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Model of time-varying linear systems and Kolmogorov equations

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In the paper an approximate model of time-varying linear systems using a sequence of time-invariant systems is suggested. The conditions for validity of the approximation are proven with a theorem. Examples comparing the numerical solution of the original system and the analytical solution of the model are given. For the system under the consideration a new criterion giving sufficient conditions for robust Lagrange stability is suggested. The criterion is proven with a theorem. Examples are given showing stable and non stable solutions of a time-varying system and the results are compared with the numerical Runge-Kutta solution of the system. In the paper an important application of the described method of solution of linear systems with time-varying coefficients, namely analytical solution of the Kolmogorov equations is shown.

Key words: time-varying, linear system, stability, dynamical systems, Kolmogorov, model.

1. Model of time-varying linear systems

1.1. Introduction

It is known that the analysis, synthesis and implementation of the dynamical and control systems, described with linear differential equations with time-dependent parameters are much more difficult then the application of time-invariant systems. In some cases, the conditions of which are described and proven with a theorem in the paper, the time-varying system can be replaced by approximate model containing a sequence of time invariant linear systems, derived from the original system using the methods of the functional analysis. In the present paper a new criterion for Lagrange stability is suggested, which uses only $g(t-\tau)$, which is the response of one-dimensional time-invariant system to the Dirac function $\delta(t)$ [1], or the Green matrix $\boldsymbol{G}(t-\tau)$ of a time-invariant system, approximating the original system. The difference between the original and the approximating system is considered as perturbation.

An important application of the described above method is the analytical solution of the Kolmogorov equations, shown in the paper.

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1.2. Analytical model of time-varying linear systems

In the general case the linear time-varying control systems can be described in a matrix form

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{A}(t,P_1)\boldsymbol{X} + \boldsymbol{B}(t,P_2)\boldsymbol{U}(t), \tag{1}$$

where P_1 and P_2 are sets of parameters. If P_3 is the set of initial values of the variables x_i which forms the matrix X, then the set of all parameters P can be defined as Decart product of P_1, P_2, P_3 : $P = P_1 \times P_2 \times P_3$. $\boldsymbol{U}(t)$ is the matrix of the control inputs, $\boldsymbol{B}(t,P_2)$ is the matrix of coefficients. When the system (1) is one-dimensional of the order n and $\boldsymbol{U}(t) = u(t)$ in the general case it can always be transformed into system (1) [1]., thus in the sequel only system (1) will be considered.

The matrix $\boldsymbol{A}(t,P_1)$ can always be represented in the following way

$$\begin{aligned} \boldsymbol{A}(t,P_1) &= \boldsymbol{A}_C(P_1) + \boldsymbol{A}^*(t,P_1) \\ a_{ij}(t,P_1) &= a_{i\,jc}(P_1) + a^*_{ij}(t,P_1) \end{aligned} \tag{2}$$

where the matrix $\boldsymbol{A}_C(P_1)$ is time-invariant and is defined in a such way that the norm of the difference between the original and the approximating systems

$$||\boldsymbol{A}^*(t,P_1)|| = ||\boldsymbol{A}(t,P_1) - \boldsymbol{A}_C(P_1)|| \tag{3}$$

is minimized, which will improve the convergence of the solution, as will be shown. An easy (but not always optimal) way of calculating $\boldsymbol{A}_C(P_1)$ is by finding the mean values of $a_{ij}(t,P_1)$ for the necessary time interval and considering $a_{ijc}(P_1)$ equal to them. Taking into account (3) the equation (1) can be transformed:

