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Exponential stability of nonlinear neutral type systems

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Nonlinear neutral type systems with distributed and discrete delays are considered. Explicit exponential stability conditions are established. The main tool is a combined usage of the recent norm estimates for the matrix resolvents, the Urysohn theorem and estimates for fundamental solutions of the linear parts.

Key words: neutral type systems, nonlinear systems, stability

1. Introduction and preliminaries

This paper is devoted to stability analysis of nonlinear neutral type systems with delay. The stability of neutral type equations was investigated by many authors. The classical results can be found in [8, 9]. The basic method for the stability analysis is the direct Lyapunov method. By that method many very strong results are obtained. Mainly, scalar equations [1, 11, 12] and linear systems [2, 5] were considered. But finding Lyapunov's type functionals for nonlinear neutral type systems is usually difficult. The papers [3, 14] should be mentioned. In these papers, systems with discrete delays have been investigated. The global exponential stability of periodic solutions for impulsive neutral-type neural networks with delays have been explored in [13].

In the present paper we suggest the explicit exponential stability conditions for a class of nonlinear neutral type systems with distributed and discrete delays.

The paper is organized as follows. It consists of 8 sections. In this section we investigate the linear parts of the systems considered below. In Section 2, the main result is formulated. The proof of the main result is presented in Section 4. In Sections 4 and 5 we establish auxiliary results, which then are used to illustrate the main result. In Section 6 we explore systems whose linear parts have discrete delays. Systems whose linear parts have distributed delays are investigated in Section 7. The illustrative example is presented in Section 8.

Let $\mathbb{C}^n$ be a complex Euclidean n-dimensional space with a scalar product $(.,.)_n$, the unit matrix I, and the Euclidean norm $\|.\|_n = \sqrt{(.,.)_n}$. For a linear operator A in $\mathbb{C}^n$ (matrix), $\|A\|_n = \sup_{x\in\mathbb{C}^n} \|Ax\|_n/\|x\|_n$ is the spectral norm.

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Denote by $C(\chi) = (\chi, \mathbb{C}^n)$ the space of continuous functions defined on a set $\chi \subset \mathbb{R}$ with values in $\mathbb{C}^n$ and the finite norm $\|w\|_{C(\chi)} = \sup_{t\in\chi} \|w(t)\|_n$. $C^1(\chi) = C^1(\chi, \mathbb{C}^n)$ is the space of continuously differentiable functions defined on χ with values in $\mathbb{C}^n$ and the norm $\|w\|_{C^1(\chi)} = \|w\|_{C(\chi)} + \|\dot{w}\|_{C(\chi)}$, where $\dot{w}$ is the derivative of w; $L^2(\chi) = L^2(\chi, \mathbb{C}^n)$ is the space of functions defined on with values in $\mathbb{C}^n$ and the finite norm

$$\|w\|_{L^2(\chi)} = [\int_\chi \|w(t)\|_n^2 dt]^{1/2}.$$

First, consider the linear problem

$$\dot{y}(t) - \int_0^\eta d\widetilde{R}(\tau)\dot{y}(t-\tau) = \int_0^\eta dR(\tau)y(t-\tau) \ (\dot{y}(t) = \frac{dy}{dt};\ 0 < \eta = const < \infty;\ t \geqslant 0), \quad (1)$$

$$y(t) = \phi(t) \text{ for } -\eta \leqslant t \leqslant 0, \quad (2)$$

where $\phi \in C^1(-\eta, 0)$ is given; $R(s) = (r_{ij}(s))_{i,j=1}^n$ and $\widetilde{R}(s) = (\widetilde{r}_{ij}(s))_{i,j=1}^n$ are real $n \times n$-matrix-valued functions defined on $[0, \eta]$ whose entries have bounded variations

$$var(r_{ij}) = \int_0^\eta |dr_{jk}| < \infty \text{ and } var(\widetilde{r}_{ij}) < \infty.$$

The integrals in (1) are understood as the Lebesgue - Stieltjes integrals.

We define the variation of $R(.)$ as the matrix

$$Var(R) = (var(r_{ij}))_{i,j=1}^n,$$

and denote $V\ (R) := \|Var(R)\|_n$. So $V\ (R)$ is the spectral norm of matrix $Var\ (R)$. Similarly, $V\ (\widetilde{R})$ is defined. It is assumed that $R(\eta)$ is invertible and

$$V(\widetilde{R}) < 1. \quad (3)$$

A solution of problem (1), (2) is an absolutely continuous vector function $y(t)$ defined on $[-\eta, \infty)$ and satisfying (1) and (2).

The matrix-valued function

$$K(z) = Iz - z\int_0^\eta exp(-zs)d\widetilde{R}(s) - \int_0^\eta exp(-zs)dR(s)\ (z \in \mathbb{C})$$

is the characteristic matrix-valued function to equation (1) and the zeros of $det\ K(\lambda)$ are the characteristic values of $K(.)$; $\lambda \in \mathbb{C}$ is a regular value of $K(.)$, if $det\ K(\lambda) \neq 0$. *Everywhere below it is assumed that all the characteristic values of $K(.)$ are in the* *open left half-plane* C_-. Below we give some conditions that provide the location of the characteristic values in C_-.

Due to Theorem 3.1.1 from [9, p. 114], under conditions (3) equation (1) is asymptotically stable and L^2-stable (that is, the inequality (16) below holds), if all the characteristic values of $K(.)$ are in C_-. Moreover, the integral

$$G(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} K^{-1}(i\omega) d\omega \; (t \geqslant 0) \tag{4}$$

exists and the function $G(t)$ defined by (4) for $t \geqslant 0$ and by $G(t) = 0$ for $-\eta \leqslant t < 0$ is called the fundamental solution to (1). From (4) it follows that $G(t)$ is a solution to (1) and $G(0) = I$, cf. [9].

