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Tracking control of an underactuated rigid body with a coupling input force

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This paper presents a set of basic problems concerning the control of an underactuated dynamic system. Exemplary system of a planar rigid body with a coupling input force is described. Lie brackets method is used to show accessibility of the system. A tracking problem is solved with computed torque algorithm. The coupling force makes the convergence to zero of all state variables errors impossible. After numerical simulation, stability of the system is mentioned.

Key words: underactuated system, tracking, coupling force, computed torque

1. Introduction

The dynamic system described by a set of second order ordinary differential equations in form

$$\ddot{q}(q,\dot{q},t) = f_1(q,\dot{q},t) + f_2(q,\dot{q},t)u(t) \tag{1}$$

is called underactuated if control input $u(t)$ cannot produce accelerations $\ddot{q}$ in arbitrary direction. This happens when $\mathrm{rank}[f_2] < \dim[q]$. A system controlled by less number of inputs than the number of degrees of freedom is called trivially underactuated. There are many typical systems of that type, e.g. acrobot [2], pendubot, autonomous underwater vehicles [10, 11] or hovercrafts [6]. Underactuated systems are usually nonlinear, so the problem of controlability may be solved using the Lie theory. In [3, 5] one finds this theory described. The tracking problem is usually solved using the passive velocity field method [9], backstepping techniques [10] or the computed torque algorithm [13]. Usually control inputs of underactuated systems are described as independent which means that the force appears at most in one right hand side of equations [1, 6]. Impossibility of such separation makes design of a control law difficult and motivate this research.

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2. Description of the system

Consider a planar rigid body in $\mathbb{R}^2$ space (Fig. 1). The object consists of mass m and inertia I_c around the center of mass (point C). Introducing the global coordinate system O_{xy}, we denote by x and y the position of C, φ denotes angle between the object symmetry line and X axis (counter clockwise direction). The vector of force F acts on the object in a point away from C by distance a. The angle β indicates force direction. We assume that the object is moving in a viscous environment where the force is proportional to velocity with a constant drag coefficient (c for linear motion and c_φ for rotation). The equations of motion for the system are as follows

$$m\ddot{x}(t)+c\dot{x}(t)=|\vec{F}(t)|\cos\big(\varphi(t)+\beta(t)\big) \tag{2}$$

$$m\ddot{y}(t)+c\dot{y}(t)=|\vec{F}(t)|\sin\big(\varphi(t)+\beta(t)\big) \tag{3}$$

$$I_c\ddot{\varphi}(t)+c_\varphi\dot{\varphi}(t)=|\vec{F}(t)|a\sin\big(\beta(t)\big) \tag{4}$$

with initial conditions $x(0)=x_0$, $\dot{x}(0)=v_{x0}$, $y(0)=y_0$, $\dot{y}(0)=v_{y0}$, $\varphi(0)=\varphi_0$, $\dot{\varphi}(0)=\omega_0$.

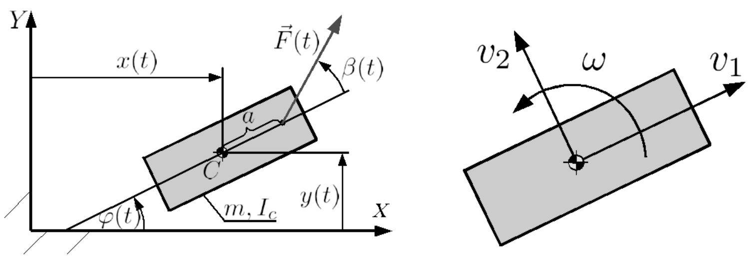

Figure 1. Object in the global coordinate system and its velocities in the local coordinate system.

It can be seen from (2-4) that the force binds the object movements in the X and Y directions and its rotation. The $F\cos\beta$ acts only in the longitudinal direction, while $F\sin\beta$ acts laterally causing rotation. Generalized coordinates [3] allows for decoupling, however, physical interpretation is missed.

