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Dynamic decoupling of left-invertible MIMO LTI plants

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In the paper problems with dynamic decoupling of the left-invertible multi-input multi-output dynamic (MIMO) linear time invariant (LTI) plants are considered. It is presented an universal and efficient algorithm for synthesis of control system for proper, square, right and left invertible plants which can be both unstable and/or non-minimumphase.

Key words: dynamic decoupling, left-invertible, pole assignment, polynomial matrix equations.

1. Introduction

The main characteristic of the multi-input multi-output (MIMO) plants is the coupling of theirs inputs and outputs. This feature can make the process of designing the control system seriously difficult. Therefore in a design of the control systems decoupling methods are used which goal is to lead a system to the situation when a specific group of inputs affect a specific group of outputs and no element of this input group have influence on any other output component of the system. After decoupling the transfer function matrix of the system is diagonal (triangular, block diagonal), thus the system is divided into small subsystems, which can be analyzed irrespectively of each other. Decoupling makes the control system much more easier to design and control.

However dynamic decoupling of MIMO systems is one of the most difficult problem in construction of multivariable control systems especially for non-square plants which can have non minimum phase transmission zeros. It is well known in the decoupling theory that some poles of the decoupled (compensated) system, related to the so called interconnection transmission zeros of the plant, are fixed. These can generate uncontrollable and/or unobservable parts of the closed-loop system. Cancelations of such non minimum phase zeros (unstable hidden modes) make the system unstable [9, 11, 20].

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An idea of dynamic decoupling for multivariable (MIMO) systems has been considered by many authors since 60s beginning with [13]. Solution to Morgan problem was given in [4] and its stability conditions in [16]. General decoupling problem with stability for square and right-invertible plants has been intensively studied in the past (see e.g. [3, 5, 16, 18, 23]) and it still arouse considerable interest [8, 14, 17, 19, 22]. However most of the proposed methods allows one to exist in the decoupled system some fixed poles which (if there are unstable) can result in its instability. Moreover there are as well often limited to the square or right invertible plants with minimum phase zeros only. Yet the everyday practice shows that often there is a need to maintain and control processes with more outputs than inputs. It may happen eg. during system failure when one lose some actuators and the precise control of all outputs is not possible.

As in this case an independent control of each output is not possible then usually methods of control after actuator faults assume that the rank of the input matrix does not change [15]. It is not possible to synthesize the decoupler for diagonal (row-by-row) decoupling but as it was shown in [20] there is a chance to synthesize block decoupled.

This paper complements the results obtained in [2, 20] and presents an universal algorithm for dynamic decoupling designed for linear m-input l-output both invertible $(m = l)$ and right invertible $(m > l)$ and left invertible $(l > m)$ plants described by proper rational full rank transfer matrix $\boldsymbol{T}(s)$. Plants which be unstable, non-minimum phase or both.

The presented algorithm guarantees free location of all poles of the system and guarantees that all designed elements (parts of the system) are proper (or strictly proper), so they are able to be physically realizable.

The paper is organized as follows. The problem statement and decoupling concept have been brought in section 2. and 3. Conditions for decoupling of left-invertible plants with $l > m$ are presented in section 4. An universal algorithm for decoupling square and right invertible plants extended by steps for left-invertible plants is given in section 5. An example which demonstrates the effectiveness of the proposed algorithm is given in section 6. Finally the paper is concluded in section 7.

Notations

The notation used here is fairly standard. R denotes the field of real numbers, $R[s]$ the ring of polynomials in s with real coefficients, $R[s]^{m\times n}$ set of $m\times n$ polynomial matrices with coefficients from the field R. Lower case bold letters denote vectors $\boldsymbol{x}$, capital bold letters $\boldsymbol{A}$ denote real matrices; $\boldsymbol{P}(s)$ denotes polynomial matrix. A polynomial matrix $\boldsymbol{P}(s)$ is said to be unimodular if $\boldsymbol{P}^{-1}(s)$ exists and is also polynomial matrix; it means that its determinant det$\boldsymbol{P}(s)$ is a real number. A right divisor of a polynomial matrix $\boldsymbol{P}(s)$ is a polynomial matrix $\boldsymbol{R}(s)$ such that $\boldsymbol{P}(s) = \boldsymbol{P}_1(s)\boldsymbol{R}(s)$ for some polynomial matrix $\boldsymbol{P}_1(s)$. Two polynomial matrices are relatively right prime (*r.r.p.*) if they have no common right divisors except unimodular matrices. The analogous definitions are made for left divisors. Detailed description of definitions and basic operations on polynomial matrices may be found in [10, 12].