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{A}_C(P_1)\boldsymbol{X} + \boldsymbol{F}(t,P_2) + \mu\boldsymbol{A}^*(t,P_1)\boldsymbol{X} \tag{4}$$

where $\boldsymbol{F}(t,P_2) = \boldsymbol{B}(t,P_2)\boldsymbol{U}(t)$ and the time-varying part is multiplied by the 'big' parameter μ [2]. Obviously, when $\mu = 1$ equation (4) is identical to (1). The time-varying part $\mu\boldsymbol{A}^*(t,P_1)\boldsymbol{X}$ can be considered as perturbation. The sets P, P_1, $|P_2$, P_3 are taken into account only when it is necessary – usually for the estimation of the norms. For the application of the functional analysis [3], [4], [7] it is necessary to represent (4) in operator form

$$Q(\boldsymbol{X},\mu) = 0, \quad Q(\boldsymbol{X}_0,\mu_0) = 0, \tag{5}$$

transferring $d\boldsymbol{X}/dt$ to the right part of (4). It is convenient to put Č$\mu = \mu_0 \sim= 0$ and in this case $\boldsymbol{X}_0$ can be found as solution of

$$\frac{d\boldsymbol{X}_0}{dt} = \boldsymbol{A}_C(P_1)\boldsymbol{X}_0 + \boldsymbol{F}(t,P_2) \tag{6}$$

using the Green matrix $\boldsymbol{G}(t-\tau)$ [5] for the time invariant part of (4)

$$\boldsymbol{X}_0(t) = \boldsymbol{G}(t)\boldsymbol{X}(0) + \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{F}(\tau)d\tau \tag{7}$$

with initial conditions: $\boldsymbol{X}(0) = \boldsymbol{X}_0(0)$. The operators C_{11}, C_{01}, C_{01}^{-1}, which will be used further, can be found as continuous Gateaux derivatives of $Q(\boldsymbol{X},\mu)$ [4], [6], [7]:

$$C_{pq} = \frac{1}{p!q!} Q^{p+q}_{\mu^p X^q}(\boldsymbol{X}_0,\mu_0)$$

$$C_{01} = \frac{\partial}{\partial \boldsymbol{X}} Q(\boldsymbol{X}_0,\mu_0) = \frac{\partial}{\partial c} Q(\boldsymbol{X}_0 + c\boldsymbol{H},\mu_0)_{c=0} = -\frac{d\boldsymbol{H}}{dt} + \boldsymbol{A}_C\boldsymbol{H} \tag{8a}$$

$$C_{11}(\boldsymbol{H}) = \frac{\partial^2}{\partial\mu\partial c} Q(\boldsymbol{X}_0 + c\boldsymbol{H},\mu_0)_{c=0} = \boldsymbol{A}^*(t)\boldsymbol{H} \tag{8b}$$

$$\Gamma_{01} = C_{01}^{-1} = \left[\frac{\partial}{\partial c} Q(\boldsymbol{X}_0 + c\boldsymbol{H},\mu_0)_{c=0}\right]^{-1} \tag{8c}$$

The operator C_{01}^{-1} has the following integral form

$$H = C_{01}^{-1}(y) = -\int_0^t \boldsymbol{G}(t-\tau)y(\tau)d\tau. \tag{9}$$

For the assessment of the norms in the space $C[a,b]$ the following relations and inequalities are useful [6]:

$$||C_{11}|| = \sup_{||\boldsymbol{H}||=1} ||C_{11}(\boldsymbol{H})|| = \sup_{||\boldsymbol{H}||=1} ||\boldsymbol{A}^*(t)\boldsymbol{H}|| \leqslant ||\boldsymbol{A}^*(t)|| \tag{10}$$

$$||C_{01}| \leqslant \sup_{||y||=1} \left\{ ||y|| \left\| \int_0^t |\boldsymbol{G}(t-\tau)|d\tau \right\| \right\} = \left\| \int_0^t |\boldsymbol{G}(t-\tau)|d\tau \right\|. \tag{11}$$

The concrete norm (11) can be chosen between those, given in [5] and taking into account, that after integrating the matrix it consist of elements $\int_0^t |g_{ik}(t-\tau)|d\tau$.