Introduce a state space vector $q=[x(t),y(t),\varphi(t),v_1(t),v_2(t),\omega(t)]^T$, where v_1 and v_2 are object velocities in the local coordinate and ω is rotation velocity. We rewrite equations (2-4) into one dynamic equation of motion in a matrix form of the first order ordinary differential equations

$$\dot{q}(t)=f(q)+g_1u_1(t)+g_2u_2(t) \tag{5}$$

where the vector field f is called a drift function, and has a form

$$f(q) = \begin{bmatrix} v_1 \cos\varphi - v_2 \sin\varphi \\ v_1 \sin\varphi + v_2 \cos\varphi \\ \omega \\ -\frac{c}{m} v_1 + v_2 \omega \\ -\frac{c}{m} v_2 + v_1 \omega \\ -\frac{c_\varphi}{I_C} \omega \end{bmatrix} \tag{6}$$

u_1 and u_2 are the controls

$$u_1(t) = F(t) \cos\beta(t) \tag{7}$$

$$u_2(t) = F(t) \sin\beta(t) \tag{8}$$

where $F(t) \in \mathbb{R}$ denote the magnitude of the force and $\beta(t) \in \mathbb{R}$ denotes its angular orientation. g_1 and g_2 are constant control vector fields

$$g_1 = \left[0, 0, 0, \frac{1}{m}, 0, 0\right]^T \tag{9}$$

$$g_2 = \left[0, 0, 0, 0, \frac{1}{m}, \frac{a}{I_c}\right]^T. \tag{10}$$

The force u_1 affects only the rate of change of the v_1 velocity. The force u_2 changes the velocities v_2 and ω. The rates of change of $x(t)$, $y(t)$ and $\varphi(t)$ are not directly affected by the u_1 and u_2 but by a part of the drift function that cause inertial behavior of the system. The drift function may also contain gyroscopic and centripetal components.

3. Controlability

Control law design should be followed by checking accessiblility and controlability of the system [3]. Solution of the problem

$$\dot{q}(t) = g_0(q) + \sum_{i=1}^{w} g_i(q) u_i(t) \tag{11}$$

is $(q(t), u_1(t), \ldots, u_w(t))$ where time $t \in [0, T]$ for arbitrary $T > 0$ and $q(t)$ is a piecewise smooth curve in state manifold $\mathcal{M} = \mathbb{R}^n$. Admissible control inputs $u_1(t), \ldots, u_w(t)$ are locally integrable functions: $[0, T] \to U$. A set

$$S = (\mathcal{M}, g_0, g_1, \ldots, g_w, U) \tag{12}$$

is called a control affine system. We define three types of a reachable set of the system S from state q_0:

$$\mathcal{R}_S(q_0, T) = \{q(T) \in \mathcal{M} \mid \text{such that there exists solution for } q(0) = q_0\} \tag{13}$$

$$\mathcal{R}_S(q_0, \leqslant T) = \bigcup_{t \in [0.T]} \mathcal{R}_S(q_0, t) \tag{14}$$

$$\mathcal{R}_S(q_0) = \bigcup_{t \geqslant 0} \mathcal{R}_S(q_0, t). \tag{15}$$

Set (13) defines a set of all possible system states q at the time T when starting from the initial condition q_0 at the time zero. Set (14) is a sum of all possible reachable sets in the whole time interval $[0, T]$. Set (15) holds all possible reachable states of the system in all the time.

Definition 2 [8] *A system defined by* (12) *is:*

a) accessible from the state q_0 if the internal of set $\mathcal{R}_S(q_0)$ is not empty,

b) strongly accessible from the state q_0 if the internal of set $\mathcal{R}_S(q_0, T)$ is not empty for all $T > 0$,

c) locally controllable from the state q_0 if this state lies inside the reachable set $\mathcal{R}_S(q_0)$,

d) small-time locally controllable from the state q_0 if there exists time $T > 0$ such that the state q_0 is inside the set $\mathcal{R}_S(q_0, \leqslant T)$ for each time in $[0, T]$,

e) globally controllable from the state q_0 if the reachable set $\mathcal{R}_S(q_0)$ is equal to the manifold $\mathcal{M}$.