2. Problem statement

We consider a controllable and observable LTI MIMO model of the plant defined by the state and output equations

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$, $\boldsymbol{u}(t) \in R^m$ and $\boldsymbol{y}(t) \in R^l$ $(l > m)$, are the state, input and output vectors respectively. In the polynomial matrix approach transfer matrices of all elements of the system are defined by pairs of polynomial matrices either *r.r.p.* for plants, or *r.l.p.* for other elements. Applying this approach, the plant model (1) can be transformed to the relatively prime matrix fraction description in the frequency s domain as follows

$$\boldsymbol{y} = \boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)\boldsymbol{u} \tag{2}$$

where

$$\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}. \tag{3}$$

Assuming dynamic block decoupling of the designed control system we group output and a vector of exogenous signals into k blocks according to the partitions

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{y}_1(t) \\ \vdots \\ \boldsymbol{y}_i(t) \\ \vdots \\ \boldsymbol{y}_k(t) \end{bmatrix}, \quad \boldsymbol{q}(t) = \begin{bmatrix} \boldsymbol{q}_1(t) \\ \vdots \\ \boldsymbol{q}_i(t) \\ \vdots \\ \boldsymbol{q}_k(t) \end{bmatrix} \tag{4}$$

where

$$\boldsymbol{y}_i(t) \in R^{l_i}, \quad \sum_{i=1}^{k} l_i = l, \quad \boldsymbol{q}_i(t) \in R^{m_i}, \quad \sum_{i=1}^{k} m_i = m. \tag{5}$$

We want to design a decoupled system in which each part (loop) $i = 1, 2, \ldots, k$ of a system defined by pairs of signals $\boldsymbol{q}_i(t)$, $\boldsymbol{y}_i(t)$ could be controlled independently of other parts $j \neq i$. Moreover, each part of the system should be designed with individually supposed dynamic properties according to the requirements.

3. Decoupling concept

The goal of decoupling of the LTI dynamic system can be achieved in a control system structure presented in Fig. 1. which contains: the dynamic feedforward compensator and (if necessary) the Luenberger observer with feedback matrix $\boldsymbol{F}$.

Feedback law, employed to decouple the system (the linear state variable feedback along with dynamic feedforward) is described by

$$\boldsymbol{u}(s) = \boldsymbol{G}^{-1}(s)\boldsymbol{L}_0(s)\boldsymbol{f}(s) + \boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)\boldsymbol{q}(s) \tag{6}$$

where

$$\boldsymbol{f}(s) = \boldsymbol{F}(s)\boldsymbol{x}_p(s) \triangleq \boldsymbol{F}\boldsymbol{x}(t) \tag{7}$$

$\boldsymbol{x}_p(s)$ is a partial state vector of the plant, $\boldsymbol{G}(s) \in R[s]^{m\times m}$, $\boldsymbol{L}(s) \in R[s]^{m\times l}$, $\boldsymbol{L}_0(s) \in R[s]^{m\times m}$, - polynomial matrices such that $\boldsymbol{G}^{-1}(s)\boldsymbol{L}_0(s)$ and $\boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)$ are proper and $\boldsymbol{F}(s)\boldsymbol{A}_1^{-1}(s)$ is strictly proper. Without any lose of generality the matrix $\boldsymbol{L}_0(s)$ may be taken as $\boldsymbol{L}_0(s) = \boldsymbol{I}_m$.

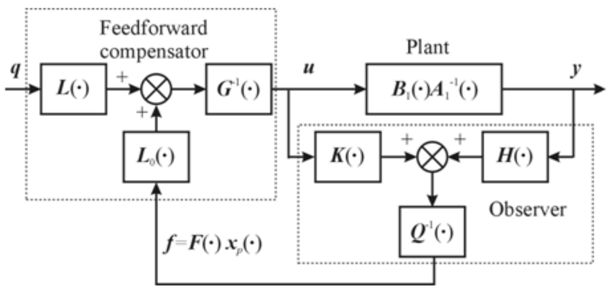

Figure 1. Structure of the decoupled control system with inaccessible plant's state vector.

According to this scheme the considered decoupling systems are suitably defined in s-domain by: proper and possible low-order transfer matrix $\boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)$ for the dynamic feedforward compensator and strictly proper (or proper) transfer matrices $\boldsymbol{Q}^{-1}(s)\boldsymbol{H}(s)$ and $\boldsymbol{Q}^{-1}(s)\boldsymbol{K}(s)$ for the full (or reduced) order Luenberger observer along with a feedback matrix $\boldsymbol{F}$. All of the above-mentioned polynomial matrix fractions should be *r.l.p.* with nonsingular, row-reduced, denominator matrices.

The main problem is to find a method for block decoupling of the control system (between the signals $\boldsymbol{q}$ and $\boldsymbol{y}$) so as to obtain the transfer matrix $\boldsymbol{T}_{\boldsymbol{yq}}(s)$ free of cancelation of unstable hidden modes. For the applied decoupling law this transfer matrix takes the form

$$\boldsymbol{T}_{\boldsymbol{yq}}(s) = \boldsymbol{B}_1(s)[\boldsymbol{G}(s)\boldsymbol{A}_1(s) - \boldsymbol{F}(s)]^{-1}\boldsymbol{L}(s) = \boldsymbol{N}(s)\boldsymbol{D}^{-1}(s) \tag{8}$$

with

$$\boldsymbol{N}(s) = \text{block diag}[\boldsymbol{N}_{ij}(s)] \in R[s]^{l\times m} \tag{9}$$

and

$$\boldsymbol{D}(s) = \text{block diag}[\boldsymbol{D}_{ii}(s)] \in R[s]^{m\times m} \tag{10}$$

where $i = 1,2,\dots,k$ and $j = 1,2,\dots,k$ according to the partition (4).

