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Determination of positive realizations with reduced numbers of delays or without delays for discrete-time linear systems

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A new modified state variable diagram method is proposed for determination of positive realizations with reduced numbers of delays and without delays of linear discrete-time systems for a given transfer function. Sufficient conditions for the existence of the positive realizations of given proper transfer function are established. It is shown that there exists a positive realization with reduced numbers of delays if there exists a positive realization without delays but with greater dimension. The proposed methods are demonstrated on a numerical example.

Key words: state diagram method, determination, linear, discrete-time, delay, realization

1. Introduction

Determination of the state space equations for given transfer matrix is a classical problem, called realization problem, which has been addressed in many papers and books [1, 3, 16, 17-23]. It is well-known that [1, 2, 16] that to find a realization for a given transfer function first we have to find a state matrix for given denominator of the transfer function. An overview on the positive realization problem is given in [1-3]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [4-9, 12] and the positive realization problem for discrete-time systems with delays in [8-10]. The fractional positive linear systems have been addressed in [14, 15, 18]. The realization problem for fractional linear systems has been analyzed in [11] and for positive 2D hybrid systems in [13]. A method based on similarity transformation of the standard realization to the discrete positive one has been proposed in [12]. Conditions for the existence of positive stable realization with system Metzler matrix for transfer function has been established in [6]. The problem of determination of the set of Metzler matrices for given stable polynomials has been formulated and partly solved in [19]. A new modified state variable diagram method for determination of pos-

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itive realizations with reduced number of delays for given proper transfer matrices of continuous-time linear systems has been proposed in [22].

In this paper an extension of the method proposed in [22] for discrete-time linear systems will be proposed and it will be shown that there exists a positive realization without delays if there exists a positive realization with reduced number of delays of a given transfer function.

The paper is organized as follows. In section 2 some preliminaries concerning positive discrete-time linear systems with delays are recalled and the problem formulation is given. Basic lemmas of the proposed method are given in section 3. The new modified state variable diagram method is proposed in section 4 and in section 5 a method for computation of positive realizations without delays. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix, $\Re^{n\times m}(z,z^{-1})$ – the set of $n\times m$ rational matrices in z and z^{-1}, $\Re^{n\times m}[z,z^{-1}]$ – the set of $n\times m$ polynomial matrices in z and z^{-1}.

2. Preliminaries and the problem formulation

Consider the discrete-time linear system with q_1 delays in state and q_2 delays in inputs

$$x_{i+1}=\sum_{j=0}^{q_1}A_jx_{i-j}+\sum_{k=0}^{q_2}B_ku_{i-k} \tag{1a}$$

$$y_i=Cx_i+Du_i \tag{1b}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are the state, input and output vectors, respectively and $A_j\in\Re^{n\times n}$, $j=0,1,\ldots,q_1$, $B_k\in\Re^{n\times m}$, $k=0,1,\ldots,q_2$, $C\in\Re^{p\times n}$ and $D\in\Re^{p\times m}$. Initial conditions for (1) are given by

$$x_j \quad \text{for} \quad j=-q_1,1-q_1,\ldots,0. \tag{2}$$

Definition 1 *The system* (1) *is called the (internally) positive if* $x_i\in\Re_+^n$, *and* $y_i\in\Re_+^p$ *for* $i\in Z_+$ *for every* $x_j\in\Re_+^n$, $j=-q_1,1-q_1,\ldots,0$, *and all inputs* $u_i\in\Re_+^m$, $i=-q_2,1-q_2,\ldots,0$.

Theorem 1 *The system* (1) *is positive if and only if*

$$A_j\in\Re_+^{n\times n},\ \ j=0,1,\ldots,q_1,\ \ B_k\in\Re_+^{n\times m},\ \ k=0,1,\ldots,q_2,\ \ C\in\Re_+^{p\times n},\ \ D\in\Re_+^{p\times m} \tag{3}$$

Proof. The proof is given in [10].

The transfer matrix of the system (1) is given by

$$T(z,z^{-1}) = C[I_n z - A_0 - A_1 z^{-1} - \dots - A_{q_1} z^{-q_1}]^{-1}[B_0 + B_1 z^{-1} + \dots + B_{q_2} z^{-q_2}] + D. \quad (4)$$

Definition 2 *Matrices* (3) *are called a positive realization of a given transfer matrix* $T(z,z^{-1}) \in \Re^{p\times m}(z,z^{-1})$ *if they satisfy the equality* (4).