2. Statement of the main result

Introduce the set $\Omega(r) = \{f \in C(-\eta,\infty) : \|f\|_{C(-\eta,\infty)} \leqslant r\}$ for a given number $0 < r \leqslant \infty$. Consider the equation

$$\dot{x}(t) - \int_0^{\eta} d\widetilde{R}(s)\dot{x}(t-s) - \int_0^{\eta} dR(t,s)x(t-s) = [Fx](t) \; (t \geqslant 0), \tag{5}$$

where F is a continuous mapping $\Omega(r) \to C(-\eta,\infty)$, satisfying the following condition: there is a nondecreasing function $\nu(t)$ defined on $[0,\eta]$, such that

$$\|[Ff](t)\|_n \leqslant \int_0^{\eta} \|f(t-s)\|_n d\nu(s) \;\; (t \geqslant 0; f \in \Omega(r)). \tag{6}$$

A (mild) solution of problem (6), (2) is a continuous function $x(t)$ defined on $[-\eta,\infty)$, such that

$$x(t) = z(t) + \int_0^{t} G(t-t_1)[Fx](t_1)dt_1 \; (t \geqslant 0), \tag{7}$$

$$x(t) = \phi(t) \; (-\eta \leqslant t \leqslant 0), \tag{8}$$

where $G(t)$ is the fundamental solution of the linear equation (1) and $z(t)$ is a solution of the problem (1), (2).

We will say that the zero solution to equation (5) is exponentially stable, if there are positive constants $r_0 \leqslant r$, $\hat{m}$ and ε, such that for any ϕ with

$$\|\phi\|_{C^1(-\eta,0)} \leqslant r_0, \tag{9}$$

problem (5), (2) has at least one solution, and any solution $x(t)$ of that problem satisfies

$$\|x(t)\|_n \leqslant \hat{m}e^{-\varepsilon t}\|\phi\|_{C^1(-\eta,0)} \quad (t \geqslant 0),$$

provided (9) holds.

Put

$$v_0 = \frac{2\,V(R)}{1 - V(\widetilde{R})} \text{ and } \theta(K) := \sup_{-v_0 \leqslant \omega \leqslant v_0} \|K^{-1}(i\omega)\|_n.$$

Now we are in a position to formulate our main result.

Theorem 1 *Let the conditions* (3), (6) *and*

$$var(\nu)\theta(K) < 1 \tag{10}$$

hold. Then the zero solution to (5) *is exponentially stable.*

This theorem is proved in the next section. Below we give the estimates for $\theta(K)$ and examples.

For instance, (5) can take the form

$$\begin{aligned} &\dot{x}(t) - \int_0^\eta \widetilde{A}(\tau)\dot{x}(t-s)d\tau - \sum_{k=1}^{\widetilde{m}} \widetilde{A}_k \dot{x}(t-\widetilde{h}_k) = \\ &\int_0^\eta A(s)x(t-s)ds + \sum_{k=0}^{m} A_k x(t-h_k) + [Fx](t) \quad (t \geqslant 0), \end{aligned} \tag{11}$$

where $m, \widetilde{m}$ are finite integers; $0 = h_0 < h_1 < \ldots < h_m \leqslant \eta$ and $0 < \widetilde{h}_1 < \ldots < \widetilde{h}_{\widetilde{m}} \leqslant \eta$ are constants, $A_k, \widetilde{A}_k$ are constant matrices and $A(s), \widetilde{A}(s)$ are integrable on $[0,\eta]$. Put

$$V_1 = \int_0^\eta \|A(s)\|_n ds + \sum_{k=0}^{m} \|A_k\|_n, \; \widetilde{V}_1 = \int_0^\eta \|\widetilde{A}(s)\|_n ds + \sum_{k=1}^{\widetilde{m}} \|\widetilde{A}_k\|_n,$$

assuming that V_C and $\widetilde{V}_C$ are finite. It is not hard to show that in this case $V(R) \leqslant V_1$ and $V(\widetilde{R}) \leqslant \widetilde{V}_1$.

3. Proof of Theorem 1

For an $f \in C([-\eta,T],\mathbb{C}^n), T < \infty$, put

$$Ef(t) = \int_0^\eta dR(s)f(t-s), \widetilde{E}f(t) = \int_0^\eta d\widetilde{R}(s)f(t-s) \quad (0 \leqslant t \leqslant T).$$

Lemma 1 *The inequalities*

$$\|Ef\|_{C([-\eta,T],\mathbb{C}^n)\to C([0,T],\mathbb{C}^n)} \leqslant \sqrt{n}\ V\ (R) \tag{12}$$

and

$$\|Ef\|_{L^2([-\eta,T],\mathbb{C}^n)\to L^2([0,T],\mathbb{C}^n)} \leqslant V\ (R) \tag{13}$$

are valid.

Proof Let $f(t) = (f_k(t))_{k=1}^n \in C([-\eta,T],\mathbb{C}^n)$. For each coordinate $(Ef)_j(t)$ of $Ef(t)$ we have

$$|(Ef)_j(t)| = |\int\limits_0^\eta \sum_{k=1}^n f_k(t-s)dr_{jk}(s)| \leqslant \sum_{k=1}^n \int\limits_0^\eta |dr_{jk}| \max_{0\leqslant s\leqslant\eta} |f_k(t-s)| =$$

$$\sum_{k=1}^n var(r_{jk}) \max_{0\leqslant s\leqslant\eta} |f_k(t-s)|.$$

Hence,

$$\sum_{j=1}^n |(Ef)_j(t)|^2 \leqslant \sum_{j=1}^n \left(\sum_{k=1}^n var(r_{jk})\|f_k\|_{C(-\eta,T)}\right)^2 =$$

$$\|Var\ (R)\ z_C\|_n^2 \leqslant (var(R)\|z_C\|_n)^2\ \ (0\leqslant t\leqslant T),$$

where $z_C = (\|f_k\|_{C[-\eta,T]})_{k=1}^n$, $\|.\|_{C(-\eta,T)} = \|.\|_{C([-\eta,T],\mathbb{C})}$. But

$$\|z_C\|_n^2 = \sum_{k=1}^n \|f_k\|^2_{C(-\eta,T)} \leqslant$$

$$n\max_k \|f_k\|^2_{C(-\eta,T)} \leqslant n\sup_t \sum_{k=1}^n \|f_k(t)\|_n^2 = n\|f\|^2_{C([-\eta,T],\mathbb{C}^n)}.$$

So

$$\|Ef\|_{C([0,T],\mathbb{C}^n)} \leqslant \sqrt{n}var\ (R)\|f\|_{C([-\eta,T],\mathbb{C}^n)}$$

and thus inequality (12) is proved.