The algorithm starts with determination of the numerator matrix of the system. It is taken as a block diagonal matrix $\boldsymbol{N}(s) = \text{block diag}[\boldsymbol{N}_{ii}(s), i = 1,2,\dots,k]$, where particular blocks $\boldsymbol{N}_{ii}(s)$ are the greatest common left divisors (*g.c.l.d.*) of columns of i-th row-block of $\boldsymbol{B}_1(s)$ caused by the partition (4)

$$\boldsymbol{B}_1(s) = \begin{bmatrix} \boldsymbol{B}_{11}(s) \\ \vdots \\ \boldsymbol{B}_{1i}(s) \\ \vdots \\ \boldsymbol{B}_{1k}(s) \end{bmatrix}. \tag{11}$$

Then $\boldsymbol{B}_1(s)$ takes the form

$$\boldsymbol{B}_1(s) = \boldsymbol{N}(s)\boldsymbol{B}(s). \tag{12}$$

In general the decoupled system does not have to be stable but it should be free of any unstable cancelations, unobservable and/or uncontrollable, unstable poles. However if the polynomial matrix $\widetilde{\boldsymbol{G}}(s) \in R[s]^{l\times l}$, which is a *g.c.l.d.* of all columns $\boldsymbol{B}(s)$ defined by the relation

$$\boldsymbol{B}(s) = \widetilde{\boldsymbol{G}}(s)\widetilde{\boldsymbol{B}}(s) \tag{13}$$

is not unimodular and if its zeros lie in the unstable region of the complex plane, the (unobservable) poles of decoupled system corresponding to these zeros are fixed and unstable [9]. These so called 'interconnection' transmission zeros can not be eliminated by an feedforward compensator of zero order. So in such case a dynamic compensator have to be used. To remove these unobservable poles we can use the compensation scheme together with an additional dynamic feedforward compensator obtained by augmenting the plant model with a serial dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$. This element has to be connected to the input of the original plant presented in Fig. 2 and finally shifted into the structure of dynamic feedforward compensator [1, 9].

After calculations of the element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$ the standard procedure with an augmented plant can be used and a decoupled system $\boldsymbol{T}_{yq}(s)$ without fixed poles caused by

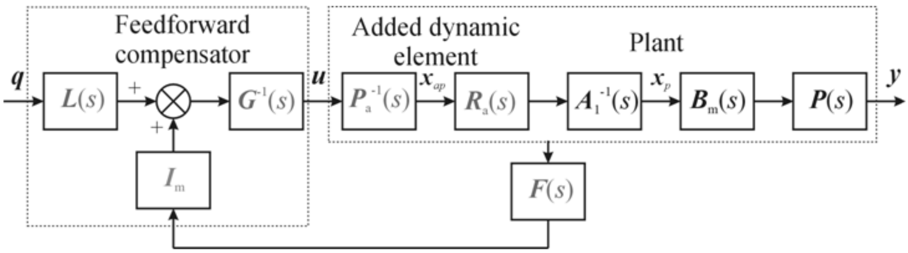

Figure 2. Structure of the decoupled system for the augmented plant.

$\widetilde{\boldsymbol{G}}(s)$ is automatically obtained. An algorithm which may be used to calculate this additional dynamics was analyzed in [1, 9] and has been modified in [6] to make it more reliable and efficient.

4. Decoupling of left-invertible plants

The problem with decoupling of a non-square plant with $l > m$ is to find a method which allows us to obtain the transfer matrix (8) free of cancelation of unstable hidden modes (uncontrollable and/or unobservable poles of $\boldsymbol{T}_{\boldsymbol{yq}}(s)$). In order to do it we adopt the following lemma and theorem given in [9, 20].

Theorem 2 *A left-invertible plant with the transfer matrix* (8) *of rank m can be block decoupled according to the partition* (4) *by use of linear state variable feedback and dynamic feedforward if and only if* $\operatorname{rank} \boldsymbol{B}_{1i}(s) = m_i$, $i = 1, 2, \ldots, k$.

If assumption of theorem 1 is satisfied then, there exists k unimodular matrices $\boldsymbol{U}_i(s) \in R[s]^{l_i \times l_i}$ such that

$$\boldsymbol{U}_i(s)\boldsymbol{B}_{1i}(s) = \begin{bmatrix} \boldsymbol{B}_{mi}(s) \\ \boldsymbol{0} \end{bmatrix} \tag{14}$$

with $\boldsymbol{B}_{mi}(s)$ of full rank. Then parting $\boldsymbol{U}_i^{-1}(s) = \begin{bmatrix} \boldsymbol{P}_i(s) & \boldsymbol{R}_i(s) \end{bmatrix}$ one can define

$$\boldsymbol{B}_{1i}(s) = \begin{bmatrix} \boldsymbol{P}_i(s) & \boldsymbol{R}_i(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{mi}(s) \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{P}_i(s)\boldsymbol{B}_{mi}(s). \tag{15}$$

After defining

$$\boldsymbol{P}(s) = \begin{bmatrix} \boldsymbol{P}_1(s) & & \\ & \ddots & \\ & & \boldsymbol{P}_k(s) \end{bmatrix} \in R[s]^{l\times m} \tag{16}$$

and

$$\boldsymbol{B}_m(s) = \begin{bmatrix} \boldsymbol{B}_{m1}(s) & & \\ & \ddots & \\ & & \boldsymbol{B}_{mk}(s) \end{bmatrix} \in R[s]^{m\times m} \tag{17}$$

the transfer matrix (2) takes the form

$$\boldsymbol{T}(s) = \boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s) = \boldsymbol{P}(s)\boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s) \tag{18}$$

which inner square part $\boldsymbol{T}_m(s) = \boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s)$ may be decoupled by using any known decoupling method.

