10.2478/v10170-011-0031-1

Archives of Control Sciences
Volume 22(LVIII), 2012
No. 4, *pages 399–416*

On the application of control models technique to investigation of some ecological and economic problems

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The paper discusses a method of auxiliary controlled models and the application of this method to solving some problems of identification and robust control for differential equations. The objects that the method is suggested to be used are two systems of nonlinear differential equations describing some ecological and economic processes. Two solving algorithms, which are stable with respect to informational noises and computational errors, are presented. The algorithms are tested by model examples.

Key words: controlled models, identification, robust control

1. Introduction

The paper addresses discussion of the uniform approach to two types of problems: a dynamical inverse problem and a robust control problem. Two types of models are considered: a dynamical model referring to main economic and climatic indices [1] and a model describing the interaction between climate and biosphere [2, 3]. For other classes of models (fault detection problem, phase field equations, feed bioreactor, equations describing pollution propagation), this approach was discussed in [4–7]. Each of the problems mentioned above is described briefly.

Problems of determining of some parameters through the information on equation's solutions are often called reconstruction (identification) problems. Therewith it is assumed that the input information (results of measurements of current states of a dynamical system) is available. As the parameters are unknown, they should be reconstructed. One of the methods of solving similar problems was suggested in [8–11]. This method

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This work was supported in part by RFBR (project 12-01-00175a), by the Program on Basic Research of the Presidium of the Russian Acad. Sci. 38P (project 12-P-1-1038) and by the Program for support of leading scientific schools of Russia (6512.2012.1).

Received 29.05.2012.

bases on the idea of the theory of ill-posed problems and actually reduces an identification problem to a control problem for an auxiliary dynamical system-model. Regularization of the problem under consideration is locally realized during the process of choosing a positional control in the system-model. The method was applied to a number of problems described by some classes of ordinary differential equations as well as by equations with distributed parameters. Different system's characteristics varying in time were under reconstruction, for example, unknown discontinuous inputs, initial and boundary data, distributed disturbances, coefficients of an elliptic operator and so on. In the present paper, we illustrate this method on example considering a dynamical model connecting main economic and climatic indices.

Problems of robust control have aroused considerable interest in control theory. In a general way, these problems can be characterized as follows. Let a dynamical system be given. A control and unobserved disturbance simultaneously act on the system. A range of changing admissible disturbances is rather wide and is somehow a priori described. A signal on system's current states is received in the process of system's motion. It is required to construct a feedback control law guaranteeing a desired mode for system's trajectory regardless of a concrete disturbance. We illustrate one of the approach to solve similar problems [12, 9] by example considering a model describing the interaction between climate and biosphere [2, 3].

2. Dynamical inverse problem

A dynamical model connecting main economic and climatic indices was suggested in [1]. This model is oriented to develop an economic strategy directed to deceleration of global warming. The main goal of the analysis of the model is to provide the means for tackling the following question: whether the reduction of emissions of greenhouse gases is justified from the economical viewpoint or not. The model takes into account global processes: it is assumed that the structure of economy is the same for all countries; the climate change is characterized by the average value of the temperature on Earth's surface and so on. This model contains three types of parameters.

1. Constant parameters (their list is presented in tables 2.3 and 2.4 on page 21 [1]).

2. Functions that are considered (for simplicity of the analysis) as exogenous with respect to the model and are a priori given.

3. Inner functions that are connected to one another and to exogenous parameters by means of some algebraic and differential equations. The list of these functions is presented in table 2.3. (see [4]), and the model equations are presented in table 2.2. The list of functions is as follows:

   $\mu(t)$ is a rate of emissions reduction with respect to uncontrollable emissions,

$E(t)$ is emissions of greenhouse gases GHGs (CO_2 (carbonic acid gas) and chlorine-fluorine carbons only),

$M_1(t) = (M(t) - 590)$ is an excess of mass of GHGs in the atmosphere in comparison with the pre-industrial period,

$T(t)$ is an average atmospheric temperature (on Earth's surface),

$T_1(t)$ is an average deep-ocean temperature,

$I(t)$ is a gross investment,

$K(t)$ is a capital stock,

$F(t)$ is an atmospheric radiative forcing from GHGs,

$O(t)$ is a forcing of exogenous GHGs (i.e., of gases, which are considered as uncontrollable; there are all GHGs, besides CO_2 (carbonic acid gas) and chlorine-fluorine carbons),

$A(t)$ is a level of technology,

$\sigma(t)$ is ratio of GHGs emissions to global output,

$L(t)$ is a population at time t, also equal to labor inputs,

$Q(t)$ is a gross world product.

Schematically, the connections between the inner functions can be pictured in the following way:

$$\begin{array}{ccccccc}
T^{**} & \longrightarrow & \Omega & & & & \\
\uparrow & & \uparrow & \searrow & & & \\
F & & \mu & & Q & & \\
\uparrow & & \downarrow & \swarrow & \uparrow & \nwarrow & \\
M_1^{*} & \longleftarrow & E & & K^{*} & \longleftarrow & I
\end{array}$$

Here, the functions marked by the asterisk are solutions of linear differential equations of the first order. The function $T(t)$ is a solution of the linear differential equation of the second order.