Now consider the norm in space L^2. We have

$$\int\limits_0^T |(Ef)_j(t)|^2 dt \leqslant \int\limits_0^T (\sum_{k=1}^n \int\limits_{-\eta}^0 |f_k(t-s)||dr_{jk}(s)|)^2 dt =$$

$$\int\limits_{-\eta}^0 \int\limits_{-\eta}^0 \sum_{i=1}^n \sum_{k=1}^n |dr_{jk}(s)||dr_{ji}(s_1)| \int\limits_0^T |f_k(t-s)f_i(t-s_1)|dt.$$

By the Schwarz inequality

$$\left(\int_0^T |f_k(t-s)f_i(t-s_1)|dt\right)^2 \leqslant \int_0^T |f_k(t-s)|^2 dt \int_0^T |f_i(t-s_1)|^2 dt \leqslant$$

$$\int_{-\eta}^T |f_k(t)|^2 dt \int_{-\eta}^T |f_i(t)|^2 dt.$$

Thus

$$\int_0^T |(Ef)_j(t)|^2 dt \leqslant \sum_{i=1}^n \sum_{k=1}^n var(r_{jk}) var(r_{ji}) \|f_k\|_{L^2(-\eta,T)} \|f_i\|_{L^2(-\eta,T)} =$$

$$(\sum_{k=1}^n var(r_{jk}) \|f_k\|_{L^2(-\eta,T)})^2 \quad (\|f_k\|_{L^2(-\eta,T)} = \|f_k\|_{L^2([-\eta,T],\mathbb{C})})$$

and therefore

$$\sum_{j=1}^n \int_0^T |(Ef)_j(t)|^2 dt \leqslant \sum_{j=1}^n (\sum_{k=1}^n var(r_{jk}) \|f_k\|_{L^2(-\eta,T)})^2 =$$

$$\|Var\ (R)\ z_2\|_n^2 \leqslant (var(R)\|z_2\|_n)^2$$

where z_2 is the vector with the coordinates $\|f_k\|_{L^2(-\eta,T)}$. But $\|z_2\|_n = \|f\|_{L^2([-\eta,T],\mathbb{C}^n)}$. So (13) is also proved. $\square$

Lemma 2 *The equality* $\sup_{-\infty\leqslant\omega\leqslant\infty} \|K^{-1}(i\omega)\|_n = \theta(K)$ *is valid. Moreover,* $\theta(K) \geqslant \|R^{-1}(\eta)\|_n$.

Proof Without loss of generality assume that $R(0-) = 0$. We have $-K(0) = R(\eta)$. Thus $\|K^{-1}(0)\|_n = \|R^{-1}(\eta)\|_n \geqslant \frac{1}{\|R(\eta)\|_n} \geqslant \frac{1}{V\ (R)}$. In addition,

$$\|K(i\omega)x\|_n \geqslant (|\omega|(1-V\ (\widetilde{R})) - V\ (R))\|x\|_n \geqslant V(R)\|x\|_n \ \ (\omega \in \mathbb{R}, |\omega| \geqslant v_0;\ x \in \mathbb{C}^n).$$

So $\|K^{-1}(i\omega)\|_n \leqslant \frac{1}{V(R)} \leqslant \|K^{-1}(0)\|_n$ $(|\omega| \geqslant v_0)$. Thus the maximum of $\|K^{-1}(i\omega)\|_n$ is attained on $[-v_0, v_0]$. As claimed. $\square$

Lemma 3 *Let condition* (6) *hold with* $r = \infty$*. Then* $\|Fw\|_{L^2(-\eta,T)} \leqslant var(\nu)\|w\|_{L^2(-\eta,T)}$ $(w \in L^2(-\eta,T))$ *for any* $T > 0$.

Proof In the space $L^2(-\eta,\infty)$ of scalar functions w introduce the operator $\hat{E}_\nu$ by

$$(\hat{E}_\nu w)(t) = \int_0^\eta w(t-\tau)d\nu(\tau).$$

Then using the previous lemma, we have $\|\hat{E}_\nu w\|_{L^2(0,\infty)} \leqslant var(\nu)\|w\|_{L^2(-\eta,T)}$. Now (6) with $w(t) = \|f(t)\|_n$ implies the required result. □

Furthermore, use the operator $\hat{G}$ defined on $L^2(0,\infty)$ by

$$\hat{G}f(t) = \int_0^t G(t-t_1)f(t_1)dt_1 \ \ (f \in L^2(0,\infty)),$$

and assume that

$$var\ (\nu)\ \|\hat{G}\|_{L^2(0,\infty)} < 1. \tag{14}$$

Lemma 4 *Let conditions* (3), (14) *and* (6) *with $r=\infty$ hold. Then problem* (5), (2) *has a solution. Moreover, any solution $x(t)$ of that problem satisfies the inequality*

$$\|x\|_{L^2(-\eta,\infty)} \leqslant (1 - var(\nu)\|\hat{G}\|_{L^2(0,\infty)})^{-1}\|z\|_{L^2(-\eta,\infty)}. \tag{15}$$

Proof Take a finite $T > 0$ and define the mapping Φ by

$$\Phi w(t) = z(t) + \int_0^t G(t-t_1)[Fw](t_1)dt_1 \ \ (0 \leqslant t \leqslant T; w \in L^2(0,T)),$$

and $\Phi w(t) = \phi(t)$ for $-\eta \leqslant t \leqslant 0$. Then by Lemma 3 we have

$$\|\Phi w\|_{L^2(-\eta,T)} \leqslant \|\phi\|_{L^2(-\eta,0)} + \|z\|_{L^2(0,T)} + \|\hat{G}\|_{L^2(0,\infty)} var(\nu)\|w\|_{L^2(-\eta,T)}.$$

As it was above mentioned, under condition (3), the linear equation (1) is L^2-stable. That is,

$$\|z\|_{L^2(-\eta,\infty)} \leqslant c_1\|\phi\|_{C^1(-\eta,0)} \ \ (c_1 = const). \tag{16}$$