The positive realization problem under consideration can be stated as follows. Given a proper transfer matrix $T(z,z^{-1}) \in \Re^{p\times m}(z,z^{-1})$, find a positive realization with reduced numbers of delays (3) of $T(z,z^{-1})$.

In this paper sufficient conditions for solvability of the problem will be established and new methods for determination of a positive realization with reduced number of delays and without delays will be proposed.

3. Problem solution

Transfer matrix (4) can be written in the following form

$$T(z,z^{-1}) = \frac{C(H_{ad}(z,z^{-1}))[B_0 + B_1 z^{-1} + \dots + B_{q_2} z^{-q_2}]}{\det H(z,z^{-1})} + D = \frac{N(z,z^{-1})}{d(z,z^{-1})} + D \quad (5)$$

where

$$H(z,z^{-1}) = [I_n z - A_0 - A_1 z^{-1} - \dots - A_{q_1} z^{-q_1}] \in \Re^{p\times m}[z,z^{-1}], \quad (6)$$

$$N(z,z^{-1}) = C(H_{ad}(z,z^{-1}))[B_0 + B_1 z^{-1} + \dots + B_{q_2} z^{-q_2}], \quad d(z,z^{-1}) = \det H(z,z^{-1}). \quad (7)$$

From (5) we have

$$D = \lim_{z\to\infty} T(z,z^{-1}) \quad (8)$$

since $\lim_{z\to\infty} H^{-1}(z,z^{-1}) = 0$. The strictly proper transfer matrix is given by

$$T_{sp}(z,z^{-1}) = T(z,z^{-1}) - D = \frac{N(z,z^{-1})}{d(z,z^{-1})}. \quad (9)$$

Therefore, the positive realization problem is reduced to finding the matrices

$$\begin{aligned} &A_j \in \Re_+^{n\times n}, \ \ j = 0,1,\dots,\bar{q}_1, B_k \in \Re_+^{n\times m}, \ \ k = 0,1,\dots,\bar{q}_2, \\ &C \in \Re_+^{p\times n}, \ \ (\bar{q}_1 < q_1, \ \ \bar{q}_2 < q_2) \end{aligned} \quad (10)$$

for a given strictly proper transfer matrix (9).

To simplify the notation we shall consider a single-input single-output (SISO) system described by the equation (1) for $m = p = 1$.

Let a given strictly proper, irreducible transfer function has the form

$$T_{sp}(z,z^{-1}) = \frac{n(z,z^{-1})}{d(z,z^{-1})} \tag{11a}$$

where

$$\begin{aligned} n(z,z^{-1}) &= b_{n-1}(z^{-1})z^{n-1} + \ldots + b_1(z^{-1})z + b_0(z^{-1}), \\ b_k(z^{-1}) &= b_{k,m}z^{-m} + \ldots + b_{k,1}z^{-1} + b_{k,0}, \quad k = 0,1,\ldots,n-1 \end{aligned} \tag{11b}$$

$$\begin{aligned} d(z,z^{-1}) &= z^n - a_{n-1}(z^{-1})z^{n-1} - \ldots - a_1(z^{-1})z - a_0(z^{-1}), \\ a_k(z^{-1}) &= a_{k,m}z^{-m} + \ldots + a_{k,1}z^{-1} + a_{k,0}, \quad k = 0,1,\ldots,n-1. \end{aligned} \tag{11c}$$

The solution of the positive realization problem for (11) is based on the following two lemmas [4].

Lemma 1 *Let $p_k = p_k(z^{-1})$ for $k = 1,2,\ldots,2n-1$ be some polynomials in z^{-1} with nonnegative coefficients and*

$$P(z^{-1}) = \begin{bmatrix} 0 & 0 & \ldots & 0 & p_n \\ p_1 & 0 & \ldots & 0 & p_{n+1} \\ 0 & p_2 & \ldots & 0 & p_{n+2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & p_{n-1} & p_{2n-1} \end{bmatrix} \tag{12}$$

Then

$$\begin{aligned} &\det[I_n z - P(z^{-1})] = \\ &\qquad s^n - p_{2n-1}z^{n-1} - p_{n-1}p_{2n-2}z^{n-2} - \ldots - p_2p_3\ldots p_{n-1}p_{n+1}z - p_1p_2\ldots p_n. \end{aligned} \tag{13}$$

Proof is given in [4].