If we pass from the discrete model suggested by the authors to the "continuous" one, then the equations of the model Σ take the form:

$$\begin{aligned}
\dot{T}(t) &= c_1T(t) + c_2T_1(t) + c_3F(t), \quad t \in [0, \vartheta] \\
\dot{T}_1(t) &= c_4(T(t) - T_1(t)) \\
\dot{M}_1(t) &= \beta E(t) - \delta_M M_1(t) \\
\dot{K}(t) &= -\delta_K K(t) + I(t),
\end{aligned} \tag{1}$$

where t is time, ϑ is a terminal time moment,

$$F(t) = 4.1 \cdot \log_2\left(1 + \frac{M_1(t)}{590}\right) + O(t),$$

$$E(t) = (1-\mu(t))\sigma(t)Q(t),$$
$$Q(t) = (1-b_1\mu(t)^{b_2})/(1+\theta_1 T(t)^{\theta_2})A(t)K(t)^{\gamma}L(t)^{1-\gamma}.$$

Initial state of Σ, $\{T(0), T_1(0), M_1(0), K(0)\}$, is assumed to be known and a priori given. Functions $\mu(t)$ and $I(t)$ are considered as control parameters determining a strategy of global control of climate and economy. The numerical analysis of the model is performed in [4]. The direct problem is solved, namely, possible strategies (rules of forming $\mu(t)$ and $I(t)$) are specified, and system's dynamics is computed.

The comparative analysis of simulation results for different structures is performed. In addition, the analysis of sensitivity of the results with respect to some model parameters is delivered.

Our aim differs from the aim of [4]. We deal with the inverse problem. It consists of the following. Let us assume that some function $I(t)$ is known. Neglecting small values ($b_1 = 0,0686$, $\vartheta_1 = 0,00144$), we transform the system (1) to the form

$$\begin{aligned}
\dot{T}(t) &= c_1 T(t) + c_2 T_1(t) + c_5 \cdot \log_2\left(1 + \frac{M_1(t)}{590}\right) + c_3 O(t), \quad t \in [0,\vartheta] \\
\dot{T}_1(t) &= c_4(T(t) - T_1(t)) \\
\dot{M}_1(t) &= E_1(t)(1-\mu(t)) - \delta_M M_1(t) \\
\dot{K}(t) &= -\delta_K K(t) + I(t),
\end{aligned} \tag{2}$$

where $E_1(t) = \beta\sigma(t)A(t)K(t)^{\gamma}L(t)^{1-\gamma}$. Hereinafter, we consider the system Σ of the form (2). The problem under consideration may be formulated in the following way. At frequent enough time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^{m}, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = 0, \quad \tau_m = \vartheta,$$

values of $T(\tau_i)$ and $T_1(\tau_i)$ are inaccurately measured. Results of measurements (vectors $\xi_i^h = \{\xi_{1i}^h, \xi_{2i}^h\} \in R^2$) satisfy the inequalities

$$|T(\tau_i) - \xi_{1i}^h|^2 + |T_1(\tau_i) - \xi_{2i}^h|^2 \leqslant h^2, \tag{3}$$

where $h \in (0,1)$ is a level of informational noise. It is required to design an algorithm allowing us to reconstruct (synchronously with the process) unknown $M_1(t)$ and $\mu(t)$. This is the most important formulation of the dynamical reconstruction problems being investigated in the present paper.

The scheme of algorithms for solving such problems is given in Fig. 1.

According to this scheme, an auxiliary dynamical system M (a model) is introduced. This model operates on the time interval $[0,\vartheta]$ and has an input $u^h(t)$ and an output $w^h(t)$. The process of synchronous feedback control of the systems Σ and M is organized on the interval $[0,\vartheta]$. This process is decomposed onto $(m-1)$ identical steps. At the i-th step

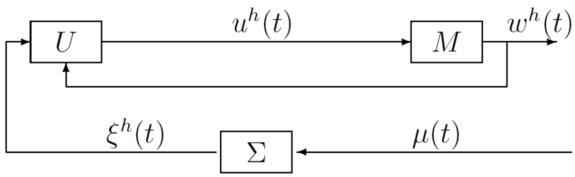

Figure 1. The scheme of solving algorithms.

which covers the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions are fulfilled. First, at the time moment τ_i according to the chosen rule U the functions

$$u^h(t) = u_i^h = U(\tau_i, \xi_i^h, w^h(\tau_i)),$$

are calculated using measurements ξ_i^h and $w^h(\tau_i)$. Then (till the moment τ_{i+1}) the control $u = u^h(t)$, $\tau_i \leqslant t < \tau_{i+1}$, is fed into the input of the model M. The values ξ_{i+1}^h and $w^h(\tau_{i+1})$ are the results of the algorithm performance at the i-th step.

Thus, the inverse problem may be formulated as follows. In the sequel, a family of partitions

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h), \quad \tau_{0,h} = 0, \quad \tau_{m_h,h} = \vartheta$$

of the interval $[0, \vartheta]$ is assumed to be fixed.

Dynamical inverse problem. It is required to indicate differential equations of the model M

$$\dot{w}^h(t) = f_1(\xi_i^h, w^h(\tau_i), u_i^h), t \in \delta_{h,i} = [\tau_{i,h}, \tau_{i+1,h}), \quad \tau_i = \tau_{i,h}, \tag{4}$$
$$w^h(0) = w_0^h, \quad w^h(t) \in R^4,$$

and the rule of choosing controls u_i^h at the moments τ_i being a mapping of the form

$$U : \{\tau_i, \xi_i^h, w^h(\tau_i)\} \to u_i^h = \{u_{i1}^h, u_{i2}^h\} \in R^2 \tag{5}$$

such that the convergence

$$\int_0^\vartheta |u_1^h(t) - M_1(t)|^2 \, dt \to 0, \quad \int_0^\vartheta |u_2^h(t) - \mu(t)|^2 \, dt \to 0 \tag{6}$$

takes place whereas h tends to 0.