Discussion of higher dimensional system is difficult if accessibility and controlability are concerned because it is hard to calculate the above defined reachable sets.

Definition 3 [4] *The Lie bracket (Jacobi bracket) of two vector fields $f(x)$ and $g(x)$ is a vector field defined as*

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \tag{16}$$

where $x \in \mathbb{R}^n$ and

$$\frac{\partial}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \\ \cdots & \cdots & \ddots & \cdots \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix}. \tag{17}$$

One can find an interesting interpretation of the Lie bracket in [8].

Definition 4 [4] *The system defined by* (12) *is locally accessible from* $q_0 \in \mathcal{M}$ *when a distribution*

$$C(q) = [g_0|g_1|\ldots|g_w|\ldots|[g_i, g_j]|\ldots|[g_k, [g_i, g_j]]|\ldots] \tag{18}$$

has the rank equal to a number of the system independent variables n (size of state space vector).

Distribution (18), called accessible algebra, includes all possible Lie brackets of the vector fields q_i but in the case of the rank condition zero vectors are not useful. One should remember properties of the Lie bracket

$$[q_i, q_i] = 0 \tag{19}$$

$$[q_i, q_j] = -[q_j, q_i]. \tag{20}$$

The number of r-th order Lie brackets is equal to $w(w+1)^r/2$.

Theorem 6 [12] *The system with drift which is locally accessible is also a small-time locally controllable system.*

Theorem 7 [12] *The system without drift which is locally accessible is globally controllable.*

The definition of small-time locally controllable used for a linear system implies the classical controllability condition [5].

For the exemplary system defined by (5), the distribution algebra is proposed as first six non-zero brackets

$$C = [f|g_1|g_2|[f, g_1]|[f, g_2]|[g_1, [f, g_2]]] \tag{21}$$

where

$$[f, g_1] = \begin{bmatrix} -\dfrac{\cos\varphi}{m} \\ -\dfrac{\sin\varphi}{m} \\ 0 \\ -\dfrac{c}{m^2} \\ -\dfrac{\omega}{m} \\ 0 \end{bmatrix}, \quad [f, g_2] = \begin{bmatrix} \dfrac{\sin\varphi}{m} \\ -\dfrac{\cos\varphi}{m} \\ -\dfrac{a}{I_C} \\ -\dfrac{v_2 a}{I_C} - \dfrac{\omega}{m} \\ \dfrac{c}{m^2} - \dfrac{v_1 a}{I} \\ \dfrac{a c_\varphi}{I_C^2} \end{bmatrix}, \quad [g_1, [f, g_2]] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{a}{m I_C} \\ 0 \end{bmatrix}.$$

Also:

$$[g_1,g_2]=[g_1,[g_1,g_2]]=[f_1,[g_1,g_2]]=[g_1,[f,g_1]]=0,$$
$$[g_2,[f,g_1]]=[g_1,[f,g_2]]$$

and $[g_2,[f,g_2]]$ is parallel to g_1.

One should check that $\text{rank}(C)=\dim(q)=6$ for $\omega\neq\frac{am}{I_C}v_2$. This implies from definition 4 and theory 7 that the system is locally accessible and small-time locally controllable for $q_0\in\mathbb{R}^n\backslash\{(\omega,v_2)$ such that $\omega=\frac{am}{I_C}v_2\}$.

4. Tracking problem

After the statement of the small-time local controllability of the exemplary system we can put a task of tracking control. Let us describe the system (2-4) using a matrix form of the second order ordinary differential equations

$$M\ddot{z}+N(\dot{z},z)=Q(t) \tag{22}$$

where

$$z=\begin{bmatrix}x(t)\\y(t)\\\varphi(t)\end{bmatrix},\ M=\begin{bmatrix}m&0&0\\0&m&0\\0&0&I_C\end{bmatrix},\ N=\begin{bmatrix}c\dot{x}\\c\dot{y}\\c_\varphi\dot{\varphi}\end{bmatrix},\ Q(t)=\begin{bmatrix}F\cos(\varphi(t)+\beta)\\F\sin(\varphi(t)+\beta)\\Fa\sin(\beta)\end{bmatrix}.$$