So Φ maps $L^2(-\eta,T)$ into itself. Taking into account that Φ is compact, due to the Schauder fixed point theorem, we prove the existence of solutions. Furthermore,

$$\|x\|_{L^2(-\eta,T)} = \|\Phi x\|_{L^2(-\eta,T)} \leqslant \|z\|_{L^2(-\eta,T)} + \|\hat{G}\|_{L^2(0,T)} var(\nu)\|x\|_{L^2(-\eta,T)}.$$

Hence we easily obtain (15). □

By the Parseval equality and Lemma 2 we have $\|\hat{G}\|_{L^2(0,\infty)} = \theta(K)$. Now the previous lemma implies the inequality

$$\|x\|_{L^2(-\eta,\infty)} \leqslant (1 - var(\nu)\theta(K))^{-1}\|z\|_{L^2(-\eta,\infty)}.$$

Thus according to (16),

$$\|x\|_{L^2(-\eta,\infty)} \leqslant c_2\|\phi\|_{C^1(-\eta,0)} \ \ (c_2 = const). \tag{17}$$

From (5), and Lemmas 1 and 3 it follows that

$$\|\dot{x}\|_{L^2(0,\infty)} \leqslant V(\widetilde{R})\|\dot{x}\|_{L^2(-\eta,\infty)} + (V(R) + var(\nu))\|x\|_{L^2(-\eta,\infty)}.$$

Or

$$\|\dot{x}\|_{L^2(0,\infty)} \leqslant V(\widetilde{R})(\|\dot{x}\|_{L^2(0,\infty)} + \|\dot{x}\|_{L^2(-\eta,0)}) + (V(R) + var(\nu))\|x\|_{L^2(-\eta,\infty)}.$$

So due to (3) we obtain

Corollary 1 *Under the hypothesis of Lemma 4 we have*

$$\|\dot{x}\|_{L^2(0,\infty)} \leqslant (1 - V(\widetilde{R}))^{-1}[(V(R) + var(\nu))\|x\|_{L^2(0,\infty)} + V(\widetilde{R})\|\dot{\phi}\|_{L^2(-\eta,0)}].$$

The previous corollary and (17) imply the inequality

$$\|\dot{x}\|_{L^2(0,\infty)} \leqslant c_3\|\phi\|_{C^1(-\eta,0)} \ \ (c_3 = const). \tag{18}$$

We need the following simple result.

Lemma 5 *Let* $f \in L^2(0,\infty)$ *and* $\dot{f} \in L^2(0,\infty)$. *Then* $\|f\|^2_{C(0,\infty)} \leqslant 2\|f\|_{L^2(0,\infty)}\|\dot{f}\|_{L^2(0,\infty)}$.

For the proof see [7, Lemma 7.7]. This lemma, (16) and (17) imply the next result.

Lemma 6 *Under the hypothesis of Lemma 4, the inequality*

$$\|x\|_{C(0,\infty)} \leqslant c_4\|\phi\|_{C^1(-\eta,0)} \ \ (c_4 = const) \tag{19}$$

is valid and therefore the zero solution of (5) *is globally stable in the Lyapunov sense.*

Proof of Theorem 1 Substituting

$$x(t) = y_\varepsilon(t)e^{-\varepsilon t} \tag{20}$$

with an $\varepsilon > 0$ into (5), we obtain the equation

$$\dot{y}_\varepsilon - \varepsilon y_\varepsilon - E_{\varepsilon,\widetilde{R}}\dot{y}_\varepsilon + \varepsilon E_{\varepsilon,\widetilde{R}} y_\varepsilon = E_{\varepsilon,R} y_\varepsilon + F_\varepsilon y_\varepsilon, \tag{21}$$

where

$$(E_{\varepsilon,\widetilde{R}}f)(t) = \int_0^\eta e^{\varepsilon\tau} d_\tau \widetilde{R}(t,\tau) f(t-\tau), (E_{\varepsilon,R}f)(t) = \int_0^\eta e^{\varepsilon\tau} d_\tau R(t,\tau) f(t-\tau)$$

and $[F_\varepsilon f](t) = e^{\varepsilon t}[F(e^{-\varepsilon t} f)](t)$. By (6) with $r = \infty$ we have

$$\|[F_\varepsilon f](t)\|_n \leqslant e^{\varepsilon t} \int_0^\eta e^{-\varepsilon(t-s)} \|f(t-s)\|_n d\nu \leqslant e^{\varepsilon\eta} \int_0^\eta \|f(t-s)\|_n d\nu.$$

Taking ε sufficiently small and applying our above arguments to equation (21), according to (19), we obtain

$$\|y_\varepsilon\|_{C(0,\infty)} \leqslant c_\varepsilon \|\phi\|_{C^1(-\eta,0)} \quad (c_\varepsilon = const). \tag{22}$$

Now (20) implies

$$\|x(t)\|_{C(0,\infty)} \leqslant c_\varepsilon \|\phi\|_{C^1(-\eta,0)} e^{-\varepsilon t} \quad (t \geqslant 0). \tag{23}$$

So in the case $r = \infty$, the theorem is proved.

Now let $r < \infty$. By the Urysohn theorem [4, p. 15], there is a continuous scalar-valued function ψ_r defined on $C(0,\infty)$, such that

$$\psi_r(f) = 1 \quad (\|f\|_{C(0,\infty)} < r) \text{ and } \psi_r(f) = 0 \quad (\|f\|_{C(0,\infty)} \geqslant r).$$

Put $[F_r f](t) = \psi_r(f)[Ff](t)$. Clearly, F_r satisfies (6) for all $f \in C(-\eta,\infty)$. Consider the equation

$$\dot{x} - \widetilde{E}\dot{x} = Ex + F_r x. \tag{24}$$

The solution of (24) denote by x_r. For a sufficiently small ε, according to (23), we have $\|x_r(t)\|_{C(0,\infty)} \leqslant c_\varepsilon \|\phi\|_{C^1(-\eta,0)}$. If we take $\|\phi\|_{C^1(-\eta,0)} \leqslant r/c_\varepsilon$, then $F_r x_r = Fx$ and equations (5) and (24) coincide. This and (23) prove the theorem. $\square$

4. Estimates for $\theta(K)$

For an $n \times n$-matrix A, $\lambda_k(A)$ $(k = 1,\dots,n)$ denote the eigenvalues of A numerated in an arbitrary order with their multiplicities, A^* is the adjoint to A and A^{-1} is the inverse to A. $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm of A: $N_2^2(A) = Trace\, AA^*$, $A_I = (A - A^*)/2i$ is the imaginary component.