Then the transfer matrix (8) of the decoupled system takes the form

$$\boldsymbol{T}_{yq}(s) = \boldsymbol{P}(s)\boldsymbol{B}_m(s)[\boldsymbol{G}(s)\boldsymbol{A}_1(s) - \boldsymbol{F}(s)]^{-1}\boldsymbol{L}(s) = \boldsymbol{P}(s)\boldsymbol{N}_m(s)\boldsymbol{D}^{-1}(s) \tag{19}$$

with

$$\boldsymbol{N}_m(s) = \text{block diag}[\boldsymbol{N}_{ii}(s),\ i = 1,\ldots,k] \in R[s]^{m\times m}. \tag{20}$$

Stability of the decoupled system describe the following lemmas and theorem.

Lemma 2 [9] *The block diagonal matrix $\boldsymbol{D}(s) \in R[s]^{l\times l}$ that satisfies the relation* (8) *exists if there exist polynomial matrices $\bar{\boldsymbol{L}}(s) \in R[s]^{m\times(m-l)}$ and $\bar{\boldsymbol{B}}(s) \in R[s]^{m\times(m-l)}$ of full rank such that $\boldsymbol{G}(s)\boldsymbol{A}_1(s) - \boldsymbol{F}(s) - \boldsymbol{L}(s)\boldsymbol{D}(s)\boldsymbol{B}(s) = \bar{\boldsymbol{L}}(s)\bar{\boldsymbol{B}}(s)$.*

Theorem 3 [9] *The closed-loop poles of the decoupled system $\boldsymbol{T}_{yq}(s)$ realized by linear state variable feedback (l.s.v.f.) with dynamic feedforward consist of the zeros of $\left|[\boldsymbol{L}(s), \bar{\boldsymbol{L}}(s)]\right|$, which are uncontrollable, the zeros of $\left|[\boldsymbol{B}^{\mathrm{T}}(s), \bar{\boldsymbol{B}}^{\mathrm{T}}(s)]^{\mathrm{T}}\right|$, which are unobservable and the zeros of $|\boldsymbol{D}(s)|$, which are controllable and observable.*

Lemma 3 [20] *The invariant zeros of $\boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s)$ are precisely those of $\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)$.*

Applying the above method for preparing the plant model to the standard decoupling procedure [2, 6, 7] we obtain the design algorithm for the considered block decoupled control system.

5. The algorithm

In this section an universal algorithm for decoupling square, right and particularly considered in the paper left-invertible plants is presented. Using theorem 1 the algorithm allows one to check decoupling conditions for systems with more outputs than inputs. The algorithm guarantees free location of all poles of the system and guarantees that all designed elements (parts of the system) are proper (or strictly proper), so they are able to be physically realizable. Moreover it allows one to fully automate process of the synthesis of decoupled system which may be used in an adaptive control for non linear or reconfigurable systems. Detailed description of all steps of the standard decoupling procedure for square and right invertible plants $m \geqslant l$ may be found in [2, 6, 7].

Step 1

Given the plant description (1) derive its transfer matrix $\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)$ using the Wolovich's structure theorem. Permute rows of $\boldsymbol{B}_1(s)$, if it is necessary, to group plant's outputs $\boldsymbol{y}(s)$ (and $\boldsymbol{y}_0(s)$). Substitute $\boldsymbol{B}_1(s) := \boldsymbol{P}\boldsymbol{B}_1(s)$, where $\boldsymbol{P}$ is a permutation matrix.

If $m \geqslant l$ go to Step 3 else do the following steps:

Step 2.1

If assumption of theorem 1 is satisfied then according to the assumed partition (4) determine k unimodular matrices $\boldsymbol{U}_i(s) \in R[s]^{l_i \times l_i}$ such that

$$\boldsymbol{U}_i(s)\boldsymbol{B}_{1i}(s) = \begin{bmatrix} \boldsymbol{B}_{mi}(s) \\ \mathbf{0} \end{bmatrix}$$

with $\boldsymbol{B}_{mi}(s)$ of full rank.

Step 2.2

Calculate $\boldsymbol{U}_i^{-1}(s) = \begin{bmatrix} \boldsymbol{P}_i(s) & \boldsymbol{R}_i(s) \end{bmatrix}$ and define

$$\boldsymbol{P}(s) = \begin{bmatrix} \boldsymbol{P}_1(s) & & \\ & \ddots & \\ & & \boldsymbol{P}_k(s) \end{bmatrix} \in R[s]^{l \times m} \tag{21}$$

to obtain matrices $\boldsymbol{B}_m(s)$ and $\boldsymbol{P}(s)$ of the square part $\boldsymbol{T}_m(s) = \boldsymbol{B}_m(s)\boldsymbol{A}_1^{-1}(s)$.