Here (see Fig. 1) $u^h(t) = \{u_1^h(t), u_2^h(t)\}$, $u_1^h(t) = u_{i1}^h$, $u_2^h(t) = u_{i2}^h$ for $t \in \delta_{h,i}$.

3. The algorithm for solving the inverse problem

Let us turn to the description of the algorithm for solving the inverse problem. From the above, it is necessary to indicate the model (4) and the strategy U (5) providing the convergence (6). Let a restriction on the rate of emissions reduction be known, i.e., we know a number $f > 0$ such that

$$|\mu(t)| \leqslant f \quad \text{for all} \quad t \in [0, \vartheta].$$

From now on, it is assumed that we know numbers $K, a_1, a_2 \in (0, +\infty)$, $a_1 < a_2$, such that each solution $x_\mu(t)$, $x_\mu(t) = \{T(t), T_1(t), M_1(t), K(t)\}$, of the equation (2) satisfies the following conditions

$$\max_{0 \leqslant t \leqslant \vartheta} \|x_\mu(t)\| \leqslant K, \quad \sup_{0 \leqslant t \leqslant \vartheta} \|\dot{x}_\mu(t)\| \leqslant K, \quad M_1(t) \in [a_1, a_2]. \tag{7}$$

Here $\|x_\mu\|$ is the Euclidean norm of the vector x_μ.

Introduce some function $\alpha(h) : (0,1) \to R^+ = \{r \in R : r \geqslant 0\}$ with the properties:

$$\alpha(h) \to 0, \quad \delta(h) \leqslant h, \quad (h^{1/6} + \omega(h))/\alpha(h) \to 0 \quad \text{as} \quad h \to 0.$$

Here $\omega(h) = \omega_E(\delta) + \omega_M(\delta)$, $\omega_E(\delta)$ and $\omega_M(\delta)$ are the modulus of continuity of functions $E_1(t)$ and $M_1(t)$, respectively. The function α plays the role of a regularizator (a smoothing functional). Let the model (4) has the form

$$\begin{aligned} \dot{w}^h(t) &= c_1\xi^h_{1i} + c_2\xi^h_{2i} + 4,1c_3\log_2\left(1 + \frac{u^h_{1i}}{590}\right) + c_3Q(\tau_i) \\ \dot{w}^h_1(t) &= c_4(\xi^h_{1i} - \xi^h_{2i}) \\ \dot{w}^h_2(t) &= E_1(\tau_i)(1 - u^h_{2i}) - \delta_M u^h_{1i} \\ \dot{w}^h_3(t) &= -\delta_K K(\tau_i) + I(\tau_i), \quad t \in [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{i,h}, \end{aligned} \tag{8}$$

and the rule U of forming the control $u^h_i = \{u^h_{i1}, u^h_{i2}\}$ is as follows

$$u^h_{1i} = 590(2^{\pi^h_i} - 1), \tag{9}$$

$$u^h_{2i} = \begin{cases} \beta^{(1)}_i \big/ \alpha(h), & \text{if} \quad |\beta^{(1)}_i| \leqslant \alpha(h) f \\ f \,\text{sign}\left(\beta^{(1)}_i\right), & \text{otherwise.} \end{cases} \tag{10}$$

The initial state of the model is

$$w^h(0) = T(0), \quad w^h_1(0) = T_1(0), \quad w^h_2(0) = M_1(0), \quad w^h_3(0) = K(0).$$

Here

$$\pi_i^h = \begin{cases} -c_i/(2h^{2/3}), & \text{if } -c_i/(2h^{2/3}) \in [b_1, b_2] \\ b_1, & \text{if } -c_i/(2h^{2/3}) < b_1 \\ b_2, & \text{if } -c_i/(2h^{2/3}) > b_2, \end{cases}$$

$$b_1 = \log_2(1 + a_1/590), \quad b_2 = \log_2(1 + a_2/590),$$

$$c_i = 8,2c_3 S_i^0,$$

$$\beta_i^{(1)} = E_1(\tau_i)(w_2^h(\tau_i) - u_{1i}^h), \quad S_i^0 = w^h(\tau_i) - \xi_{i1}^h.$$

Theorem 2 *Let* $E_1(t) > 0$, $t \in [0, \vartheta]$. *Then the convergence* (6) *take place under choosing the model equation in the form* (4), (8) *and the strategy* U *in the form* (5), (9), (10).

Proof. It can be easily seen that from results of [9] the following inequality follows

$$\int_0^\vartheta |u_1^h(t) - M_1(t)|^2 \, dt \leqslant Ch^{1/3}. \tag{11}$$

Estimate the variation of the value

$$\varepsilon(t) = |w_2^h(t) - M_1(t)|^2 + \alpha(h) \int_0^t \{|u_2^h(\tau)|^2 - |\mu(\tau)|^2\} \, d\tau.$$

It can be easily seen that for $t \in \delta_i = [\tau_i, \tau_{i+1})$ the following inequality is true:

$$\varepsilon(t) \leqslant \varepsilon(\tau_i) + \delta(h) \int_{\tau_i}^t |\dot{w}_2^h(\tau) - M_1(\tau)|^2 \, d\tau + \int_{\tau_i}^t \mu_i(\tau) \, d\tau + \alpha(h) \int_{\tau_i}^t \{|u_{i2}^h|^2 - |\mu(\tau)|^2\} \, d\tau, \tag{12}$$

$$\mu_i(t) = 2(w_2^h(\tau_i) - M_1(\tau_i))(\dot{w}_2^h(t) - \dot{M}_1(t)), \quad t \in \delta_i.$$