Lemma 2 *Let $R_n(z^{-1})$ be the n-th row of the adjoint matrix $[I_n z - P(z^{-1})]_{ad}$. Then*

$$R_n(z^{-1}) = \begin{bmatrix} p_1p_2\ldots p_{n-1} & p_2p_3\ldots p_{n-1}z & p_3p_4\ldots p_{n-1}z^2 & \ldots & p_{n-1}z^{n-2} & z^{n-1} \end{bmatrix} \tag{14}$$

Proof is given in [4].

In particular case for $p_1 = p_2 = \ldots = p_{n-1}$ from Lemmas 1 and 2 we obtain

$$P(z^{-1}) = \begin{bmatrix} 0 & 0 & \ldots & 0 & p_2 \\ p_1 & 0 & \ldots & 0 & p_3 \\ 0 & p_1 & \ldots & 0 & p_4 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & p_1 & p_{n+1} \end{bmatrix} \tag{15}$$

then

$$\det[I_n s - P(z^{-1})] = z^n - p_{n+1}z^{n-1} - \ldots - p_3 p_1^{n-2} z - p_2 p_1^{n-1} \tag{16}$$

and

$$R_n(z) = [\ p_1^{n-1} \quad p_1^{n-2}z \quad \ldots \quad p_1 z^{n-2} \quad z^{n-1}\ ]. \tag{17}$$

It is assumed that for the given denominator (11c) there exist polynomials

$$p_k = p_k(z^{-1}) = p_{k,q_1} z^{-q_1} + \ldots + p_{k,1} z^{-1} + p_{k,0}, \quad k = 0,1,\ldots,2n-1 \tag{18}$$

with nonnegative coefficients $p_{k,j}$, $j = 0,1,\ldots,q_1$ such that

$$\begin{aligned} &a_{n-1}(z^{-1}) = p_{2n-1}, \quad a_{n-2}(z^{-1}) = \\ &\quad p_{n-1}p_{2n-2},\ldots,a_1(z^{-1}) = p_2 p_3 \ldots p_{n-1} p_{n+1}, \quad a_0(z^{-1}) = p_1 p_2 \ldots p_n. \end{aligned} \tag{19}$$

In particular case if the matrix $P(z^{-1})$ has the form (15) then (19) takes the form

$$a_k(z^{-1}) = p_1^{n-k-1} p_{k+2}, \quad k = 0,1,\ldots,n-1. \tag{20}$$

Note that if the assumption (19) is satisfied then for the given denominator $d(z,z^{-1})$ of (11a) we may find the matrix (12) and next the corresponding matrices $A_j \in \Re_+^{n\times n}$, $j = 0,1,\ldots,\bar{q}_1$ since

$$I_n z - P(z^{-1}) = I_n z - \sum_{j=0}^{\bar{q}_1} A_j z^{-j}. \tag{21}$$

The matrix C is chosen in the form

$$C = [\ 0 \quad \ldots \quad 0 \quad 1\ ] \in \Re^{1\times n}. \tag{22}$$

Taking into account (14), (21) and (7) we obtain

$$\begin{aligned} &C[I_n z - P(z^{-1})]_{ad}[B_0 + B_1 z^{-1} + \ldots + B_{q_2} z^{-\bar{q}_2}] = R_n(z^{-1})[B_0 + B_1 z^{-1} + \ldots + B_{\bar{q}_2} z^{-\bar{q}_2}] \\ &\quad = [p_1 p_2 \ldots p_{n-1} \quad p_2 p_3 \ldots p_{n-1} z \quad p_3 p_4 \ldots p_{n-1} z^2 \ \ldots \ p_{n-1} z^{n-2} \quad z^{n-1}] \\ &\quad \times [B_0 + B_1 z^{-1} + \ldots + B_{q_2} z^{-\bar{q}_2}] = n(z,z^{-1}) \end{aligned} \tag{23}$$