1.3. Approximate model of the time-varying system

Theorem 7 *The sufficient conditions for the approximation of the solution $\boldsymbol{X}(t)$ of the original time-varying system (1) with the first m elements of the series*

$$\boldsymbol{X}(t) = \boldsymbol{X}_0(t) + \boldsymbol{X}_1(t) + \boldsymbol{X}_2(t) + \dots + \boldsymbol{X}_m(t) + \boldsymbol{X}_{m+1}(t) + \dots, \tag{12}$$

namely:

$$\boldsymbol{X}(t) = \boldsymbol{X}_0(t) + \int_0^t \boldsymbol{G}(t-\tau_1)\boldsymbol{A}^*(\tau-1)\boldsymbol{X}_0(\tau_1)d\tau_1 +$$

$$\int_0^t \boldsymbol{G}(t-\tau_2)\boldsymbol{A}^*(\tau_2)\int_0^{\tau_2} \boldsymbol{G}(\tau_2-\tau_1)\boldsymbol{A}^*(\tau_1)\boldsymbol{X}_0(\tau_1)d\tau_1 d\tau_2 + \dots \tag{13}$$

$$\int_0^t \boldsymbol{G}(t-\tau_m)\boldsymbol{A}^*(\tau_m)\int_0^{\tau_m} \boldsymbol{G}(\tau_m-\tau_{m-1})\ldots d\tau_1\ldots d\tau_m$$

with error less than a priori given value δ *are:*

$$||P_{11}|| \leqslant ||C_{01}^{-1}||\cdot||C_{11}|| < 1, \tag{14}$$

$$\frac{||\boldsymbol{X}_0||\,(||P_{11}||)^{m+1}}{1-||P_{11}||} < \delta, \tag{15}$$

$$P_{11} = C_{01}^{-1}(C_{11}). \tag{16}$$

Proof If the operator $Q(\boldsymbol{X},\mu)$ allows the finding of operators C_{10}, C_{11}, C_{01} as Gateaux derivatives (8) the following presentation is possible

$$Q(\boldsymbol{X},\mu) = Q(\boldsymbol{X}_0,\mu_0)+(\mu-\mu_0)C_{10}+(\mu-\mu_0)C_{11}(\boldsymbol{X}-\boldsymbol{X}_0)+C_{01}(\boldsymbol{X}-\boldsymbol{X}_0). \tag{17}$$

If the infinite series of the solution

$$\boldsymbol{X} = \boldsymbol{X}_0+(\mu-\mu_0)\boldsymbol{X}_1+(\mu-\mu_0)^2\boldsymbol{X}_2+\ldots \tag{18}$$

is replaced in (17), the parts containing $(\mu-\mu_0)$, $(\mu-\mu_0)^2\ldots$, $(\mu-\mu_0)^m$, form the following equations

$$\boldsymbol{X}_1 = -C_{01}^{-1}[C_{10}] \quad C_{10} = \boldsymbol{A}^*(t)\boldsymbol{X}_0, \tag{19a}$$

hence $C_{10} = C_{11}(\boldsymbol{X}_0)$, taking into account (8b)

$$\boldsymbol{X}_m = -C_{01}^{-1}[C_{11}(\boldsymbol{X}_{m-1})], \quad m = 1,2,3,\ldots, \tag{19b}$$

which corresponds to the integral presentation (13) because the inverse operator C_{01}^{-1} has the form (9).

From (19b) it follows that

$$||\boldsymbol{X}_m|| \leqslant ||P_{11}||\cdot||\boldsymbol{X}_{m-1}|| \tag{20}$$

The convergence of the series (12) for Č$\mu_0 \sim= 0$, Č$\mu = 1$ will exist if the norm of its sum

$$||\boldsymbol{X}-\boldsymbol{X}_0|| \leqslant ||\boldsymbol{X}_1||+||\boldsymbol{X}_2||+\cdots+||\boldsymbol{X}_m||+\cdots \leqslant \frac{||P_{11}||\cdot||\boldsymbol{X}_0||}{1-||P_{11}||} \tag{21}$$

is finite, the sufficient condition of which corresponds to (14). Of course, the norm of the first element $\boldsymbol{X}_0$ should be also finite, if the matrix $\boldsymbol{A}_C$ is defined correctly and the process in its nature is not unstable. The majorant estimation of the norm of the error $\boldsymbol{X}_{err}$ if only the first m elements of the series (12) are taken into account is

$$||\boldsymbol{X}_{err}|| \leqslant ||\boldsymbol{X}_{m+1}||+||\boldsymbol{X}_{m+2}+\cdots \leqslant \frac{\boldsymbol{X}_0\,(P_{11})^{m+1}}{1-||P_{11}||} \tag{22}$$

which confirms the sufficient condition (14). Obviously if the sufficient condition (14) is proven and $\boldsymbol{X}_0$ is finite there always exist m for which the condition (15) will be satisfied. So the theorem is proven. □