Consider the value $\mu_i(t)$. We have for $t \in \delta_i$

$$\mu_i(t) = \sum_{j=1}^4 \lambda_{ji}(t), \quad t \in \delta_i, \tag{13}$$

where

$$\lambda_{1i}(t) = 2\beta_i^{(2)} \left(E_1(\tau_i) - E_1(t)\right),$$

$$\lambda_{2i}(t) = 2\beta_i^{(2)}\left(\mu(t) - u_{2i}^h\right)E_1(\tau_i),$$

$$\lambda_{3i}(t) = 2\beta_i^{(2)}\left(E_1(t) - E_1(\tau_i)\right)\mu(t),$$

$$\lambda_{4i}(t) = 2\delta_M\beta_i^{(2)}\left(M_1(t) - u_{1i}^h\right),$$

$$\beta_i^{(2)} = w_2^h(\tau_i) - M_1(\tau_i).$$

Estimate each term in the right-hand part of the equality (13). From (7) and (11) it follows that

$$\lambda_{1i}(t) \leqslant C_1\omega(\delta), \quad \lambda_{3i}(t) \leqslant C_2\omega(\delta), \quad t \in \delta_i.$$

$$\sum_{i=0}^{m_h-1}\int\limits_{\tau_i}^{\tau_{i+1}}\lambda_{4i}(t)\,dt \leqslant C_3\int\limits_0^{\vartheta}|M_1(t) - u_1^h(t)|\,dt \leqslant C_4h^{1/6}. \tag{14}$$

In addition, the estimate

$$\sum_{i=0}^{m_h-1}\int\limits_{\tau_i}^{\tau_{i+1}}\lambda_{6i}(t)\,dt \leqslant C_5(h^{1/6} + w_M(\delta))$$

is valid. Then we have

$$\lambda_{2i}(t) \leqslant \lambda_{5i}(t) + \lambda_{6i}(t),$$

$$\lambda_{5i}(t) = 2\beta_i^{(1)}(\mu(t) - u_{2i}^h), \quad \lambda_{6i}(t) = 2(u_{1i}^h - M_1(\tau_i))(\mu(t) - u_{2i}^h).$$

Note that

$$u_{2i}^h = \arg\min\{-2\beta_i^{(1)}u + \alpha u^2 : |u| \leqslant f\}.$$

Therefore, in virtue of (3) we obtain

$$\lambda_{5i}(\tau_{i+1}) + \alpha(h)\int\limits_{\tau_i}^{\tau_{i+1}}\{|u_{i2}^h|^2 - |\mu(\tau)|^2\}\,d\tau = \tag{15}$$

$$\int\limits_{\tau_i}^{\tau_{i+1}}\left\{\left[2\beta_i^{(1)}u_{i2}^h + \alpha(h)|u_{i2}^h|^2\right] - \left[2\beta_i^{(1)}\mu(\tau) + \alpha(h)|\mu(\tau)|^2\right]\right\}d\tau \leqslant 0.$$

Taking into account (13)–(15) and the inequality $\delta(h) \leqslant h$, we have for all $i \in [1 : m_h]$ the following estimate

$$\varepsilon(\tau_i) \leqslant C(h^{1/6} + \omega(\delta(h))) \leqslant C(h^{1/6} + \omega(h)).$$

Further argument corresponds to the standard scheme (see, for example, [8–11]). The theorem is proved. □

4. Problem of robust control

The model describing the interaction between climate and biosphere was suggested in [2, 3] and was tested on real data. Applying modern methods of the mathematical theory of optimal (program) control, the authors presented a qualitative analysis of the model. They oriented to the problem of finding an optimal profile of CO_2 emission maximizing a cumulative emission (here temperature climate changes should be taken into account). Our aim differs from the aims of [2, 3]. We attract our attention to analyzing the problem of robust control. According to the mathematical model Σ of the interaction between climate and biosphere, we take the following system of equations [2]

$$\begin{aligned} \dot{T}(t) &= \mu \ln\{C(t)/C_0\} - \alpha T(t), \quad t \in [0, \vartheta] \\ \dot{C}(t) &= -P_t(C,T) + (1-\varepsilon)m(t)N(t) + \delta_t(T)S(t) + u(t) - Q_{oc}(t), \\ \dot{N}(t) &= P_t(C,T) - m(t)N(t), \\ \dot{S}(t) &= \varepsilon m(t)N(t) - \delta_t(T)S(t). \end{aligned} \tag{16}$$

Here t is time,

$T(t)$ is the average atmospheric temperature (on Earth's surface),

$C(t)$ is the total amount of carbon in the atmosphere,

$N(t)$ and $S(t)$ are the amounts of carbon in the biota and in the soil, respectively,

$Q_{oc}(t) = \sigma((C(t) - C_0) - \nu(D(t) - D_0))$,

$P_t(C,T) = P_0(1 + a_1 T)(1 + a_2(C - C_0))$,

$\delta_t(T) = \delta_0(1 + a_3 T)$, $m(t) = m_0(1.087 + a_4 t)$,

$D(t)$ is the carbon content in the ocean.