In this paper it is proposed to use the computed torque technique to control in closed-loop system with PD feedback. In this method necessary forces are calculated by substituting state variable $z(t)$ by desired trajectory functions. To achieve feedback reaction for disturbances, forces generated by a system state error and its rate of change are attached. Control forces $\tau(t)=[\tau_x,\tau_y,\tau_\varphi]^T$ are then calculated as

$$\tau(t)=M\ddot{z}_d+N(z,\dot{z})+K_v(\dot{z}_d-\dot{z})+K_p(z_d-z) \tag{23}$$

where $z_d(t)$ denotes a desired system trajectory in the configuration space, K_v and K_p are diagonal matrix of constants. Substituting new forces (23) into the right hand side of equation (22) with a new error variables matrix $e(t)=z_d(t)-z(t)$ leads to the formula

$$M\ddot{e}(t)+K_v\dot{e}(t)+K_pe(t)=0. \tag{24}$$

This formula proves exponential convergence of the errors to zero providing M, K_v and K_p are positive defined.

The problem concerns presenting of necessary forces (23) by available right hand side of equations of motion $Q(t)$. One can try to search step by step for values of F and β minimizing $||\tau - Q||^2$. This optimization problem will not be solved in this paper. The second possibility is to neglect the rotation control, what entails simple relationships

$$F(t) = \sqrt{\tau_x^2 + \tau_y^2} \tag{25}$$

$$\beta(t) = Arg\left(\tau_x + \tau_y\sqrt{-1}\right) - \varphi(t) \tag{26}$$

where equation (26) returns the angle between the objects symmetry line and the vector given by the coordinates $[\tau_x, \tau_y]$.

Let us propose two tracking problems: circular motion with tangential orientation described by desire state function

$$z_d(t) = \begin{bmatrix} R\cos(\theta t) \\ R\sin(\theta t) \\ \theta t + \frac{\pi}{2} \end{bmatrix} \tag{27}$$

where R is the circle radius and θ is the angular velocity; and eight curve motion described by function

$$z_d(t) = \begin{bmatrix} 0.5R\sin(2\theta t) \\ R\sin(\theta t) \\ Arg(\cos 2\theta t + \sqrt{-1}\cos\theta t) \end{bmatrix}. \tag{28}$$

5. Numerical simulation

Let us show example effects of numerical simulation for the tracking task introduced in previous section using inputs defined by equations (23, 25, 26). Tab. 1 presents system and control parameters, Fig. 2 shows the results for circular trajectory. From equation (24) one can proof that

$$\lim_{t\to\infty}(x(t) - x_d(t)) = 0 \tag{29}$$

$$\lim_{t\to\infty}(y(t) - y_d(t)) = 0. \tag{30}$$

Additionally, for a set of initial conditions, it is possible to achieve the condition

$$\lim_{t\to\infty}(\varphi(t) - \varphi_d(t)) = \text{const}. \tag{31}$$

Negligence of rotation control causes non-zero rotation error values and implies the need of stability analysis. The problem of rotation behavior is more visible at tracking of the eight curve (Fig. 3).

Table 5. System and control parameters

	symbol	value
System parameters	m	$2kg$
	I_C	$0.1kgm^2$
	a	$0.4m$
	c	$0.6\frac{Ns}{m}$
	c_φ	$0.1Nms$
Initial conditions	$x(0), y(0), \varphi(0)$	$1m, 0m, \frac{\pi}{2}$
	$\dot{x}(0), \dot{y}(0), \dot{\varphi}(0)$	$0\frac{m}{s}, 0\frac{m}{s}, 0\frac{1}{s}$
Feedback parameters	K_p	$diag(5.2\frac{N}{m})$
	K_v	$diag(\sqrt{10.4}\frac{Ns}{m})$
Circular trajectory parameters	R	$2m$
	θ	$\frac{\pi}{4}$