4. Modified state variables diagram method

First the modified state variables diagram method of determination of positive realizations will be presented on the following strictly proper transfer function

$$
\begin{aligned}
T_{sp}(z,z^{-1}) &= \frac{b_2(z^{-1})z^2+b_1(z^{-1})z+b_0(z^{-1})}{z^3-a_2(z^{-1})z^2-a_1(z^{-1})z-a_0(z^{-1})} \\
&= \frac{(3z^{-2}+z^{-1}+2)z^2+(z^{-2}+3z^{-1}+2)z+z^{-4}+2z^{-3}+z^{-2}}{z^3-(2z^{-2}+3z^{-1}+1)z^2-(z^{-3}+3z^{-2}+2z^{-1})z-(z^{-5}+2z^{-4}+3z^{-3}+2z^{-2})}.
\end{aligned} \tag{24}
$$

The proposed method is based on Lemmas 1 and 2. It is assumed that there exist polynomials (18) with nonnegative coefficients $p_{k,j}$, $j=0,1,\ldots,q_1$ satisfying (19) and

$$
\begin{aligned}
&b_{n-1}(z^{-1})=\bar{b}_{n-1}(z^{-1}), \quad b_{n-2}(z^{-1})=p_{n-1}\bar{b}_{n-2}(z^{-1}),\ldots, \\
&b_1(z^{-1})=p_2p_3\ldots p_{n-1}\bar{b}_1(z^{-1}), \quad b_0(z^{-1})=p_1p_2\ldots p_{n-1}\bar{b}_0(z^{-1})
\end{aligned} \tag{25}
$$

for some polynomials with nonnegative coefficients $\bar{b}_{n-2}(z^{-1}),\ldots,\bar{b}_1(z^{-1}),\bar{b}_0(z^{-1})$. For (24) we have

$$
\begin{aligned}
&p_1(z^{-1})=z^{-2}, \quad p_2(z^{-1})=z^{-1}+1, \quad p_3(z^{-1})=z^{-2}+z^{-1}+2, \\
&p_4(z^{-1})=z^{-2}+2z^{-1}, \quad p_5(z^{-1})=2z^{-2}+3z^{-1}+1
\end{aligned} \tag{26}
$$

since

$$
\begin{aligned}
\det[I_3z-P(z^{-1})] &= \begin{vmatrix} z & 0 & -z^{-2}-z^{-1}-2 \\ -z^{-2} & z & -z^{-2}-2z^{-1} \\ 0 & -z^{-1}-1 & z-2z^{-2}-3z^{-1}-1 \end{vmatrix} \\
&= z^3-(2z^{-2}+3z^{-1}+1)z^2-(z^{-3}+3z^{-2}+2z^{-1})z-(z^{-5}+2z^{-4}+3z^{-3}+2z^{-2}) \\
&= z^3-a_2(z^{-1})z^2-a_1(z^{-1})z-a_0(z^{-1})
\end{aligned}
$$

$$
\begin{aligned}
a_2(z^{-1}) &= p_5(z^{-1}), \\
a_1(z^{-1}) &= p_2(z^{-1})p_4(z^{-1}), \\
a_0(z^{-1}) &= p_1(z^{-1})p_2(z^{-1})p_3(z^{-1})
\end{aligned} \tag{27a}
$$

and

$$
\begin{aligned}
&b_2(z^{-1})=3z^{-2}+z^{-1}+2, \\
&b_1(z^{-1})=p_2(z^{-1})\bar{b}_1(z^{-1})=(z^{-1}+1)(z^{-1}+2)=z^{-2}+3z^{-1}+2, \\
&b_0(z^{-1})=p_1(z^{-1})p_2(z^{-1})\bar{b}_0(z^{-1})=(z^{-3}+z^{-2})(z^{-1}+1)=z^{-4}+2z^{-3}+z^{-2}
\end{aligned} \tag{27b}
$$

where

$$
\bar{b}_1(z^{-1})=z^{-1}+2, \quad \bar{b}_0(z^{-1})=z^{-1}+1. \tag{27c}
$$

From (24) written in the form

$$\frac{b_2(z^{-1})z^{-1} + b_1(z^{-1})z^{-2} + b_0(z^{-1})z^{-3}}{1 - a_2(z^{-1})z^{-1} - a_1(z^{-1})z^{-2} - a_0(z^{-1})z^{-3}} = \frac{Y}{U}$$

we have

$$\begin{aligned} Y = \; & z^{-1}\{b_2(z^{-1})U + a_2(z^{-1})Y + z^{-1}[b_1(z^{-1})U + a_1(z^{-1})Y + z^{-1}(b_0(z^{-1})U + a_0(z^{-1})Y)]\} \\ & = z^{-1}\{\bar{b}_2(z^{-1})U + p_5(z^{-1})Y + z^{-1}[p_2(z^{-1})\bar{b}_1(z^{-1})U + p_2(z^{-1})p_4(z^{-1})Y \\ & \quad + z^{-1}(p_1(z^{-1})p_2(z^{-1})\bar{b}_0(z^{-1})U + p_1(z^{-1})p_2(z^{-1})p_3(z^{-1})Y)]\}. \end{aligned} \tag{28}$$

Using (28) we may draw the modified state variable diagram shown in Fig. 1.