Values of model parameters are give in [2, p. 30]. Assuming that $\dot{N} = \dot{S} = 0$, i.e., the carbon amounts in the biota and in the soil do not change, we should deal with system (16) in the form

$$\begin{aligned} \dot{T}(t) &= \mu \ln\{C(t)/C_0\} - \alpha T(t), \\ \dot{C}(t) &= -P_t(C,T) + (1-\varepsilon)m(t)N + \delta_t(T)S + u(t) - Q_{oc}(t). \end{aligned} \tag{17}$$

In the sequel, we consider the system Σ in the form (17).

The problem under consideration is stated as follows. At discrete, frequent enough, time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = t_0, \quad \tau_m = \vartheta,$$

the value of $T(\tau_i)$ is inaccurately measured. Results of measurements (elements $\xi_i^h \in R$) satisfy the inequalities

$$|T(\tau_i) - \xi_i^h| \leqslant h, \tag{18}$$

where $h \in (0,1)$ is a level of informational noise.

Condition 1. *Some mode of changing the annual temperature $T = T_*(t)$ and the total amount of carbon in the atmosphere $C = C_*(t)$ is assumed to be given:*

$$x_*(t) = \{T_*(t), C_*(t)\}.$$

Upper and lower bounds $u(t)$ and $Q_{oc}(t)$ for changing the atmospheric emission are known. Namely, we have numbers e_1, e_2, $0 < e_1 < e_2$ and g_1, g_2, $0 < g_1 < g_2$ such that

$$u(t) \in [e_1, e_2], \quad Q_{oc}(t) \in [g_1, g_2] \quad \text{for all} \quad t \in T.$$

The number $\varepsilon > 0$ is given. It is required to construct an algorithm of feedback control of the system (17) providing fulfilment of the following condition. Whatever the unknown possible disturbance $Q = Q_{oc}(t) \in [g_1, g_2]$ may be, the distance between $x^h(t)$ and $x_*(t)$ at all moments $t \in [0, \vartheta]$ should not exceed the value of ε provided the values of h and δ are sufficiently small.

Here

$$x^h(\cdot) = \{T(\cdot), C(\cdot)\} = \{T(\cdot; U(\cdot;\xi), Q_{oc}(\cdot)), C(\cdot; U(\cdot;\xi), Q_{oc}(\cdot))\}$$

is the trajectory of Σ generated by the unknown disturbance $Q_{oc}(t) \in [g_1, g_2]$ and the control $u(t) = u^h(t;\xi) = U(\tau_i, \xi_i) \in [e_1, e_2]$, which is formed according to the feedback principle.

The scheme of algorithms for solving the problem is given in Fig. 2. An auxiliary

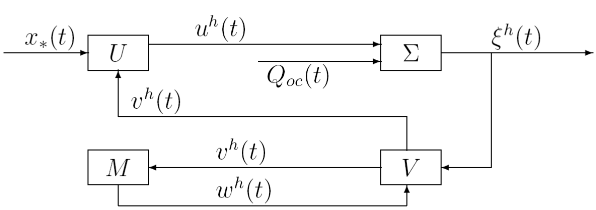

Figure 2. The scheme of solving algorithm.

dynamical system M (a model) is introduced. This model operates on the time interval $[0, \vartheta]$ and has the input $v^h(t)$ and the output $w^h(t)$. The process of synchronous feedback control of the systems Σ and M is organized on the interval $[0, \vartheta]$. This process is decomposed onto $(m-1)$ identical steps. i-th step is carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$ and the following actions are fulfilled. First, at the time moment τ_i according to the chosen rules U and V, the functions

$$u^h(t) = u_i^h = U(\tau_i, v_i^h, x_*(\tau_i)), \tag{19}$$

$$v^h(t) = v_i^h = V(\tau_i, \xi_i^h, w^h(\tau_i)), \quad t \in \delta_i \tag{20}$$

are calculated using measurements ξ_i^h and $w^h(\tau_i)$. Then (till the moment τ_{i+1}) the control $u = u^h(t)$, $\tau_i \leqslant t < \tau_{i+1}$, is fed onto the input of the system Σ and the control $v = v^h(t)$, $\tau_i \leqslant t < \tau_{i+1}$, into the input of the model M. The values ξ_{i+1}^h and $w^h(\tau_{i+1})$ are the results of the algorithm performance at the i-th step. Thus, the complexity of solving of these problems is reduced to appropriate choice of the model M and the functions U and V.

In the sequel, a family of partitions

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h), \quad \tau_{0,h} = 0, \quad \tau_{m_h,h} = \vartheta$$

of the interval $[0, \vartheta]$ is assumed to be fixed. So, the problem may be formulated as follows.

Problem of robust control. It is required to indicate differential equations of the model M

$$\dot{w}^h(t) = f_1(\xi_i^h, w^h(\tau_i), v_i^h), \tag{21}$$

$$t \in \delta_{h,i} = [\tau_{i,h}, \tau_{i+1,h}), \quad \tau_i = \tau_{i,h},$$

$$w^h(0) = w_0^h, \quad w^h(t) \in R,$$

and the rule of choosing controls u_i^h and v_i^h at the moments τ_i being a mapping of the form (19), (20) such that for $h \in (0, h_*(\varepsilon))$, $\delta = \delta(h) \in (0, \delta(h_*(\varepsilon))$ the following inequality holds

$$\max_{t\in[0,\vartheta]} \|x^h(t) - x_*(t)\|_{R^2} \leqslant \varepsilon. \tag{22}$$