Substitute $\boldsymbol{B}_1(s) := \boldsymbol{B}_m(s)$.

Step 3

Define $\boldsymbol{N}(s) = \text{block diag}[\boldsymbol{N}_{ii}(s), i = 1,2,...,k]$, where $\boldsymbol{N}_{ii}(s)$ are *g.c.l.d.* of the columns of i-th row-block of $\boldsymbol{B}_1(s)$. Calculate $\boldsymbol{B}(s) \in R[s]^{l\times m}$ such that $\boldsymbol{B}_1(s) = \boldsymbol{N}(s)\boldsymbol{B}(s)$.

Determine $\widetilde{\boldsymbol{G}}(s) \in R[s]^{l\times l}$, a *g.c.l.d.* of all columns of the matrix $\boldsymbol{B}(s) = \widetilde{\boldsymbol{G}}(s)\widetilde{\boldsymbol{B}}(s)$.

If $\widetilde{\boldsymbol{G}}(s)$ is unimodular (or stable) go to Step 4, else to calculate the additional dynamic element do the following steps:

Step 3.1

Convert the left to right fractions $\widetilde{\boldsymbol{G}}^{-1}(s)\boldsymbol{E}_i = \widetilde{\boldsymbol{R}}_i(s)\widetilde{\boldsymbol{J}}_{ii}^{-1}(s)$ for $i = 1,2,\ldots,k$ with $\boldsymbol{E}_i$ defined by $\boldsymbol{I}_l = [\boldsymbol{E}_1,\boldsymbol{E}_2,\ldots,\boldsymbol{E}_k]$. Define $\widetilde{\boldsymbol{R}}(s) = [\widetilde{\boldsymbol{R}}_1(s),...,\widetilde{\boldsymbol{R}}_k(s)]$ and $\widetilde{\boldsymbol{J}}(s) = \text{block diag}[\boldsymbol{J}_{ii}(s), i = 1,2,\ldots,k]$.

Step 3.2

Calculate $\hat{\boldsymbol{B}}(s) \in R[s]^{l\times m}$ and $\hat{\boldsymbol{R}}(s) \in R[s]^{m\times m}$ by the left to right conversion $\widetilde{\boldsymbol{R}}^{-1}(s)\widetilde{\boldsymbol{B}}(s) = \hat{\boldsymbol{B}}(s)\hat{\boldsymbol{R}}^{-1}(s)$.

Step 3.3

Convert the right to left fraction of $\boldsymbol{A}_1(s)[\hat{\boldsymbol{R}}_{ad}(s)]^{-1} = \dot{\boldsymbol{R}}^{-1}(s)\dot{\boldsymbol{P}}(s)$ and set $\boldsymbol{R}_{\boldsymbol{a}}(s) = \dot{\boldsymbol{R}}_{ad}(s)$ and $\bar{\boldsymbol{P}}(s) = \dot{\boldsymbol{P}}(s)$. The $\hat{\boldsymbol{R}}_{ad}(s)$ and $\dot{\boldsymbol{R}}_{ad}(s)$ are adjoints of $\hat{\boldsymbol{R}}(s)$ and $\dot{\boldsymbol{R}}(s)$, respectively.

Step 3.4

Select $\boldsymbol{U}_4(s) \in R[s]^{m\times m}$ such that $\boldsymbol{R}_a(s)\boldsymbol{U}_4(s)$ is column-reduced.

Substitute $\boldsymbol{R}_a(s) := \boldsymbol{R}_a(s)\boldsymbol{U}_4(s)$.

For assumed poles derive $\boldsymbol{P}_a(s) = \boldsymbol{\Lambda}(s)$ where $\boldsymbol{\Lambda}(s) = \text{diag}[\lambda_i(s), i = 1,2,...,m]$ with $\deg[\lambda_i(s)] = \deg_{ci}[\boldsymbol{R}_a(s)]$.

Step 3.5

Derive minimal state space realization of $\boldsymbol{R}_a(s)\boldsymbol{P}_a{}^{-1}(s)$

$$\begin{aligned}
\dot{\boldsymbol{x}}_a(t) &= \boldsymbol{A}_a\boldsymbol{x}_a(t) + \boldsymbol{B}_a\boldsymbol{u}_{oa}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}_a\boldsymbol{x}_a(t) + \boldsymbol{D}_a\boldsymbol{u}_{oa}(t),
\end{aligned} \tag{22}$$

where $\boldsymbol{x}_a(t) \in R^{n_a}$, $\boldsymbol{u}_{oa}(t) \in R^m$ and $\boldsymbol{u}(t) \in R^m$ are vectors of state, input and output of this element respectively.

