transformation of proper transfer matrices.

An extension of the presented procedure for fractional linear systems [5, 6, 15] is also an open problem.

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