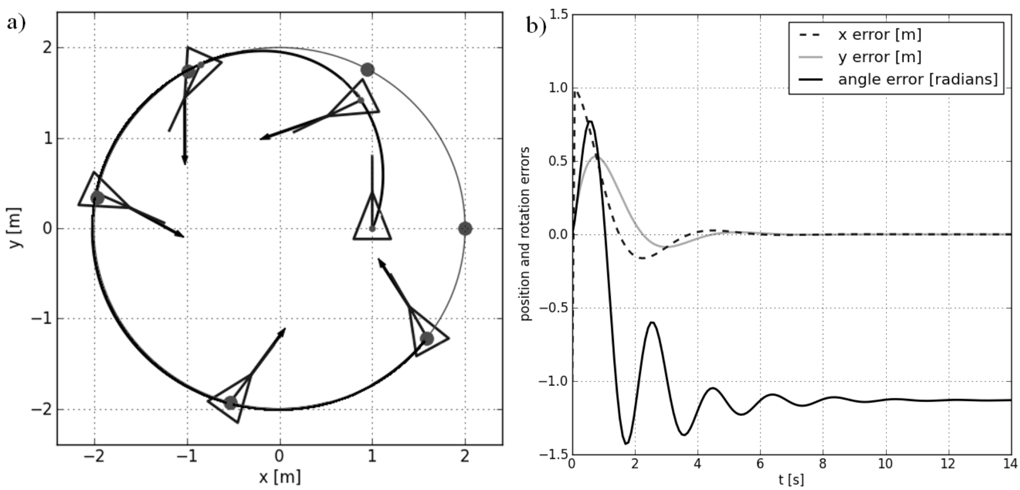

Figure 2. Numerical simulation of circle tracking: a) achieved trajectory of the object with exemplary orientations and force directions, b) evolution of position and rotation errors.

In practice, steering signals are limited – exemplary simulation of the system with a limited maximum force is presented in Fig. 4. Insufficient maximum force value causes a non zero periodic position error. The force angle limit may cause quasi-periodic or chaotic behavior of errors. One can watch simulation effects in video format at Internet [7].

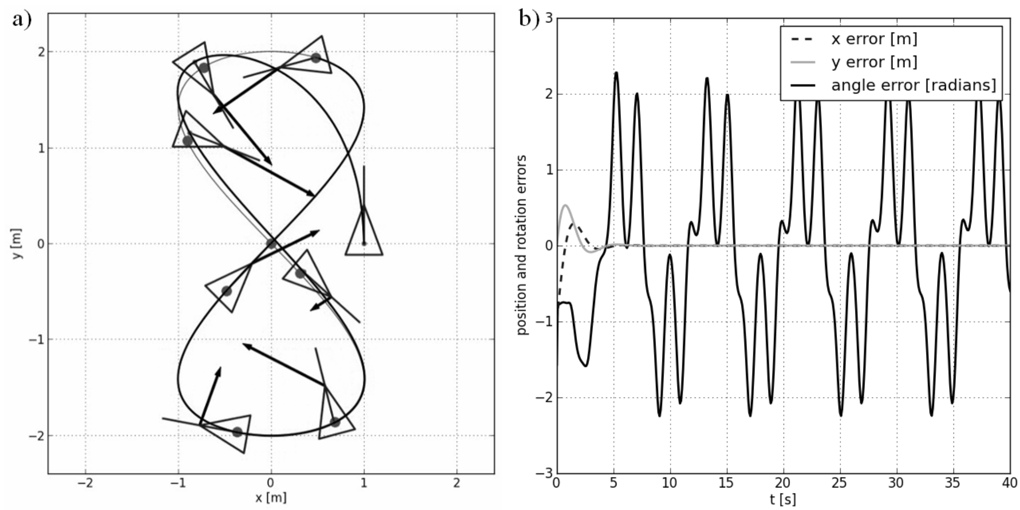

Figure 3. Numerical simulation of eight curve tracking: a) reference trajectory, achieved trajectory and some exemplary orientations of the object, b) evolution of position and rotation errors.