Step 3.6

Connect (in series) additional dynamic element (22) with the plant (1)

$$\dot{\boldsymbol{x}}_{oa}(t) = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{BC}_a \\ \boldsymbol{0} & \boldsymbol{A}_a \end{bmatrix} \boldsymbol{x}_{oa}(t) + \begin{bmatrix} \boldsymbol{BD}_a \\ \boldsymbol{B}_a \end{bmatrix} \boldsymbol{u}_{oa}(t) \tag{23}$$

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{DC}_a \end{bmatrix} \boldsymbol{x}_{oa}(t) + \boldsymbol{DD}_a \boldsymbol{u}_{oa}(t), \tag{24}$$

where vector $\boldsymbol{x}_{oa}(t)$ comes from substitution

$$\boldsymbol{x}_{oa}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix}. \tag{25}$$

Step 3.7

Using the Wolovich's structure theorem derive the *r.r.p.* transfer matrix fraction $\boldsymbol{B}_1(s)\boldsymbol{A}_1^{-1}(s)$ for obtained state space description (23) of the augmented plant.

Go to Step 2.

Step 4

If $m = l$ go to Step 5 else according to theorem 2 to eliminate any unobservable poles of the decoupled system calculate $\bar{\boldsymbol{B}}(s)$ to obtain $[\mathbf{B}^{\mathrm{T}}(s), \bar{\boldsymbol{B}}^{\mathrm{T}}(s)]^{\mathrm{T}}$ unimodular. To do that:

Derive an unimodular matrix $\boldsymbol{U}(s)$ such that $\boldsymbol{B}(s)\boldsymbol{U}(s) = [\ \boldsymbol{I}_l \quad \boldsymbol{0}\ ]$. Let $\boldsymbol{U}^{-1}(s) = [\ \boldsymbol{U}_1^T(s) \quad \boldsymbol{U}_2^T(s)\ ]^T$, where $\boldsymbol{U}_1(s) \in R[s]^{l\times m}$, $\boldsymbol{U}_2(s) \in R[s]^{(m-l)\times m}$. Substitute $\bar{\boldsymbol{B}}(s) = \boldsymbol{U}_2(s)$.

Step 5

To determine degrees of the diagonal elements of $\boldsymbol{D}(s)$, which decide about the number of controllable and observable poles of the decoupled system:

Perform the right to left conversion of $\boldsymbol{A}_1(s)\begin{bmatrix} \boldsymbol{B}(s) \\ \bar{\boldsymbol{B}}(s) \end{bmatrix}^{-1} = \widetilde{\boldsymbol{Q}}^{-1}(s)\widetilde{\boldsymbol{P}}(s)$ to obtain with row-reduced.

Determine $\nu_j = \deg_{rj}[\widetilde{\boldsymbol{Q}}(s)]$ for $j = 1, 2, \ldots, m$ and define $\nu = \max\{\nu_j\}$.

Given the ν_j and ν derive $\hat{\boldsymbol{P}}(s) = \mathrm{diag}[s^{\nu-\nu_i}]\widetilde{\boldsymbol{P}}(s)$.

Let $\hat{\boldsymbol{P}}(s) = [\hat{\boldsymbol{P}}^F(s), \hat{\boldsymbol{P}}^L(s)]$, where $\hat{\boldsymbol{P}}^F(s) \in R[s]^{m\times l}$ and $\hat{\boldsymbol{P}}^L(s) \in R[s]^{m\times(m-l)}$.

Define $\hat{\boldsymbol{P}}^F(s) = [\boldsymbol{P}_1^F(s) \vdots \mathbf{P}_2^F(s) \vdots \ \ldots \ \vdots \boldsymbol{P}_k^F(s)]$ where $\hat{\boldsymbol{P}}_i^F(s) \in R[s]^{m\times l_i}$, $i = 1, 2, \ldots, k$.

Step 6

For $i = 1,2,\ldots,k$ and $j = 1,2,\ldots,l_i$ determine degrees $\bar{d}^i_j$ for diagonal elements $d^i_j(s)$ of $\boldsymbol{D}_{ii}(s)$ from the constraint $\deg d^i_j(s) = \max\{\deg_{cj}\hat{\boldsymbol{P}}^F_i(s) - \nu, 0\}$.

Step 7

According to theorem 2 determine set of the controllable and observable poles of the decoupled system. Using freely chosen (stable) values of poles for block decoupled system set the Hurwitz matrix $\boldsymbol{S}(s) = \mathrm{diag}[\bar{s}_i(s)]$, $i = 1,2,\ldots,l$ and calculate $\boldsymbol{D}(s) = \boldsymbol{U}^{-1}(s)\boldsymbol{S}(s)\boldsymbol{V}^{-1}(s)$ where $\boldsymbol{V}(s) = \mathrm{block\,diag}[\boldsymbol{V}_i(s)]$ and $\boldsymbol{U}(s) = \boldsymbol{I}_l$.

Step 8

Perform the right to left conversion of $\boldsymbol{A}_1(s)\begin{bmatrix} \boldsymbol{D}(s)\boldsymbol{B}(s) \\ \bar{\boldsymbol{B}}(s) \end{bmatrix}^{-1} = \boldsymbol{\Phi}_D^{-1}(s)\boldsymbol{\Phi}_N(s)$ to obtain $\boldsymbol{\Phi}_N(s) \in R[s]^{m\times m}$ with row-reduced.

Determine $\mu_j = \deg_{rj}[\boldsymbol{\Phi}_D(s)]$, $j = 1,2,\ldots,m$ and define $\mu = \max\{\mu_j\}$.