5. The algorithm for solving the robust control problem

Let us turn to the description of the algorithm for solving the robust control problem. It follows from the above, that it is necessary to indicate the model (21) and the strategies U and V (19), (20) assuring the inequality (22). Assume the numbers $K, a_1, a_2 \in (0, +\infty)$, $a_1 < a_2$, to be known and such, that each solution $x(t)$, $x(t) = \{T(t), C(t)\}$, of the equation (17) satisfies the following conditions

$$\max_{0\leqslant t\leqslant\vartheta} \|x(t)\| \leqslant K, \quad \sup_{0\leqslant t\leqslant\vartheta} \|\dot{x}(t)\| \leqslant K, \quad C(t) \in [a_1, a_2].$$

Introduce some function $\gamma(h) : (0,1) \to R^+$ with the properties:

$$\gamma(h) \to 0, \quad \delta(h) \leqslant h, \quad (h^{1/6} + \omega_C(h))/\gamma(h) \to 0 \quad \text{as} \quad h \to 0.$$

Here $\omega_C(h)$ is the modulo of continuity of function $C(t)$. Let the model (21) have the form

$$\dot{w}^h(t) = \mu \ln\{v_i^h/C_0\} - \alpha\xi_i^h, \quad t \in [\tau_i, \tau_{i+1}), \quad w^h(0) = \xi_0^h \tag{23}$$

and the rules U and V of forming the controls u_i^h and v_i^h are as follows

$$v_i^h = C_0 \exp(\pi_i^h), \tag{24}$$

$$u_i^h = \begin{cases} e_1 & \text{if} \quad C_*(\tau_i) - v_i^h > 0 \\ e_2, & \text{otherwise.} \end{cases} \tag{25}$$

Here

$$\pi_i^h = \begin{cases} -c_i/(2h^{2/3}), & \text{if } -c_i/(2h^{2/3}) \in [b_1, b_2] \\ b_1, & \text{if } -c_i/(2h^{2/3}) < b_1 \\ b_2, & \text{if } -c_i/(2h^{2/3}) > b_2, \end{cases}$$

$$b_1 = \ln(a_1/C_0), \quad b_2 = \ln(a_2/C_0), \quad c_i = 2\mu(w^h(\tau_i) - \xi_i^h).$$

Let the following condition be fulfilled.

Condition 2. *There exists a measurable (by Lebesgue) function* $\phi(t) \in [e_1 - g_2, e_2 - g_1]$ *such that the prescribed mode* $x_*(t) = \{T_*(t), C_*(t)\}$ *satisfies the relations*

$$\begin{aligned} \dot{T}_*(t) &= \mu \ln\{C_*(t)/C_0\} - \alpha T_*(t), \\ \dot{C}_*(t) &= -P(C_*, T_*) + (1-\varepsilon)m(t)N + \delta(T_*)S + \phi(t), \end{aligned}$$

with the initial state

$$T_*(0) = T(0), \qquad C_*(0) = C(0).$$

Theorem 3 *The inequality* (22) *holds under choosing the model equation in the form* (21), (23) *and the strategies* U *and* V *in the form* (19), (20), (24), (25).

Proof. While choosing the control v_i^h in the form (24), one can obtain the estimate

$$\int_0^{\vartheta} |v^h(t) - C(t)|^2 \, dt \leqslant C h^{1/3} \tag{26}$$

by analogy with (11). Estimate the variation of

$$\varepsilon(t) = |T(t) - T_*(t)|^2 + |C(t) - C_*(t)|^2.$$

It can be easily seen that for $t \in \delta_i = [\tau_i, \tau_{i+1})$ the following inequalities are true:

$$\nu_1(t) = |C(t) - C_*(t)|^2 \leqslant \nu_1(\tau_i) + \lambda(t;\tau_i) + L(t,\tau_i) + (t-\tau_i)O_h(t-\tau_i), \tag{27}$$

$$\nu_2(t) = |T(t) - T_*(t)|^2 \leqslant \nu_2(\tau_i) +$$

$$+ (T(\tau_i) - T_*(\tau_i)) \int_{\tau_i}^{t} [\mu(\ln C(t) - \ln C_*(t)) + \alpha(T(\tau_i) - T_*(\tau_i))] + (t - \tau_i) O_h^{(1)}(t - \tau_i),$$

$$\sum_{i=0}^{m_h - 1} O_h^{(1)}(t - \tau_i) \leqslant K_2,$$

$$\sum_{i=0}^{m_h - 1} O_h(\tau_{i+1} - \tau_i) \leqslant K_1,$$

where

$$\lambda(t;\tau_i) = (C(\tau_i) - C_*(\tau_i)) \int_{\tau_i}^{t} [P_\tau(C_*, T_*) - P_\tau(C, T) + (\delta_\tau(T) - \delta_\tau(T_*))S,$$

$$L(t,\tau_i) = (C(\tau_i) - C_*(\tau_i)) \int_{\tau_i}^{t} [u_i^h - Q_{oc}(\tau) - \varphi(\tau)]\, d\tau. \tag{28}$$

Here constants K_1, K_2 do not depend on h, u^h, Q_{oc}, and φ. It can be easily seen that

$$\lambda(t;\tau_i) \leqslant c_1(t - \tau_i)\nu_1(\tau_i) + c_2 \int_{\tau_i}^{t} \varepsilon(\tau)\, d\tau. \tag{29}$$

Note that, in virtue of the inequality $C(t) \geqslant a_1 > 0$,

$$|\ln C(t) - \ln C_*(t)| \leqslant \frac{|C(t) - C_*(t)|}{a_1}.$$

In this case

$$\nu_2(t) \leqslant (1 + c_3(t - \tau_i))\nu_2(\tau_i) + c_4 \int_{\tau_i}^{t} \varepsilon(\tau) d\tau. \tag{30}$$