The solution (7), (12), (13) of the original time-varying system (1) is illustrated in Fig.1 for the case of 3 elements $-\boldsymbol{X}(t) \approx \boldsymbol{X}_0(t) + \boldsymbol{X}_1(t) + \boldsymbol{X}_2(t)$. There L.S. is the time invariant part of (4) with Green matrix $\boldsymbol{G}(t-\tau)$.

An alternative to the described in the paper approximation method are the methods of order reduction of large linear systems [9] or application [10] of Grobner basis algorithm, but there are no explicit rules for their application, which is achieved in the described here method, according to Fig.1. On the other hand the direct application of numerical method [11]-[13] to the system (1) does not give an analytical solution and therefore is not appropriate for synthesis of linear systems or algorithms for fast real time control or optimization.

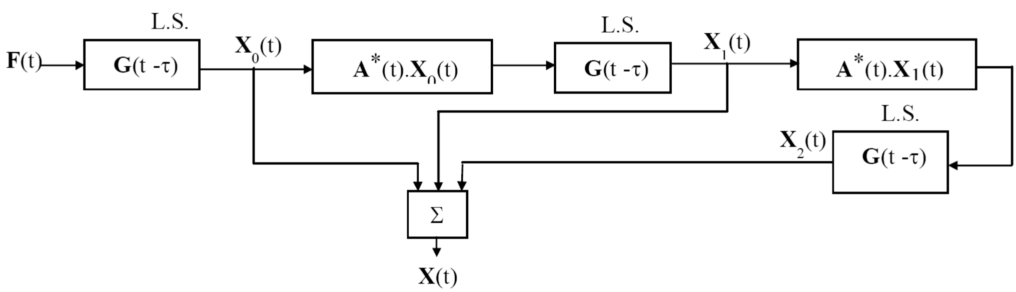

Figure 1: Approximate solution $X(t) \approx X_0(t) + X_1(t) + X_2(t)$ of the time-varying system (1).

Example 3 *In the following example of one-dimensional linear system* (23) *the conditions for convergence of the approximate model* (12), (13) *will be investigated analytically and numerically:*

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x - \mu A^*(t)x = 0.76. \tag{23}$$

The solution can be found without transforming (23) *into a system of two differential equations of first order, which is trivial. Using simple analogy operators* C_{10}, C_{01}, C_{11} *for* $\mu_0 = 0$, $\mu = 1$ *can be obtained from* (8), (8a), (8b), (8c) *as derivatives of equation* (23), *taking into account that* C_{10}, C_{11} *form external signal to the system described by operator* C_{01}

$$C_{10} = -A^*(t)x_0, \quad C_{11}(H) = -A^*(t)H, \tag{24}$$

$$C_{01}(H) = \frac{d^2H}{dt} + 3\frac{dH}{dt} + 2H. \tag{25}$$

The Green function, corresponding to the operator C_{01} is

$$g(t-\tau)=e^{-(t-\tau)}-e^{-2(t-\tau)} \tag{26}$$

and can be found as Laplace original of the transfer function of the time-invariant part of (23)*. For zero initial condition x_0 can be found from* (23) *and $\mu=0$*

$$x_0(t)=\int_0^t[0.76g(t-\tau)]d\tau=0.38(e^{-2t}-2e^{-t}+1). \tag{27}$$

For the Banach space $C[a,b]$ the norms of the operators C_{01} and C_{11} are:

$$||C_{01}^{-1}||\leqslant\left|\left|\int_0^t|g(t-\tau)|d\tau\right|\right|= \tag{28}$$

$$=\max_{0\leqslant t\leqslant T}|\frac{e^{-2t}-2e^{-t}+1}{2}<0.5, \tag{29}$$

where equality to 0.5 corresponds to $T\to\infty$. If it is considered that $A^*(t)=e^{-t}$, then

$$||C_{11}||\leqslant||A^*(t)||=\max_{0\leqslant t\leqslant T}\left|e^{-t}\right|\leqslant 1. \tag{30}$$