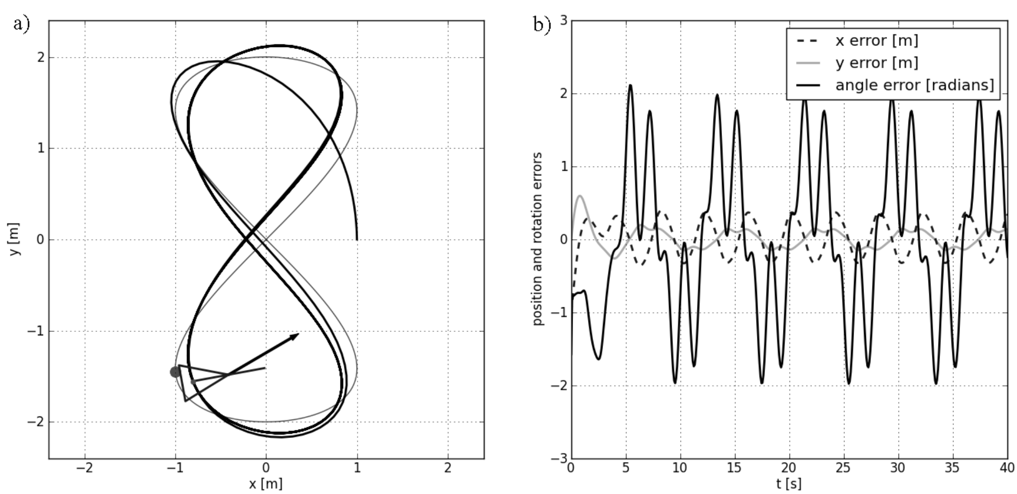

Figure 4. Numerical simulation of eight curve tracking with the maximum force limitation to 4 newtons: a) reference and achieved trajectory, b) evolution of position and rotation errors.

6. 6. Stability analysis

Let us analyze stability of rotational motion of the system when steering without rotation control. For initial conditions corresponding to the desire trajectory $z(0) = z_d(0)$, $\dot{z}(0) = \dot{z}_d(0)$ equation of rotational motion has the form

$$I\ddot{\varphi} + c_{\varphi}\dot{\varphi} = \tau_y a \cos\varphi - \tau_x a \sin\varphi. \tag{32}$$

After substituting control functions (23), desired circular trajectory functions and new error variable $p = \varphi - \theta t - \frac{\pi}{2}$ the equation (32) takes the form

$$\begin{aligned} \dot{p} &= v_p \\ \dot{v}_p &= -\frac{c_\varphi}{I} v_p - \frac{ac\Theta R}{I} \sin p - \frac{am\Theta^2 R}{I} \cos p - \frac{c_\varphi \theta}{I}. \end{aligned} \tag{33}$$

Equilibrium points of the system exists when $(m^2\Theta^2 + c^2)a^2R^2/c_\varphi^2 \geqslant 1$ and are described by $(p_0^{(k)}, v_{p0})$, where $v_{p0} = 0$, $p_0^{(k)} = k\pi - \arctan\frac{m\Theta}{c} - \arcsin\left(c_\varphi/acR\cos k\pi\sqrt{\frac{m^2\Theta^2}{c+1}}\right)$ and $k \in \mathbb{Z}$.

For stability analysis of the first equilibrium $(p_0^{(1)}, v_{p0})$ Lyapunov's linearization method was used. Linear approximation of the system around the first equilibrium has a form

$$\begin{bmatrix} \dot{p} \\ \dot{v}_p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{ac\Theta R}{I}\cos p_0^{(1)} - \frac{am\Theta^2 R}{I}\sin p_0^{(1)} & -\frac{c_\varphi}{I} \end{bmatrix} \begin{bmatrix} p \\ v_p \end{bmatrix} \tag{34}$$

and its characteristic equation

$$\lambda^2 + \frac{c_\varphi}{I}\lambda + \frac{\Theta}{I}\sqrt{(a^2m^2\theta^2 + a^2c^2)R^2 - c_\varphi{}^2} = 0. \tag{35}$$