The following quantity plays a key role in the sequel:

$$g(A) = (N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2)^{1/2}.$$

It is not hard to check that $g^2(A) \leqslant N^2(A) - |Trace\, A^2|$. In Section 2.2 of [6] it is proved that

$$g^2(A) \leqslant 2N_2^2(A_I) \text{ and } g(e^{i\tau}A + zI) = g(A) \tag{25}$$

for all $\tau \in \mathbb{R}$ and $z \in \mathbb{C}$. If A_1 and A_2 are commuting matrices, then

$$g(A_1 + A_2) \leqslant g(A_1) + g(A_2). \tag{26}$$

From Corollary 2.1.2 [6], it follows that for any invertible $n \times n$-matrix A, the inequality

$$\|A^{-1}\|_n \leqslant \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A)} \tag{27}$$

is true, where $\rho(A)$ is the smallest modulus of the eigenvalues of A: $\rho(A) = \min_{k=1,\dots,n} |\lambda_k(A)|$.

Put

$$B(z) = \int_0^\eta z\; exp(-zs)d\widetilde{R}(s) + \int_0^\eta exp(-zs)dR(s).$$

So $K(z) = zI - B(z)$. By (25) $g(B(z)) = g(K(z))$. Thanks to (27), for any regular value z of $K(.)$, the inequality

$$\|[K(z)]^{-1}\|_n \leqslant \Gamma(K(z)) \; (z \in \mathbb{C}) \tag{28}$$

is valid, where

$$\Gamma(K(z)) = \sum_{k=0}^{n-1} \frac{g^k(B(z))}{\sqrt{k!}\rho^{k+1}(K(z))}$$

and $\rho(K(z))$ is the smallest absolute value of the eigenvalues of $K(z)$:

$$\rho(K(z)) = \min_{k=1,\dots,n} |\lambda_k(K(z))|.$$

If $B(z)$ is a normal matrix, then $g(B(z)) = 0$, and $\|[K(z)]^{-1}\|_n \leqslant \rho^{-1}(K(z))$. For example, that inequality holds, if $K(z) = zI - \widetilde{A}ze^{-z\eta} - Ae^{-z\eta}$, where A and $\widetilde{A}$ are Hermitian matrices.

Furthermore, from (28) the inequality

$$\theta(K) \leqslant \Gamma_0(K), \text{ where } \Gamma_0(K) := \sup_{-v_0 \leqslant \omega \leqslant v_0} \Gamma(K(i\omega)) \tag{29}$$

follows. Thus due to Theorem 1 we arrive at the following result.

Theorem 2 *Let all the zeros of K be in C_- and the conditions* (3), (6) *and*

$$var(\nu)\Gamma_0(K) < 1 \tag{30}$$

hold. Then the zero solution to equation (5) *is exponentially stable.*

Denote

$$\hat{g}(B) := \sup_{\omega\in[-v_0,v_0]} g(B(i\omega)) \text{ and } \hat{\rho}(K) := \inf_{\omega\in[-v_0,v_0]} \rho(K(i\omega)).$$

Then we have

$$\Gamma_0(K) \leqslant \hat{\Gamma}(K), \text{ where } \hat{\Gamma}(K) := \sum_{k=0}^{n-1} \frac{\hat{g}^k(B)}{\sqrt{k!}\hat{\rho}^{k+1}(K)}. \tag{31}$$

Now Theorem 2 implies

Corollary 2 *Let all the zeros of K be in C_- and the conditions* (3)*, and $q\hat{\Gamma}(K) < 1$ hold. Then the zero solution to equation* (5) *is exponentially stable.*

Thanks to the definition of $g(A)$, for all $\omega \in \mathbb{R}$ one can write

$$g(B(i\omega)) \leqslant N_2(B(i\omega)) \leqslant \sqrt{n}\|B(i\omega)\|_n \leqslant \sqrt{n}(|\omega|V(\tilde{R}) + V(R)). \tag{32}$$

The sharper estimates for $g(B(i\omega))$ under additional conditions are given below.

5. Auxiliary results

In this section we investigate scalar quasi-polynomials. The results obtained in this section will be used in the rest of the paper. Consider the function

$$k(z) = z\left(1 - \int_0^\eta e^{-\tau z} d\tilde{\mu}\right) + \int_0^\eta e^{-\tau z} d\mu \ \ (z \in \mathbb{C}), \tag{33}$$

where $\mu = \mu(\tau)$ and $\tilde{\mu} = \tilde{\mu}(\tau)$ are nondecreasing functions defined on $[0,\eta]$, with

$$0 < var(\tilde{\mu}) < 1 \text{ and } var(\mu) < \infty. \tag{34}$$

Put

$$v_1 = \frac{2\, var(\mu)}{1 - var(\tilde{\mu})}.$$

Lemma 7 *The equality* $\inf_{-\infty\leqslant\omega\leqslant\infty} |k(i\omega)| = \inf_{-v_1\leqslant\omega\leqslant v_1} |k(i\omega)|$ *is valid.*

Proof We have $-k(0) = var(\mu)$. In addition,

$$|k(i\omega)| \geqslant |\omega|(1 - var\,(\tilde{\mu})) - var\,(\mu) \geqslant var(\mu) \ \ (\omega \in \mathbb{R}, |\omega| \geqslant v_1).$$

Thus the minimum of $|k(i\omega)|$ is attained on $[-v_1, v_1]$. As claimed. □

Put

$$\xi(z) = z + \int_0^\eta exp(-zs) d\mu(s) \ (z \in \mathbb{C}).$$

Lemma 8 *Let* $\eta\, var(\mu) < \pi/4$. *Then all the zeros of* $\xi(z)$ *are in* C_- *and* $\inf_{\omega\in\mathbb{R}} |\xi(i\omega)| \geqslant \hat{d}$, *where*

$$\hat{d} := \int_0^{\eta} cos(2var(\mu)\tau)d\mu(\tau).$$

Proof Put $v_2 = var(\mu)$ and

$$\xi(m,z) = z + (1-m)v_2 + m\int_0^{\eta} exp(-z\tau)d\mu(\tau)\ (m \in\ [0,1]).$$