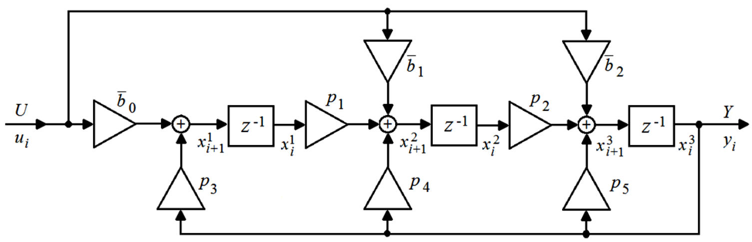

Figure 1. State variable diagram for (28).

The variables x^1_i, x^2_i, x^3_i are chosen as the outputs of the integral elements. Using the modified state variables diagram we may write the following equations.

$$\begin{aligned} x^1_{i+1} &= p_3x^3_i + \bar{b}_0u_i \\ x^2_{i+1} &= p_1x^1_i + p_4x^3_i + \bar{b}_1u_i \\ x^3_{i+1} &= p_2x^2_i + p_5x^3_i + \bar{b}_2u_i \end{aligned} \tag{29a}$$

which can be written in the form

$$x_{i+1} = Ax_i + Bu_i \tag{29b}$$

where

$$A = \begin{bmatrix} 0 & 0 & p_3 \\ p_1 & 0 & p_4 \\ 0 & p_2 & p_5 \end{bmatrix} = P(z^{-1}) = A_0 + A_1 z^{-1} + A_2 z^{-2},$$

$$A_0 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \tag{29c}$$

$$B = \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}.$$

Assuming $C = [\ 0 \quad 0 \quad 1\ ]$ and taking into account (29b) we have

$$T_{sp}(z, z^{-1}) = C[I_3 z - A]^{-1} B = [\ 0 \quad 0 \quad 1\ ] \begin{bmatrix} z & 0 & -p_3 \\ -p_1 & z & -p_4 \\ 0 & -p_2 & z - p_5 \end{bmatrix}^{-1} \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}$$
$$= \frac{[\ p_1 p_2 \quad p_2 z \quad z^2\ ]}{z^3 - p_5 z^2 - p_2 p_4 z - p_1 p_2 p_3} \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \frac{p_1 p_2 \bar{b}_0 + p_2 \bar{b}_1 z + \bar{b}_2 z^2}{z^3 - a_2(z^{-1}) z^2 - a_1(z^{-1}) z - a_0(z^{-1})}. \tag{30}$$

Comparison of (30) and (24) yields

$$\bar{b}_2(z^{-1}) = b_2(z^{-1}) = 3z^{-2} + z^{-1} + 2, \quad \bar{b}_1(z^{-1}) = z^{-1} + 2, \quad \bar{b}_0(z^{-1}) = z^{-1} + 1 \tag{31a}$$

and

$$\bar{B} = \begin{bmatrix} \bar{b}_0(z^{-1}) \\ \bar{b}_1(z^{-1}) \\ \bar{b}_2(z^{-1}) \end{bmatrix} = B_0 + B_1 z^{-1} + B_2 z^{-2} \tag{31b}$$

where

$$B_0 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}. \tag{31c}$$

The desired positive realization of (24) is given by (29c), (31c) and $C = [0\ \ 0\ \ 1]$, $D = [0]$.