Linearization of the system is asymptotically stable around the first equilibrium from Routh-Hurwitz criterion. Linearization and characteristic equation of the system around the second equilibrium $(p_0^{(2)}, v_{p0})$ are similar to (34) and (35), however Routh-Hurwitz criterion argues its instability. Other equilibria do not require analysis because of system periodicity. The error of presented object's rotation while tracking a circular curve is therefore globally asymptotically stable for a trajectory satisfying condition

$$R > \frac{c_\varphi}{a\sqrt{m^2\theta^2 + c^2}}. \tag{36}$$

The equation of rotational motion of the system while tracking eight curve using computed torque algorithm with PD feedback has a form

$$I\ddot{\varphi} + c_\varphi\dot{\varphi} = a\Theta R\left((c\cos\Theta t - m\Theta\sin\Theta t)\cos\varphi - (c\cos 2\Theta t - 2m\Theta\sin 2\Theta t)\right)\sin\varphi. \tag{37}$$

After substituting new error variable - angle between the object's center line and tangent to the trajectory $p = \varphi - Arg(\cos 2\Theta t + \sqrt{-1}cos\Theta t)$ the equation cannot be simplified for stability analysis purposes. The error variable approximation $p \approx \varphi - (\pi - 2.26\cos\Theta t - 0.24cos3\Theta t + 0.16cos5\Theta t)$ cause extremely great length of the equation, therefore numerical simulation was used for tracking error analysis. Fig. 5 presents range of the rotation error changing with R and Θ patch parameters, also points of loss of stability are included.

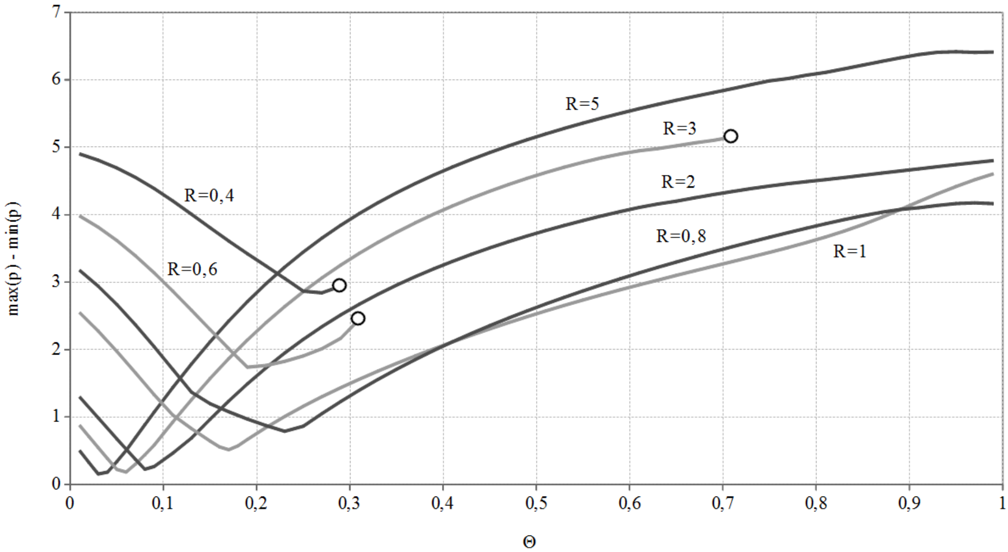

Figure 5. Range of rotation error varying with parameters of the eight curve tracked patch. Empty circles denotes loss of stability.

7. Conclusion

In this paper, a problem of control of an underactuated system was shown. Based on an exemplary planar rigid body subjected to an eccentric external force, the accessibility and controllability was presented. Usage of the computed torque algorithm for the tracking problem was proposed. Two trajectories, circular and eight-shaped, were used for examples of numerical simulation. Because of the coupled input force components, we cannot take control on all state variables. Therefore after proposition of control of the system position without rotation control, a computed torque technique with proportional-derivative feedback was investigated. This new method reduce the complexity of method in comparison with full state control algorithms. Stability analysis using the first Lyapunov method allows to obtain a stability criterion for circular motion. Eight curve path tracking stability could be checked only by numerical simulation.

Future research will be aimed to control of an underactuated vehicle with caster wheels and vision based feedback signals.

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