Then we obtain

$$L(t;\tau_i) \leqslant (C(\tau_i) - v_i^h) \int_{\tau_i}^{t} [u_i^h - Q_{oc}(\tau) - \varphi(\tau)]\, d\tau +$$

$$+ (v_i^h - C_*(\tau_i)) \int_{\tau_i}^{t} [u_i^h - Q_{oc}(\tau) - \varphi(\tau)]\, d\tau.$$

Taking into account (24), by analogy with [12], we derive the inequality

$$(v_i^h - C_*(\tau_i)) \int_{\tau_i}^{t} [u_i^h - Q_{oc}(\tau) - \varphi(\tau)]\, d\tau \leqslant 0. \tag{31}$$

Moreover, by (26) it follows that

$$\sum_{i=0}^{m_h-1} |C(\tau_i) - v_i^h| \int_{\tau_i}^{\tau_{i+1}} |u_i^h - Q_{oc}(\tau) - \varphi(\tau)|\, d\tau \leqslant \tag{32}$$

$$\leqslant c_5 \Big\{ \int_0^{\vartheta} |C(t) - v^h(t)|\, dt + \delta(h)\omega_C(\delta(h)) \Big\} \leqslant$$

$$\leqslant c_6 \{\delta(h)\omega_C(\delta(h)) + h^{1/6}\}.$$

Taking into account (13)–(32) and the inequality $\delta(h) \leqslant h$, we have for $t \in [\tau_i, \tau_{i+1}]$ the following estimate

$$\varepsilon(\tau_i) \leqslant (1 + c_7(t - \tau_i))\varepsilon(\tau_i) + c_8 \int_{\tau_i}^{t} \varepsilon(\tau) d\tau.$$

In virtue of Gronwall's inequality, we conclude that

$$\varepsilon(t) \leqslant (1 + c_9(t - \tau_i))\varepsilon(\tau_i), \quad t \in [\tau_i, \tau_{i+1}].$$

Further argument corresponds to the standard scheme (see, for example, [12]). The theorem is proved. □

6. Results of computer modeling

The algorithms described above were tested on computers.

Example 1. In Figures 3–4, the results of computer modeling of the dynamic inverse problem are presented for the following case:

$$\begin{array}{ll} c_1 = c_2 = c_3 = 1, & \sigma = 1 + 0.5t, \\ c_4 = 0.5, & Q(t) = 5sin(t), \\ \delta_k = 0.65, & L(t) = 1, \\ \delta_m = 0.0833, & \mu(t) = 1 + 0.5t, \\ \beta = 0.1, & I(t) = 1 = 0.15t^2, \\ \gamma = 0.1, & A(t) = 2t^{1/2}. \end{array}$$

The parameters are as follows:

$$f = 2, \qquad a_1 = 30, \qquad a_2 = 60.$$

The initial conditions for the system are the following:

$$T(0) = 1, \qquad T1(0) = 0.5,$$
$$M(0) = 50, \quad K(0) = 10.$$

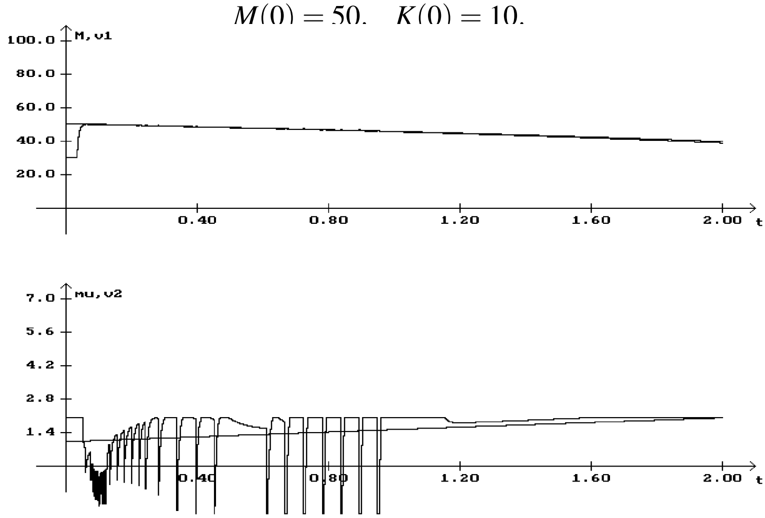

Figure 3. Result of simulation of the example 1 for $\delta = 0.001$

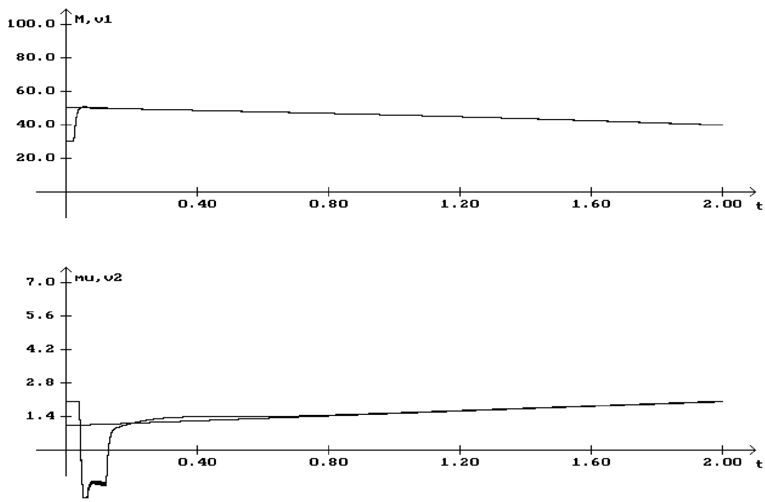

Figure 4. Result of simulation of the example 1 for $\delta = 0.0001$

Example 2. In Figures 5–7 the results of computer modeling of the problem of robust control correspond to the following data:

$$C_0 = 617, \qquad P_0 = 60,$$
$$a_1 = 0.05, \qquad N = 690,$$
$$a_2 = 0.00047, \qquad \delta_0 = 0.00005,$$
$$S = 1229, \qquad \mu = 0.172,$$
$$M(t) = 0.087(1.087 + 0.00633t).$$