We have $\xi(m,0) = var(\mu) = v_2$. So according to the previous lemma,

$$\inf_{\omega\in\mathbb{R}} |\xi(m,i\omega)| = \inf_{-2v_2\leqslant\omega\leqslant 2v_2} |\xi(m,\omega)|.$$

In particular, $\inf_{-2v_2\leqslant\omega\leqslant 2v_2} |\xi(i\omega)| = \inf_{\omega\in\mathbb{R}} |\xi(i\omega)|$. But

$$|\xi(m,i\omega)|^2 = |i\omega + v_2(1-m) + m\int_0^{\eta} e^{-\tau i\omega}d\mu(\tau)|^2 =$$

$$(\omega - m\int_0^{\eta} sin\ (\tau\omega)d\mu(\tau))^2 + (v_2(1-m) + m\int_0^{\eta} cos\ (\tau\omega)d\mu(\tau))^2 \geqslant w^2(m)\ (|\omega| \leqslant 2v_2),$$

where

$$w(m) = v_2(1-m) + m\int_0^{\eta} cos\ (2\tau v_2)d\mu(\tau).$$

The derivative of $w(m)$ is non-positive. So $w(m) \geqslant w(1) = \hat{d}$. Thus

$$\inf_{\omega\in\mathbb{R}} |\xi(m,i\omega)| = \inf_{-2v_2\leqslant\omega\leqslant 2v_2} |\xi(m,i\omega)| \geqslant \hat{d}\ \ (m \in\ [0,1]). \tag{35}$$

Furthermore, assume that $\xi(z)$ has a zero in the closed right hand plane $\overline{C}_+$. Since $\xi(0,z) = z + v$ does not have zeros in $\overline{C}_+$, $\xi(m_0, is)$ $(\omega \in \mathbb{R})$ should have a zero for some m_0, according to continuous dependence of zeros on coefficients. But according to (35) this is impossible. The proof is complete. □

Lemma 9 *Let the conditions* (34),

$$\eta v_1 < \pi/2 \tag{36}$$

and

$$d_0 := \int_0^\eta cos(v_1\tau)d\mu - v_1 \int_0^\eta sin(v_1\tau)d\tilde{\mu} > 0 \tag{37}$$

hold. Then all the zeros of $k(.)$ *are in* C_- *and*

$$\inf_{-\infty \leqslant \omega \leqslant \infty} |k(i\omega)| \geqslant d_0 > 0. \tag{38}$$

Proof Clearly, $Re\ k(i\omega) = -\omega \int_0^\eta sin(\omega\tau)d\tilde{\mu} + \int_0^\eta cos(\omega\tau)d\mu$. Hence, with $|\omega| \leqslant v_1$, we have

$$|Re\ k(i\omega)| = Re\ k(i\omega) \geqslant -v_1 \int_0^\eta sin(v_1\tau)d\tilde{\mu} + \int_0^\eta cos(v_1\tau)d\mu = d_0 > 0.$$

This proves (38). Furthermore, put $g_2(z) = -z\int_0^\eta e^{-\tau z}d\tilde{\mu}$ and $k(m,z) = \xi(z) + mg_2(z)$, $0 \leqslant m \leqslant 1$. According to Lemma 7, $\inf_{\omega\in\mathbb{R}} |k(m,i\omega)| = \inf_{-v_1\leqslant\omega\leqslant v_1} |k(m,i\omega)|$. Hence, due to (38),

$$|k(m,i\omega)| \geqslant \int_0^\eta cos(v_1\tau)d\mu - mv_1 \int_0^\eta sin(v_1\tau)d\tilde{\mu} > d_0\ (\omega \in \mathbb{R}). \tag{39}$$

Furthermore, assume that $k(z)$ has a zero in the closed right hand plane $\overline{C}_+$. By Lemma 8 $k(0,z) = \xi(z)$ does not zeros in $\overline{C}_+$. So $k(m_0,i\omega)$ $(\omega \in \mathbb{R})$ should have a zero for some $m_0 \in (0,1]$, according to continuous dependence of zeros on coefficients. But due to to (39) this is impossible. The proof is complete. □

For instance, consider the function

$$k_1(z) = z(1 - \tilde{a}e^{-\tilde{h}z}) + ae^{-hz} + b$$

with $a,b,h,\tilde{h} = const \geqslant 0$, and $0 < \tilde{a} < 1$. Then $v_1 = 2(a+b)(1-\tilde{a})^{-1}$. Furthermore, due to Lemma 9 we arrive at the following result.

Corollary 3 *Assume that the conditions*

$$hv_1 < \pi/2, \tilde{h}v_1 < \pi/2 \tag{40}$$

and

$$d_1 := a\cos(v_1h) + b - v_1\tilde{a}\ sin(v_1\tilde{h}) > 0. \tag{41}$$

Then all the zeros of $k_1(.)$ *are in* C_- *and* $\inf_{-\infty\leqslant\omega\leqslant\infty} |k(i\omega)| \geqslant d_1 > 0$.

6. Linear parts with discrete delays

Let $\widetilde{A} = (\widetilde{a}_{jk}), A = (a_{jk})$ and $C = (c_{jk})$ be $n \times n-$ matrices. Consider the equation

$$\dot{y}(t) - \widetilde{A}\dot{y}(t - \widetilde{h}) + Ay(t-h) + Cy(t) = [Fy](t) \quad (t \geqslant 0), \tag{42}$$

assuming that $\|\widetilde{A}\|_n < 1$. So $K(z) = z(I - \widetilde{A}e^{-\widetilde{h}z}) + Ae^{-hz} + C$. The entries of K are

$$k_{jk}(z) = z(1 - \widetilde{a}_{jk}e^{-\widetilde{h}z}) + a_{jk}e^{-hz} + c_{jk.}$$

Thanks to the Hadamard criterion [10], any characteristic value λ of $K(z)$ satisfies the inequality

$$|k_{jj}(z) - \lambda| \leqslant \sum_{m=1, m \neq j}^{n} |k_{jm}(z)| \quad (j = 1, \ldots, n).$$