Note that the sufficient condition for the existence of a positive realization of (24) is the existence of polynomials (26) and $\bar{b}_1(z^{-1})$, $\bar{b}_0(z^{-1})$ with nonnegative coefficients satisfying (27a) and (27b). In general case consider the strictly proper irreducible transfer function

$$T_{sp}(z,z^{-1}) = \frac{n(z,z^{-1})}{d(z,z^{-1})} \tag{32a}$$

where

$$\begin{aligned} &n(z,z^{-1}) = b_{n-1}(z^{-1})z^{n-1} + \ldots + b_1(z^{-1})z + b_0(z^{-1}), \\ &b_k(z^{-1}) = b_{k,m}z^{-m} + \ldots + b_{k,1}z^{-1} + b_{k,0}, \quad k = 0,1,\ldots,n-1 \end{aligned} \tag{32b}$$

$$\begin{aligned} &d(z,z^{-1}) = z^n - a_{n-1}(z^{-1})z^{n-1} - \ldots - a_1(z^{-1})z - a_0(z^{-1}), \\ &a_k(z^{-1}) = a_{k,m}z^{-m} + \ldots + a_{k,1}z^{-1} + a_{k,0}, \quad k = 0,1,\ldots,n-1. \end{aligned} \tag{32c}$$

It is assumed that there exist polynomials $p_k(z^{-1})$, $k = 0,1,\ldots,2n-1$ and $\bar{b}_0(z^{-1})$, $\bar{b}_1(z^{-1})$, ...,$\bar{b}_{n-2}(z^{-1})$ with nonnegative coefficients satisfying (25).
By Lemma 1 we have

$$\det[I_n z - P(z^{-1})] = \begin{vmatrix} z & 0 & \ldots & 0 & -p_n \\ -p_1 & z & \ldots & 0 & -p_{n+1} \\ 0 & -p_2 & \ldots & 0 & -p_{n+2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & -p_{n-1} & z - p_{2n-1} \end{vmatrix} \tag{33}$$
$$= z^n - p_{2n-1}z^{n-1} - p_{n-1}p_{2n-2}z^{n-2} - \ldots - p_2p_3\ldots p_{n-1}p_{n+1}z - p_1p_2\ldots p_n$$

and by Lemma 2 the n-th row of $R_n(z^{-1})$ of the adjoint matrix $[I_n z - P(z^{-1})]_{ad}$ is

$$R_n(z^{-1}) = [\; p_1p_2\ldots p_{n-1} \quad p_2p_3\ldots p_{n-1}z \quad p_3p_4\ldots p_{n-1}z^2 \quad \ldots \quad p_{n-1}z^{n-2} \quad z^{n-1} \;] \tag{34}$$

Let

$$\bar{B}(z^{-1}) = \begin{bmatrix} \bar{b}_0(z^{-1}) \\ \vdots \\ \bar{b}_{n-1}(z^{-1}) \end{bmatrix} = \begin{bmatrix} \bar{b}_0 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix} \tag{35}$$

and

$$\begin{aligned} &R_n(z^{-1})\bar{B}(z^{-1}) = \\ &p_1p_2\ldots p_{n-1}\bar{b}_0 + p_2p_3\ldots p_{n-1}\bar{b}_1 z + \ldots + p_{n-1}\bar{b}_{n-2}z^{n-2} + \bar{b}_{n-1}z^{n-1} \\ &\quad = b_0 + b_1 z + \ldots + b_{n-2}z^{n-2} + b_{n-1}z^{n-1} = n(z,z^{-1}). \end{aligned} \tag{36}$$

Assuming $C = [\ 0 \quad \dots \quad 0 \quad 1\ ] \in \Re_+^{1\times n}$ and using (33), (36) we obtain

$$T_{sp}(z,z^{-1}) = C[I_n z - A(z^{-1})]^{-1}\bar{B}(z^{-1})$$
$$= [\ 0 \quad \dots \quad 0 \quad 1\ ] \begin{bmatrix} z & 0 & \dots & 0 & -p_n \\ -p_1 & z & \dots & 0 & -p_{n+1} \\ 0 & -p_2 & \dots & 0 & -p_{n+2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -p_{n-1} & z - p_{2n-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{b}_0 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix} \tag{37}$$
$$= \frac{1}{z^n - a_{n-1}(z^{-1})z^{n-1} - \dots - a_1(z^{-1})z - a_0(z^{-1})} R_n(z^{-1})\bar{B}(z^{-1}) = \frac{n(z,z^{-1})}{d(z,z^{-1})}.$$