Given the μ_j and μ derive $\hat{\boldsymbol{\Phi}}_N(s) = \mathrm{diag}[s^{\mu-\mu_j}]\boldsymbol{\Phi}_N(s)$. Select an unimodular matrix $\hat{\boldsymbol{W}}(s) \in R[s]^{m\times m}$ such that $\hat{\boldsymbol{\Phi}}_N(s)\hat{\boldsymbol{W}}(s)$ is column-reduced.

Step 9

Determine degrees $\bar{l}_j = \deg[\hat{l}_j(s)]$ for $j = 1,2,\ldots,m$ from the constraint $\bar{l}_j = \max\{\deg_{cj}[\ hat\boldsymbol{\Phi}_{\boldsymbol{N}}(s)\hat{\boldsymbol{W}}(s)] - \mu, 0\}$ and set $\hat{\boldsymbol{L}}(s) = \mathrm{diag}[\hat{l}_j(s)]$ with $\hat{l}_j(s)$ chosen freely as stable (monic) polynomials suited to the assumed (according to theorem 2 uncontrollable) poles of the transfer matrix $\boldsymbol{T}_{yq}(s)$.

Step 10

Calculate $[\boldsymbol{L}(s), \bar{\boldsymbol{L}}(s)] = \hat{\boldsymbol{L}}(s)\hat{\boldsymbol{W}}(s)$ to obtain the matrices $\boldsymbol{L}(s) \in R[s]^{m\times l}$ and $\bar{\boldsymbol{L}}(s) \in R[s]^{m\times(m-l)}$, the first l and the last $m-l$ columns of $\hat{\boldsymbol{L}}(s)\hat{\boldsymbol{W}}(s)$.

Step 11

Execute right matrix division $[\boldsymbol{L}(s)\boldsymbol{D}(s)\boldsymbol{B}(s) + \bar{\boldsymbol{L}}(s)\bar{\boldsymbol{B}}(s)]\boldsymbol{A}_1^{-1}(s) = \boldsymbol{G}(s) - \boldsymbol{F}(s)\boldsymbol{A}_1^{-1}(s)$ where $\boldsymbol{G}(s) \in R[s]^{m\times m}$ is the quotient and $-\boldsymbol{F}(s) \in R[s]^{m\times m}$ is the remainder.

Step 12

Given the column structure of the plant's denominator matrix $\boldsymbol{A}_1(s)$ and the obtained matrix $\boldsymbol{F}(s)$, determine the feedback matrix $\boldsymbol{F}$ from the equation $\boldsymbol{F}(s) = \boldsymbol{F}\hat{\boldsymbol{T}}\hat{\boldsymbol{S}}(s)$, which results from the well-known Wolovich's structure theorem [21].

Step 13

If the plant's state vector is not accessible for direct measurement in order to design the full order Luenberger observer set the matrix

$$\bar{\boldsymbol{C}}_2(s) = \mathrm{diag}[\bar{c}_j(s)], \quad j = 1, 2, ..., l$$

where $\bar{c}_j(s) = \prod_{i=1}^{\bar{d}_j}(s - s_i)$. The s_i are assumed (stable) values of poles for the observer and $\bar{d}_i$ are observability indices equal to the row degrees of matrix $\boldsymbol{A}_2(s)$, denominator matrix of the *r.l.p.* matrix fraction description of the (for left invertible plants original) plant's transfer matrix.

Transform matrix $\bar{\boldsymbol{C}}_2(s)$ to the matrix $\boldsymbol{C}_2(s)$ with the same (row) structure as $\boldsymbol{A}_2(s)$.

Determine the gain matrix $\boldsymbol{L}$ of the observer from the equation

$$\boldsymbol{C}_2(s) - \boldsymbol{A}_2(s) = \widetilde{\boldsymbol{S}}(s)\widetilde{\boldsymbol{T}}\boldsymbol{L} \tag{26}$$

where $\widetilde{\boldsymbol{S}}(s)$ and $\widetilde{\boldsymbol{T}}$ are calculated during *r.l.p.* (dual) factorization of the plant's transfer matrix.

6. Example

In order to illustrate the theoretical considerations an example of design of a decoupling control system is presented. Let assume a plant (of $n = 5$ order with $m = 3$ inputs and $l = 4$ outputs) defined by the following matrices of the state and output equations (1)

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{C}=\begin{bmatrix}1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\quad \boldsymbol{D}=\boldsymbol{0}.$$

This plant can be described in the *r.r.p.* matrix fraction as follows

$$\boldsymbol{B}_1(s)=\begin{bmatrix}s-2&s-8&4\\1&s+4&-1\\0&1&0\\0&s&0\end{bmatrix},\quad \boldsymbol{A}_1(s)=\begin{bmatrix}s^2-2s&-8s-1&4s\\s-2&s^2+s-6&-s+3\\0&0&s+1\end{bmatrix}.$$

It has the poles $s_1=2$, $s_{2,3}=-0.2150\pm i1$, $s_5=-0.5698$, and one transmission zero $s_1^o=2$. So, it is unstable and nonminimum phase.

Before start of design procedure we have assumed that: the control system will be block decoupled with the partition (4) of the output and reference input taken as $l_1=1$, $l_2=1$ and $l_3=2$ which allows existing a coupling between signals $y_3(t)$ and $y_4(t)$.