Hence we have

$$\rho(K(z)) \geqslant \min_{j=1,\ldots,n} (|k_{jj}(z)| - \sum_{m=1, m\neq j}^{n} |k_{jm}(z)|\,), \tag{43}$$

provided the right-hand part is positive. Furthermore, in the case (42) we have $V(\widetilde{R}) = \|\widetilde{A}\|_n$, $V(R) = \|A\|_n + \|C\|_n$, $v_0 = 2(\|A\| + \|C\|)(1 - \|\widetilde{A}\|)^{-1}$. In addition,

$$g(K(z)) = g(B(z)) = g(-z\widetilde{A}e^{-\widetilde{h}z} + Ae^{-hz} + C).$$

Hence, by (25)

$$g(B(i\omega)) \leqslant \frac{1}{\sqrt{2}} N_2(B(i\omega) - B^*(i\omega)) \leqslant$$

$$\frac{1}{\sqrt{2}}\,[|\omega| N_2(e^{-i\widetilde{h}\omega}\widetilde{A} + e^{i\widetilde{h}\omega}\widetilde{A}^*) + N_2(e^{-ih\omega}A - e^{i\widetilde{h}\omega}A^*) + N_2(C - C^*)].$$

One can use also the relation $g(B(i\omega)) = g(e^{is}B(i\omega))$ for all real s and ω. In particular, taking $s = -\widetilde{h} + \pi/2$, we have by (25)

$$g(B(i\omega)) \leqslant \frac{1}{\sqrt{2}}[|\omega| N_2(\widetilde{A} - \widetilde{A}^*) + N_2(e^{-i(h-\widetilde{h})\omega}A + e^{i(h-\widetilde{h})\omega}A^*) + N_2(Ce^{i\widetilde{h}\omega} + e^{-i\widetilde{h}\omega}C^*)].$$

If $\widetilde{A}$ is self-adjoint, then

$$g(B(i\omega)) \leqslant \frac{1}{\sqrt{2}}[N_2(e^{-i(h-\widetilde{h})\omega}A + e^{i(h-\widetilde{h})\omega}A^*) + N_2(Ce^{i\widetilde{h}\omega} + e^{-i\widetilde{h}\omega}C^*)].$$

Hence,

$$g(B(i\omega)) \leqslant \sqrt{2}[N_2(A) + N_2(C)] \quad (\omega \in \mathbb{R}). \tag{44}$$

For example, consider the system

$$\dot{y}_j(t) - \widetilde{a}_{jj}\dot{y}_j(t-\widetilde{h}) + \sum_{k=1}^{n}(a_{jk}y_k(t-h) + c_{jk}y_k(t)) = [F_j y](t), \tag{45}$$

$(j = 1,...,n;\ t \geqslant 0)$, where $[F_j y](t)$ are coordinates of $[Fy](t)$, and suppose that

$$a_{jj}, c_{jj} \geqslant 0,\ 0 < \widetilde{a}_{jj} < 1 \ \ (j = 1,...,n). \tag{46}$$

So $\widetilde{A} = diag\ (\widetilde{a}_{jj})$. Then (43) implies

$$\rho(K(i\omega)) \geqslant \min_{j=1,...,n}\left(|k_{jj}(i\omega)| - \sum_{m=1, m\neq j}^{n}(|a_{jm}| + |c_{jm}|)\right). \tag{47}$$

Put

$$v_j = \frac{2(a_{jj} + c_{jj})}{1 - \widetilde{a}_{jj}}$$

and assume that

$$v_j max\{h, \widetilde{h}\} < \pi/2 \text{ and } d_j := a_{jj}cos(v_j h) + c_{jj} - v_j\widetilde{a}sin(v_j\widetilde{h}) > 0 \tag{48}$$

$(j = 1,...,n)$. Then by Corollary 3 all the zeros of $k_{jj}(.)$ are in C_- and

$$\inf_{-\infty\leqslant\omega\leqslant\infty}|k_{jj}(i\omega)| \geqslant d_j > 0.$$

In addition, let

$$\rho_d := \min_{j=1,...,n}(d_j - \sum_{m=1, m\neq j}^{n}(|a_{jm}| + |c_{jm}|)) > 0, \tag{49}$$

then according to (44) we get

$$\Gamma_0(K) \leqslant \Gamma_d := \sum_{k=0}^{n-1}\frac{(\sqrt{2}(N_2(A) + N_2(C)))^k}{\sqrt{k!}\rho_d^{k+1}}.$$

Now Theorem 2 yields our next result.

Corollary 4 *Let conditions* (46), (48) *and* (49) *be fulfilled. Then the zero solution to system* (45) *is exponentially stable, provided* $var(\nu)\Gamma_d < 1$.

7. Linear parts with distributed delays

Let us consider the equation

$$\dot{y}(t) - \widetilde{A} \int_0^\eta \dot{y}(t-s)d\widetilde{\mu}(s) + A \int_0^\eta y(t-s)d\mu(s) = [Fy](t) \ \ (t \geqslant 0), \tag{50}$$

where $\widetilde{A}$ and A are $n \times n-$ matrices with $\|\widetilde{A}\|_n < 1$, and $\mu, \widetilde{\mu}$ are scalar nondecreasing functions, again. Without loss of generality suppose that

$$var(\mu) = var(\widetilde{\mu}) = 1. \tag{51}$$

So in the considered case $R(s) = \mu(s)A, \widetilde{R}(s) = \widetilde{\mu}(s)\widetilde{A}$,

$$K(z) = zI - z\widetilde{A} \int_0^\eta e^{-zs} d\widetilde{\mu}(s) + A \int_0^\eta e^{-zs} d\mu(s),$$

$V(R) = \|A\|_n$, $V(\widetilde{R}) = \|\widetilde{A}\|_n$ and $v_0 = 2\|A\|_n(1 - \|\widetilde{A}\|_n)^{-1}$. Moreover,

$$\hat{g}(K) = \sup_{|\omega|<v_0} g(B(i\omega)) \leqslant N_2(A) + v_0 N_2(\widetilde{A}).$$

If the both matrices $\widetilde{A}$ and A are self-adjoint, then $B(z)$ is normal and $\hat{g}(K) = 0$. If

$$K(z) = zI - z\widetilde{A}e^{-z\widetilde{h}} + A \int_0^\eta e^{-zs} d\mu(s), \tag{52}$$

then by (25) $g(B(i\omega)) = g(ie^{i\omega\widetilde{h}}B(i\omega)) \leqslant$

$$\frac{1}{\sqrt{2}} [\, |\omega| N_2(\widetilde{A} - \widetilde{A}^*) + N_2(\int_0^\eta e^{-i\omega(s-\widetilde{h})} d\mu(s)A + \int_0^\eta e^{i\omega(s-\widetilde{h})} d\mu(s)A^*)].$$

Consequently, in the case (52) we get

$$\hat{g}(K) \leqslant \frac{v_0}{\sqrt{2}} N_2(\widetilde{A} - \widetilde{A}^*) + \sqrt{2} N_2(A).$$

Now we can directly apply Corollary 2.