So $P_{11}<0.5$ and (14) *is satisfied. The element x_1 is found analytically, according to the second element of* (13)

$$x_1(t)=0.38e^{-t}\left[t(2e^{-t}+1)+2e^{-t}+0.5e^{-2t}-2.5\right]. \tag{31}$$

The element $x_2(t)$ is found solving numerically the 3-th element of (13)*, which can be written more simply*

$$x_2(t)=\int_0^t[g(t-\tau)A^*(\tau_2)x_1(\tau 2)]\,d\tau_2. \tag{32}$$

In Tab. 1 *the numerical solution of* (23) *is compared with the approximation $x=x_0+x_1+x_2$. The approximation is excellent for the whole interval $t\in[0,\infty]$, because for $t\to\infty$, $x\to 0.38$.*

1.4. Criterion for Lagrange stability

The Lagrange stability means that the solution of the system estimated by its norm is bounded [8], i.e. limited for the time interval $t\in[0,\infty)$. The sufficient conditions for Lagrange stability of the original time-varying system (1) are described and proven in the following theorem.

Table 4: Comparison of analytical solution and Runge-Kutta solution

t, s	0.25	1	1.5	2	3	5	10
$x_0+x_1+x_2$	0.01867	0.1574	0.2417	0.3012	0.3609	0.3818	0.38009
Runge-Kutta	0.01867	0.1574	0.2416	0.3013	0.3609	0.3818	0.3801

Theorem 8 *The sufficient conditions for Lagrange stability of a time-varying linear system* (1) *with a norm of the bounded solution*

$$||\boldsymbol{X}|| \leqslant \frac{||\boldsymbol{X}_0||}{1-P_{11}} \tag{33}$$

are:

a) *There exist a time-invariant system* (6)*, approximating the original time-varying system* (1)*, the norm of the solution of which* $\boldsymbol{X}_0$ *is bounded, i.e. Lagrange stable.*

b) *There exists the inequality*

$$||P_{11}|| \leqslant ||C_{01}^{-1}|| \cdot ||C_{11}|| < 1. \tag{34}$$

Proof If the operator $Q(\boldsymbol{X},\mu)$ allows the finding of operators C_{10}, C_{11}, C_{01} as Gateaux derivatives (8) the presentation (17) is possible and if the assumed infinite series of the solution (18) is replaced in (17), the parts containing $(\mu-\mu_0)$, $(\mu-\mu_0)^2,\dots,$ $(\mu-\mu_0)^m$ form the equations (19a), (19b). Hence (20) is valid. The convergence of the series (18) for $\mu_0 = 0$, $\mu = 1$ will exist if the norm of its sum $\boldsymbol{X}$ is finite. From (21) follows:

$$||\boldsymbol{X}|| - ||\boldsymbol{X}_0|| \leqslant ||\boldsymbol{X}-\boldsymbol{X}_0|| \leqslant ||\boldsymbol{X}_1|| + ||\boldsymbol{X}_2|| + \dots$$

$$\dots + ||\boldsymbol{X}_m|| + \dots \leqslant \frac{||P_{11}|| \cdot ||\boldsymbol{X}_0||}{1-||P_{11}||}.$$

Hence

$$||\boldsymbol{X}|| \leqslant ||\boldsymbol{X}_0|| + \frac{||P_{11}|| \cdot ||\boldsymbol{X}_0||}{1-||P_{11}||} = \frac{||\boldsymbol{X}_0||}{1-||P_{11}||}. \tag{35}$$

Obviously, when the norm of the solution of the approximating system $||\boldsymbol{X}_0||$ is bounded and condition (34) is satisfied then the norm of the solution $||\boldsymbol{X}||$ of the original system is also bounded, which proves the theorem. □

The operator formulas (33)-(35) are valid and can be used not only for $t \in [0,\infty)$ but also for $t \in [0,T]$, where $T \neq \infty$. For the technical systems [14], [15] it is very interesting and important the solution to be limited by its norm for a limited time interval $t \in [0,T]$. Such property can be named limited Lagrange stability for a given time.