Theorem 2 *There exists the positive realization*

$$A_j \in \Re_+^{n\times n},\ \ j = 0,1,\dots,\bar{q}_1,\ \ B_k \in \Re_+^{n\times m},\ \ k = 0,1,\dots,\bar{q}_2,\ \ C = [0\ \dots\ 0\ \ 1] \in \Re_+^{1\times n} \tag{38}$$

of the transfer function (32) *if it is possible to find the polynomials*

$$p_1(z^{-1}),\quad p_2(z^{-1}),\dots,\quad p_{2n-1}(z^{-1}) \tag{39}$$

and

$$\bar{b}_0(z^{-1}),\quad \bar{b}_1(z^{-1}),\dots,\quad \bar{b}_{n-1}(z^{-1}) \tag{40}$$

with nonnegative coefficients such that (33) *and* (36) *are satisfied.*

Proof If the polynomials (39) have nonnegative coefficients then

$$A(z^{-1}) = P(z^{-1}) = A_{\bar{q}_1} z^{-\bar{q}_1} + \dots + A_1 z^{-1} + A_0 \tag{41}$$

and $A_j \in \Re_+^{n\times n}$, $j = 0,1,\dots,q_1$. If the coefficients of the polynomials (40) are nonnegative then

$$\bar{B}(z^{-1}) = \begin{bmatrix} \bar{b}_0(z^{-1}) \\ \vdots \\ \bar{b}_{n-1}(z^{-1}) \end{bmatrix} = B_{\bar{q}_2} z^{-\bar{q}_2} + \dots + B_1 z^{-1} + B_0 \tag{42}$$

and $B_k \in \Re_+^{n\times m}$, $k = 0,1,\dots,\bar{q}_2$. The matrices (38) are a realization of (32) since they satisfy (37). □

Procedure 1

Step 1. Knowing the coefficients $a_k(z^{-1})$, $b_k(z^{-1})$, $k = 0,1,\dots,n-1$ of (32) find the polynomials (39) and (40) with nonnegative coefficients satisfying the conditions (33) and (36).

Step 2. Knowing (39) and using (41) find the matrices A_j for $j = 0, 1, \ldots, \bar{q}_1$.

Step 3. Using the equality (36) and (42) find the polynomials $\bar{b}_0(z^{-1})$, $\bar{b}_1(z^{-1})$, $\ldots, \bar{b}_{n-1}(z^{-1})$ and the matrices B_k, $k = 0, 1, \ldots, \bar{q}_2$.

Example 1 Using Procedure 1 find a positive realization of transfer function

$$T_{sp}(z, z^{-1}) = \frac{(z^{-2} + 2z^{-1})z + (z^{-3} + z^{-2})}{z^2 - (2z^{-1} - 3)z - (z^{-3} + z^{-1})}. \tag{43}$$

Step 1. In this case we choose the polynomials

$$p_1(z^{-1}) = z^{-1}, \quad p_2(z^{-1}) = z^{-2} + 1, \quad p_3(z^{-1}) = 2z^{-1} - 3 \tag{44}$$

and

$$\bar{b}_0(z^{-1}) = z^{-2} + z^{-1}, \quad \bar{b}_1(z^{-1}) = z^{-2} + 2z^{-1} \tag{45}$$

which satisfy the conditions (33) and (36) since

$$\begin{bmatrix} p_1 & z \end{bmatrix} \begin{bmatrix} \bar{b}_0(z^{-1}) \\ \bar{b}_1(z^{-1}) \end{bmatrix} = \begin{bmatrix} z^{-1} & z \end{bmatrix} \begin{bmatrix} z^{-2} + z^{-1} \\ z^{-2} + 2z^{-1} \end{bmatrix} = (z^{-2} + 2z^{-1})z + (z^{-3} + z^{-2}). \tag{46}$$

Step 2. Using (41) and (44) we obtain

$$A(z^{-1}) = P(z^{-1}) = \begin{bmatrix} 0 & z^{-2} + 1 \\ z^{-1} & 2z^{-1} - 3 \end{bmatrix} = A_0 + A_1 z^{-1} + A_2 z^{-2} \tag{47}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{48}$$

Step 3. Taking into account that

$$\bar{B}(z^{-1}) = \begin{bmatrix} \bar{b}_0(z^{-1}) \\ \bar{b}_1(z^{-1}) \end{bmatrix} = \begin{bmatrix} z^{-2} + z^{-1} \\ z^{-2} + 2z^{-1} \end{bmatrix} = B_0 + B_1 z^{-1} + B_2 z^{-2} \tag{49a}$$

where

$$B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{49b}$$

The desired positive realization of (43) is given by (48), (49b) and

$$C = [0 \;\; 1], \;\; D = [0]. \tag{49c}$$

The proposed method can be extended to multi-input multi-output linear (MIMO) system with delays in a similar way as for continuous-time linear systems [22].