The parameters are as follows:

$$e_1 = 5, \quad g_1 = 10,$$
$$e_2 = 30, \quad g_2 = 20.$$

The initial conditions for the system are the following:

$$T(0) = 0.64, \quad C(0) = 162.$$

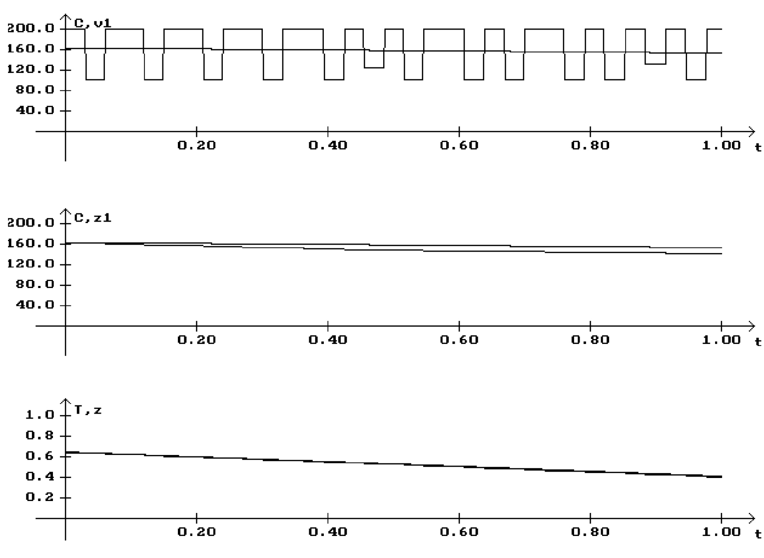

Figure 5. Result of simulation of the example 2 for $\delta = 0.03$

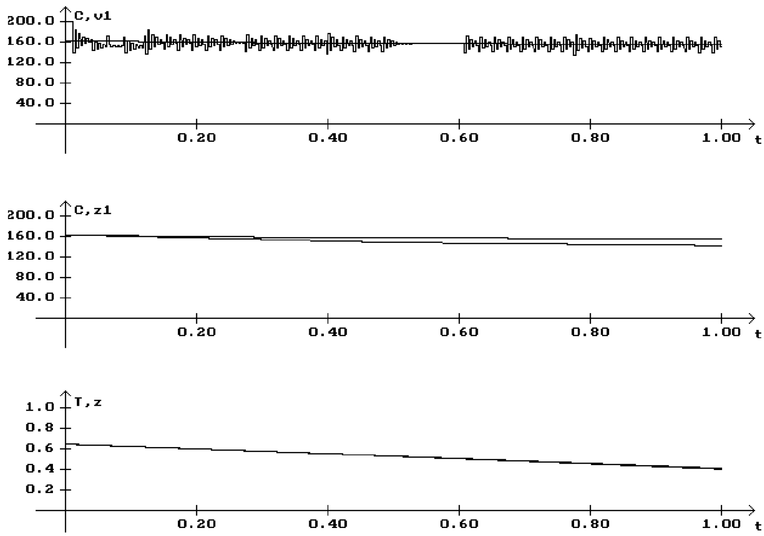

Figure 6. Result of simulation of the example 2 $\delta = 0.003$

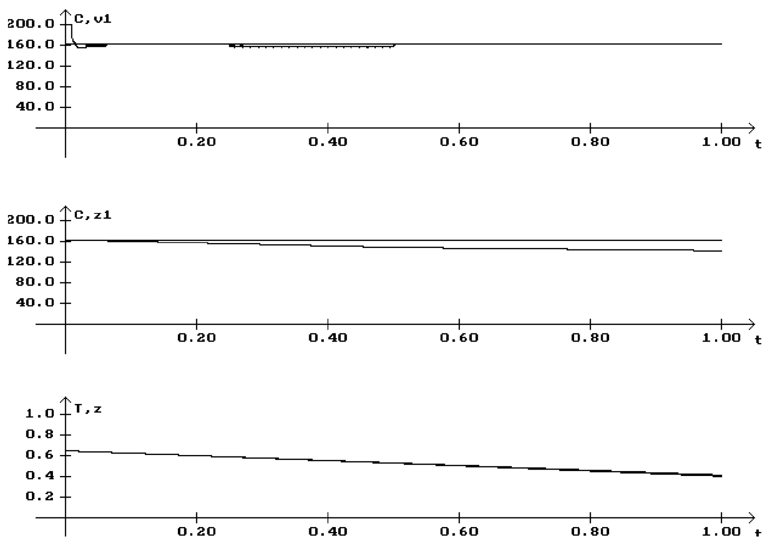

Figure 7. Result of simulation of the example 2 $\delta = 0.0003$

7. Conclusion

We consider the problems of identification and robust control for differential equations describing some ecological and economic processes. We present solving algorithms, which are based on the method of auxiliary feedback controlled models.

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