Corollary 1 *The sufficient conditions for limited Lagrange stability in the space* $C[a,b]$ *for the time interval* $t \in [0,T]$ *are:*

$$\text{a)}\quad ||\boldsymbol{X}_0|| \ \textit{ is limited}, \tag{36a}$$

$$\text{b)}\quad T < \frac{1}{||\boldsymbol{G}(t)|| \cdot ||\boldsymbol{A}^*(t)||}. \tag{36b}$$

Proof The norm (11) can be written as follows:

$$\left|\left| \int_0^t |\boldsymbol{G}(t-\tau)|d\tau \right|\right| = \max_{0 \leqslant t \leqslant T} \left| \int_0^t |\boldsymbol{G}(t-\tau)|d\tau \right| \leqslant \int_0^T \max_{0 \leqslant t \leqslant T} |\boldsymbol{G}(t)|dt \leqslant T||\boldsymbol{G}(t)||. \tag{37}$$

Hence the corollary is proven. □

According to the terminology in [8] with (36b) is assessed the 'escape time' of the solution.

Example 4 *Let investigate the stability of one-dimensional linear system* (23) *of second order. For the calculation of stability two variants are considered:*

Variant 1 *Here* $A^*(t) = e^{-t}$

Taking into account (34), (29), (30) *for* $t \in [0,\infty)$, $||P_{11}|| \leqslant 0.5$, *the condition for Lagrange stability* (34) *is satisfied.*

Variant 2 *Here* $A^*(t) = 0.4t$ *and* $||P_{11}|| \leqslant 1$ *only for the time interval* $t \in [0,5]$ *and the sufficient conditions for limited solution, i.e. limited Lagrange stability exist only for this interval. However analyzing* (23) *it is not difficult to conclude that for* $t \to infty$, $x \to infty$, *but* $||\boldsymbol{X}||$ *is limited for greater time interval than* $t \in [0,5]$, *as the numerical Runge-Kutta solution shows in Tab. 2. The principle reason for such a difference is that the applied criterion gives only sufficient conditions for stability mainly because of the majorant estimation of the norms.*

If (33) *is applied for the time interval* $t \in [0,3]$ *then the norm of the solution can be estimated:*

$$||P_{11}|| \leqslant 0.5 \cdot 1.2 = 0.6,$$

$$||x_0|| \leqslant 0.38,$$

$$||x|| \leqslant \frac{0.38}{1-0.6} = 0.95.$$

This result is correct but obviously it is majorant, compared with the numerical solution in Tab. 2, which can be expected. The corollary can also be applied calculating the norm of $g(t) = e^{-t} - e^{-2t}$ *and replacing it in* (36b). *The result is* $T < 3.16$. *The described in this section criterion concerns the stability of the linear dynamical system. On the other hand exist the problem of stability of numerical methods* [16], *which does not exist for the methods in this paper.*

Table 5: Example of unstable process.

t, s	0.25	1	3	5	7	10	12.5
Runge-Kutta	0.01867	0.154	0.445	0.776	1.49	6.05	32.64

2. Analytical method for solving Kolmogorov equations

In this subchapter will be shown an important application of the described in the previous subchapter method of solution of linear systems with time-varying coefficients, namely analytical solution of the equations of Kolmogorov [17].

Finding the analytical solution for such a system, in general, is extremely difficult. Of course it is not a problem the finding of the numerical solution of these equations. Analytical solution, however, has a number of advantages especially in the case when is sought optimization of semi-Markovian processes, because in this case optimization of a function, not optimization of the slow numerical solution of the system of differential equations will be sought.

In the general case the Kolmogorov equations are [18]:

$$\begin{aligned}
\frac{dP_1(t)}{dt} &= \sum_j \lambda_{1j} P_1(t) + \lambda_{21} P_2(t) + \cdots + \lambda_{n1} P_n(t) \\
&\vdots \\
\frac{dP_{n-1}(t)}{dt} &= \lambda_{1,n-1} P_1(t) + \lambda_{2,n-1} P_2(t) + \cdots - \sum_j \lambda_{n-1,j} P_{n-1}(t) + \lambda_{n,n-1} P_n(t) \\
\frac{dP_n(t)}{dt} &= \lambda_{1,n} P_1(t) + \lambda_{2,n} P_2(t) + \cdots - \sum_j \lambda_{n,j} P_n(t)
\end{aligned} \tag{38}$$

where $P_i(t)$ is the probability that the system at time t is in state S_i and the sum of the probabilities of the system to be in all possible states for each moment of time (including the initial time $t = 0$) is 1.