5. Positive realizations without delays

In this section it will be shown that for the transfer function (5) there exist also positive realizations without delays. First we shall consider the transfer function (24). Multiplying the numerator and denominator of (24) by z^5 we obtain

$$T_{sp}(z) = \frac{2z^7 + 3z^6 + 6z^5 + z^4 + z^3 + 2z^2 + z}{z^8 - z^7 - 3z^6 - 4z^5 - 3z^4 - 3z^3 - 3z^2 - 2z - 1}. \tag{50}$$

It is easy to check that one of the positive realizations of the transfer function (50) has the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 3 & 3 & 4 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C = [\, 0 \quad 1 \quad 2 \quad 1 \quad 1 \quad 6 \quad 3 \quad 2 \,], \quad D = [0]. \tag{51}$$

Note that the realization without delays (51) has greater dimension than the realization with delays given by (29c), (31c) and $C = [0 \;\; 0 \;\; 1]$, $D = [0]$.

In general case consider the strictly proper transfer function (32). Multiplying the numerator and the denominator of (32) by z^m we obtain the strictly proper transfer function

$$T(z) = \frac{\bar{b}_{\bar{n}-1} z^{\bar{n}-1} + \bar{b}_{\bar{n}-2} z^{\bar{n}-2} + \ldots + \bar{b}_1 z + \bar{b}_0}{z^{\bar{n}} - \bar{a}_{\bar{n}-1} z^{\bar{n}-1} - \ldots - \bar{a}_1 z - \bar{a}_0} \tag{52}$$

where $\bar{n} = mn$.

Note that if in (32b) and (32c) $a_k \geqslant 0$ and $b_k \geqslant 0$ for $k = 0, 1, \ldots, n-1$ and $a_{k,j} \geqslant 0$ and $b_{k,j} \geqslant 0$ for $k = 0, 1, \ldots, n-1$; $j = 0, 1, \ldots, m$ then also $\bar{a}_k \geqslant 0$ and $\bar{b}_k \geqslant 0$ for $k = 0, 1, \ldots, \bar{n}-1$. In this case there exists a positive realization of the transfer function (52), for example, of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ \bar{a}_0 & \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_{\bar{n}-1} \end{bmatrix} \in \Re_+^{\bar{n}\times\bar{n}}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re_+^{\bar{n}}$$

$$C = [\ \bar{b}_0 \quad \bar{b}_1 \quad \dots \quad \bar{b}_{\bar{n}-1}\ ] \in \Re_+^{1\times\bar{n}}, \quad D = [0]. \tag{53}$$

It is easy to show that if the matrices (53) are a positive realization of (52) then the matrices

$$\bar{A} = P^{-1}AP, \ \bar{B} = P^{-1}B, \ \bar{C} = CP, \ \bar{D} = D \tag{54}$$

are also a positive realization of (52) for any monomial matrix $P \in \Re_+^{\bar{n}\times\bar{n}}$ (in which in each row and column only one entry is positive and all remaining entries are zero). Therefore, the following theorem has been proved.

Theorem 3 *If the transfer function* (32) *has a positive realization with delays then it has also a positive realization without delays* (53).

These considerations can be easily extended for multi-input multi-output linear systems in a similar way as for continuous-time linear systems in [22].

6. Concluding remarks

A new modified state variable diagram method for determination of positive realizations of linear discrete-time systems with delays in state and input vectors has been proposed. Using the method it is possible to find positive realization with reduced numbers of delays. A method has been also proposed for computation of positive realizations without delays for a given proper transfer matrices. Sufficient conditions for the existence of positive realizations have been established and procedure for finding the positive realizations have been proposed. The procedures have been illustrated by numerical example. It has been shown that if there exists a positive realization with delays there also exists a positive realization without delays of the transfer function but with greater dimension (Theorem 3). The proposed method can be extended to continuous-discrete linear systems and to fractional continuous-time linear systems.

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