In the rest of this section we suppose that $\widetilde{A}$ and A commute. So the eigenvalues of K can be written as

$$\lambda_j(K(z)) = z - z \int_0^\eta e^{-zs} d\widetilde{\mu}(s) \lambda_j(\widetilde{A}) + \int_0^\eta e^{-zs} d\mu(s) \lambda_j(A),$$

and, in addition, according to (26), $g(B(i\omega)) \leqslant |\omega| g(\widetilde{A}) + g(A)$. So

$$g(B(i\omega)) \leqslant g(A,\widetilde{A}) := v_0 g(\widetilde{A}) + g(A) \quad (\omega \in [-v_0, v_0]).$$

If A is normal, then $g(A) = 0$, and $g(A,\widetilde{A}) = g(\widetilde{A})$, if $\widetilde{A}$ is normal, then $g(A,\widetilde{A}) = g(A)$. If the both $\widetilde{A}$ and A are normal commuting matrices, then $g(A,\widetilde{A}) = 0$.

Furthermore, suppose $\lambda_k(A)$ and $\lambda_k(\widetilde{A})$ $(k = 1,\dots,n)$ are positive and put

$$v_k = \frac{2\lambda_k(A)}{1-\lambda_k(\widetilde{A})}.$$

If

$$\eta v_k < \pi/2 \text{ and } d_k(\mu,\widetilde{\mu}) := \lambda_k(A) \int_0^\eta cos\,(\tau v_k) d\mu - v_k \lambda_k(\widetilde{A}) \int_0^\eta sin\,(\tau v_k) d\widetilde{\mu} > 0, \tag{53}$$

$(k = 1,\dots,n)$, then by Corollary 3 all the characteristic values of K are in C_- and

$$\inf_{\omega\in\mathbb{R}} |\lambda_j(K(i\omega))| \geqslant \widetilde{d}_{com} := \min_k d_k(\mu,\widetilde{\mu}) \; (j = 1,\dots,n).$$

So

$$\hat{\Gamma}(K) \leqslant \Gamma_{com}(K) := \sum_{k=0}^{n-1} \frac{\hat{g}^k(A,\widetilde{A})}{\sqrt{k!}\,\widetilde{d}_{com}^{k+1}}.$$

Now Corollary 2 implies

Corollary 5 *Let $\widetilde{A}$ and A be commuting matrices with positive eigenvalues. Let the conditions* (51), (53) *and $q\Gamma_{com}(K) < 1$ be fulfilled. Then the zero solution to equation* (50) *is exponentially stable.*

8. Example

Consider the system

$$\dot{x}_j(t) - a\dot{x}_j(t-h) + \sum_{k=1}^{2} c_{jk} x_k(t) = [F_j x](t) \quad (j = 1,2;\ t \geqslant 0), \tag{54}$$

where $0 < a < 1, 0 \leqslant h \leqslant 1$, c_{jk} are real and

$$[F_j x](t) = q_j x_1^{p_1}(t-h) + \int_0^1 m_j(s) x_2^{p_2}(t-s) ds \quad (q_j = const \geqslant 0;\ p_j > 1)$$

with integrable functions m_j $(j=1,2)$. For continuous scalar functions f_1, f_2 defined on $[-h,\infty)$ and a finite $r>0$ we have

$$|[F_j f](t)| \leqslant q_j r^{p_1-1}|f_1(t-h)| + r^{p_2-1}\int_0^1 |m_j(s)||f_2(t-s)|ds \ \ (|f_j(t)| \leqslant r;\, t \geqslant 0;\ j=1,2).$$

Hence, omitting simple calculations, we obtain inequality (6) with $n=2, \eta=1$ and $var(\nu) \leqslant \hat{q}_r$, where

$$\hat{q}_r^2 = (q_1 r^{p_1-1} + r^{p_2-1}\int_0^1 |m_1(s)|ds)^2 + (q_2 r^{p_1-1} + r^{p_2-1}\int_0^1 |m_2(s)|ds)^2.$$

Furthermore, we have $K(z) = z(1-ae^{-zh})I + C$ with $C = (c_{jk})$, $B(z) = -ae^{-zh}I + C$ and by (25), $g(B(z)) = g(C) \leqslant g_C = |c_{12}-c_{21}|$. Since $\widetilde{A}$ and C commute, the eigenvalues of K are

$$\lambda_j(K(z)) = z - z\,ae^{-zh} + \lambda_j(C).$$

Suppose $\lambda_k(C)$ $(k=1,2)$ are positive and put

$$v_k = \frac{2\lambda_k(C)}{1-a}.$$

If

$$hv_k < \pi/2 \text{ and } d_k := \lambda_k(C) - v_k a\, sin\,(hv_k) > 0 \ \ (k=1,2), \tag{55}$$

then by Corollary 3, the characteristic values of K are in C_-, and

$$\inf_{\omega\in\mathbb{R}} |\lambda_k(K(i\omega))| \geqslant \hat{d} := \min_{k=1,2} d_k.$$

So

$$\hat{\Gamma}(K) \leqslant \widetilde{\Gamma} := \frac{1}{\hat{d}}\left(1 + \frac{g_C}{\hat{d}}\right).$$

Thanks to Corollary 2 we can assert that the zero solution to system (45) is exponentially stable provided the conditions (55) and $\hat{q}\widetilde{\Gamma} < 1$ hold.

9. Concluding remarks

In this paper we have established the explicit exponential stability conditions for a wide class of neutral systems. As the example shows, in appropriate situations we can avoid constructing the Lyapunov functionals.

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