The intensities of the transitions from state i to state j, which for the semi-Markovian processes are functions of the time are λ_{ij}. For the system (38) is known that [18] the right side of the (38) is linear dependent, but the unknown variables should be subject to the additional condition

$$\sum_{j=1}^{n} P_i(t) = P_1(t) + P_2(t) + \cdots + P_n(t) = 1. \tag{39}$$

For this purpose one of the unknown $P+1(t),\dots,P_n(t)$ should be eliminated (for example $P_n(t)$) and as result (38) is reduced to $n-1$ order:

$$\begin{aligned}
&\frac{dP_1(t)}{dt} = -\left[\sum_j \lambda_{1j} + \lambda_{n1}\right] P_1(t) + [\lambda_{21} - \lambda_{n1}] P_2(t) + \\
&\cdots + [\lambda_{n-1,1} - \lambda_{n1}] P_n(t)\lambda_{n1} \\
&\vdots \\
&\frac{dP_{n-1}(t)}{dt} = [\lambda_{1,n-1} - \lambda_{n,n-1}] P_1(t) + [\lambda_{2,n-1} - \lambda_{n,n-1}] P_2(t) + \\
&\cdots - \left[\sum_j \lambda_{n-1,j} + \lambda_{n,n-1}\right] P_{n-1}(t) + \lambda_{n,n-1}.
\end{aligned} \tag{40}$$

In matrix form system (40) is

$$\frac{dP(t)}{dt} = \boldsymbol{A}(t)\boldsymbol{P}(t) + \boldsymbol{F}(t). \tag{41}$$

The particular type of the matrix elements $a_{ij}(t)$ can be determined relatively simply by comparing (40) and (43). The resulting system (41), (42), (43) is a linear system with variable parameters in time, finding the analytical solution of which is very difficult, even for systems of low order.

$$\boldsymbol{P}(t) = \begin{vmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_{n-1}(t) \end{vmatrix}, \boldsymbol{F} = \begin{vmatrix} \lambda_{n1}(t) \\ \lambda_{n2}(t) \\ \vdots \\ \lambda_{nn-1}(t) \end{vmatrix}, \tag{42}$$

$$\boldsymbol{A}(t) = \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1,n-1}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2,n-1}(t) \\ \vdots & \vdots & \dots & \vdots \\ a_{n-1,1}(t) & a_{n-1,2}(t) & \dots & a_{n-1,n-1}(t) \end{vmatrix}. \tag{43}$$

Without loss of generality $\boldsymbol{A}(t)$ can be represented as a sum of 2 parts, analogically to (2):

$$\boldsymbol{A}(t) = \boldsymbol{A}_C + \boldsymbol{A}^*(t), \tag{44}$$

$$a_{ij}(t) = a_{ijc}(t) + a^*_{ij}(t), \tag{45}$$

where $\boldsymbol{A}_C$ is time constant matrix, the elements of which is desirable to be equal to the average of the elements for the considered time interval. Deviation from this average values would worsen the convergence of the solution, without leading to an incorrect result. Under these assumptions (41) will be written

$$\frac{d\boldsymbol{P}(t)}{dt} = \boldsymbol{A}_C\boldsymbol{P}(t) + \boldsymbol{F}(t) + \mu\boldsymbol{A}(t)\boldsymbol{P}(t), \tag{46}$$

where the 'inconvenient' non-stationary element is multiplied in (46) by a numerical parameter μ. This will eventually lead to no change in the initial system of differential equations, since the solution of (46) will be sought in a series

$$\boldsymbol{P}(t) = \boldsymbol{X}_0 + (\mu-\mu_0)\boldsymbol{X}_1 + (\mu-\mu_0)^2\boldsymbol{X}_2 + \cdots + (\mu-\mu_0)^m\boldsymbol{X}_m \tag{47}$$