According to the assumed partition after calculations of Step 2 the transfer matrix (2) takes the form (18) with matrices

$$\boldsymbol{B}_{\mathrm{m}}(s)=\begin{bmatrix}s-2&s-8&4\\1&s+4&-1\\0&1&0\end{bmatrix},\quad\text{and}\quad \boldsymbol{P}(s)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-s&1\end{bmatrix}.$$

Then calculations were continued for a square plant with matrix $\boldsymbol{B}_1(s):=\boldsymbol{B}_m(s)$.

As the calculated in Step 3 interconnection transmission zero of the plant is stable then it is not necessary to synthesize an additional dynamic element $\boldsymbol{R}_a(s)\boldsymbol{P}_a^{-1}(s)$. The system after decoupling will posses an unobservable but stable pole $s_{uo}=-2$. Calculation of this element would result in growth of system degree the order of feedforward compensator (including additional dynamic element) would increase by 2. It would be also necessary to assume seven instead four poles for the denominator matrix of the decoupled system $D(s)$.

Assuming in Step 7 the following values of poles:

- for the first block: $s_1=-0.5$,
- for the second block: $s_2=-0.4$,
- for the third block: $s_3=-0.6$, $s_4=-0.4$,

We set matrix as:

$$D(s) = \begin{bmatrix} s+0.5 & 0 & 0 \\ 0 & s+0.4 & 0 \\ 0 & 0 & s^2+s+0.24 \end{bmatrix}$$

and obtain: -dynamic feedforward compensator $\boldsymbol{G}^{-1}(s)\boldsymbol{L}(s)$

$$\boldsymbol{L}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{G}(s) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and the feedback matrix

$$\boldsymbol{F} = \begin{bmatrix} -1 & -0.5 & -3 & 0.5 & 2 \\ -2.4 & 0 & -7.6 & -3.4 & 3.4 \\ -2 & 1 & -6.24 & 0 & 4 \end{bmatrix}.$$

The gain matrix of the full order Luenberger observer with the values of its poles assumed as $s_1 = -3$, $s_2 = -3$, $s_3 = -4$, $s_4 = -5$, $s_5 = -2$ is given as

$$\boldsymbol{L} = \begin{bmatrix} 3 & 0 & 1 & -2 \\ 0 & 0 & 5 & 1 \\ -1 & 0 & -2 & 2 \\ 0 & -6 & -8 & 1 \\ 0 & 8 & -5 & -1 \end{bmatrix}.$$

As it is shown in Fig. 3 according to our assumptions there is no influence between signals $y_1(t)$, $y_2(t)$ and signals $y_3(t)$ and $y_4(t)$. Change of the reference value for the first output $y_{o1}(t)$ at $t = 20$s influence only first output $y_1(t)$. Similarly reference input $y_{o2}(t)$ does not influence any other outputs but $y_2(t)$. State of two outputs $y_3(t)$ and $y_4(t)$ depends only on reference input $y_{o3}(t)$. So, the system is (block) decoupled and all of the assumed design objectives are achieved.

The designed control system (including the plant) has the order $n_s = n + n_o = 10$, where $n_o = n = 5$ is the order of Luenberger observer. The system has one unobservable pole $s_{uo} = -2$ as well as five uncontrollable poles $s_1 = -3$, $s_2 = -3$, $s_3 = -4$, $s_4 = -5$, $s_5 = -2$ for observer, which define stable hidden modes of the system.

7. Conclusions and final remarks

In the paper an universal algorithm for block decoupling of dynamic plants with the number of inputs less, equal or greater than the number of their outputs has been

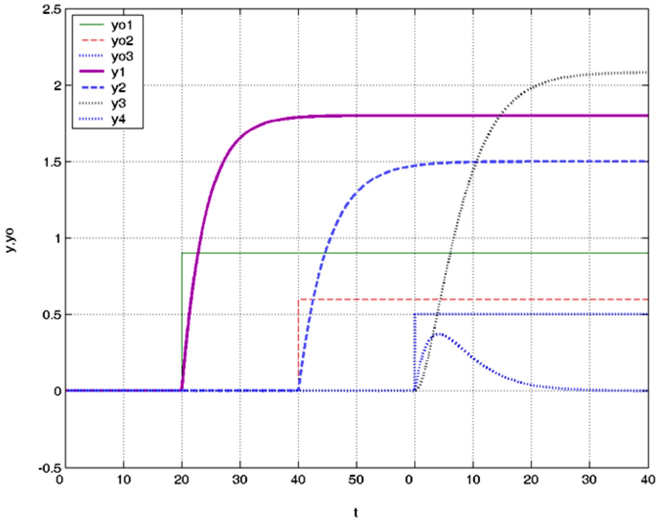

Figure 3. Results of simulation of the block decoupled control system.

presented. The proposed algorithm guarantees to achieve the assumed dynamics in each control loop and ensures internal stability and internal property for both unstable and non-minimum phase proper plants.

Results of simulation confirm the correctness of the proposed algorithm and giving possibility to automate the process of design the control system the algorithm would be used to build e.g. adaptive decoupled, reconfigurable, fault tolerant MIMO systems.

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