where Č$\mu_0 = 0$, Č$\mu = 1$. Obviously, in this conditions (46) is identical to (41). The first element of the series of the solution (47) can be determined, using the impulse response matrix of Green for the time-invariant part of (46):

$$\boldsymbol{X}_0(t) = \boldsymbol{G}(t)\boldsymbol{P}(0) + \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{F}(\tau)d\tau \tag{48}$$

where

$$\boldsymbol{P}(0) = \boldsymbol{X}_0(t=0), \quad \boldsymbol{X}_1(t=0) = 0, \ldots, \boldsymbol{X}_m(t=0) = 0. \tag{49}$$

It is known that the Green function of one-dimensional systems can be obtained as original of the transfer function in Laplace presentation. When the system is multidimensional and of high order it will be needed to find numerically the eigenvalues of the matrix, to obtain analytical form of $\boldsymbol{G}(t-\tau)$. For this purpose can be used the formula of Sylvester [5] or another method.

The other elements of the solution (47) are found from the operator equations, according to (19a), (19b):

$$C_{10} = \boldsymbol{A}^*(t)\boldsymbol{X}_0 \tag{50}$$

$$-C_{01}(\boldsymbol{X}_1) = C_{10} \tag{51}$$

$$-C_{01}(\boldsymbol{X}_2) = C_{11}(\boldsymbol{X}_1) \tag{52}$$

$$\vdots$$

$$-C_{01}(\boldsymbol{X}m) = C_{11}(\boldsymbol{X}_{m-1}) \tag{53}$$

Taking into account that. (8a), (8b), (8c), (9), the operator equations (51), (52), (53) can be represented in differential form:

$$\frac{d\boldsymbol{X}_1}{dt} - \boldsymbol{A}_C\boldsymbol{X}_1 = A*(t).X0(t) = \boldsymbol{A}^*(t)\boldsymbol{X}_0(t), \tag{54}$$

$$\frac{d\boldsymbol{X}_2}{dt} - \boldsymbol{A}_C\boldsymbol{X}_2 = A*(t).X0(t) = \boldsymbol{A}^*(t)\boldsymbol{X}_1(t). \tag{55}$$

By analogy can be found elements $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m$ in integral form, applying the general formulas (13), given in the previous subchapter. The result is:

$$\begin{aligned}
\boldsymbol{X}_1(t) &= \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{A}^*(\tau)\boldsymbol{X}_0(\tau)d\tau, \\
\boldsymbol{X}_2(t) &= \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{A}^*(\tau)\boldsymbol{X}_1(\tau)d\tau, \\
&\vdots \\
\boldsymbol{X}_m(t) &= \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{A}^*(\tau)\boldsymbol{X}_{m-1}(\tau)d\tau.
\end{aligned} \tag{56}$$

Hence the solution (47) is

$$\boldsymbol{P}(t) = \boldsymbol{X}_0(t) + \boldsymbol{X}_1(t) + \boldsymbol{X}_2(t) + \cdots + \boldsymbol{X}_m(t), \tag{57}$$

the elements of which can be calculated using formulas (48), (54), (55) or (48), (56).

The convergence of the solution and the error $\boldsymbol{X}_{err}$ if only the first m elements of the series (57) are taken into account are investigated in the previous section.

3. Conclusion

- The approximation method shows excellent accuracy, gives analytical solution and therefore can be used for synthesis of control circuits, high speed computer control and computer optimization of dynamical systems.
- The criterion for Lagrange stability can be applied relatively easily to all time-varying systems without theoretical obstacles. It can give assessment of robust Lagrange stability for a given subset of parameters.
- The shown method allows the analytical solution of a linear nonstationary system with time-variable parameters be reduced to solving a linear system with constant parameters, which task is far more simple and has long been known methods for solving (e.g. Green matrix).
- The method can be used in semi-Markovian random processes described by non-stationary system of differential equations of Kolmogorov.
- In practice finite number of the elements of the solution (19) are taken into account, but there are methods for estimation of error and convergence, so that accuracy requirements are met.

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