Rational polynomial windows
with high attenuation of sidelobes

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Abstract: In the paper the idea of rational polynomial windows optimised towards low level of Fourier spectrum's sidelobes is presented. A relevant advantage of the polynomial windows family and their modifications is their ability to easily change their properties changing only the values of the polynomial coefficients. The obtained frequency characteristics demonstrate better properties of proposed rational windows than their standard polynomial equivalents requiring only the additional division operation. Such approach does not increase the computational complexity in significant way and the great advantage of polynomial windows which is their low computational complexity is preserved.

Key words: polynomial windows; signal processing; spectrum analysis

1. Introduction

The relevance of time windows in signal processing applications is mainly related to their usage for the reduction of the spectral leakage and Gibbs phenomenon as well as their application in short-time Fourier transform (STFT). The well-known spectral leakage problem is caused by the finite duration of the collected data sequence. In order to reduce this effect the window with low level of their Fourier spectrum's sidelobes should be used to prevent the rapid cutting of the data series.

The most typical window functions used for such purposes are based on cosine series, such as Hann, Hamming or Blackman windows and their modifications [1, 2]. Apart from their properties, their calculation is usually based on the expansion of the cosine function into series of preferably at least 20-th order. A similar problem takes place for Kaiser window based on the modified Bessel function, which should also be expanded into high order series.

One of the most efficient methods of the reduction of the computational complexity of windowing operations is the application of the polynomial windows, which can be computed using Horner's scheme (only 2·N operations required for the N-th order window) and the proper choice of their polynomial coefficients allows similar or even better properties than

traditionally used windowing functions. Probably the most interesting type of the polynomial window is their family with only even exponents which can be optimized towards fast decaying sidelobes as presented in the paper [3] or minimizing the highest sidelobe level (HSLL) as in the paper [4]. The cost of the high decay speed for the first type of windows is the high level of the first sidelobes, so the decrease of the sidelobes' level forces lower decay ratio, which means resigning from forcing the window's value and its derivatives to zero at the end points ($t = \pm T/2$).

An extension of the polynomial windows is the family of rational polynomial windows where two polynomials are divided. Since both polynomials can be computed independently, their coefficients may differ leading to windows with low computational complexity and even better properties than optimized standard polynomial windows in terms of sidelobes' attenuation. In order to provide the window's value equal to 1 in its central point the first coefficients (constant elements) of both polynomials should be identical (they are set to 1 for simplification).

2. Definitions of the polynomial and rational polynomial windows

Standard symmetrical N^{th} order polynomial window with only even exponents is defined as:

$$w(t) = 1 + \sum_{n=1}^{N/2} a_{2n} \left(\frac{t}{T}\right)^{2n},$$

(1)

assuming $t \in \, < -T/2; \; T/2 >$ and its value for the central point $w(t = 0) = 1$.

Such definition can be extended into rational polynomial windows as:

$$w(t) = \frac{1 + \displaystyle\sum_{n=1}^{N/2} a_{2n} \left(\frac{t}{T}\right)^{2n}}{1 + \displaystyle\sum_{m=1}^{M/2} b_{2m} \left(\frac{t}{T}\right)^{2m}},$$

(2)

where two N-th and M-th order polynomials are used as the numerator and denominator in the window function respectively.

Analysing the properties of the polynomial windows the following parameters of their frequency characteristics should be considered:
- Width of the Main Lobe (WML),
- Highest Sidelobes Level (HSLL),
- asymptotic decay ratio (typically expressed in dB/octave).

Since the decay ratio of the polynomial windows depends directly on the number of derivatives set to 0 for $|t| = T/2$ (at both ending points of the window), it can be easily controlled

using the system of equations where the number of fulfilled conditions forces the desired decay ratio. Setting $w(\pm T/2) = 0$ leads to asymptotic sidelobes' decay of 12 dB/octave and each derivative forced to 0 increases the decay ratio by additional 6 dB/octave. If the number of equations in such system is equal to the number of polynomial coefficients, their optimization is impossible. For example for the 2nd order polynomial with $w(\pm 0.5) = 0$ we obtain $a_2 = 4$ (window denoted as $w_{0,2}$ where $w_{k+1,N}$ stands for the N-th order window with k derivatives set to 0 and $k = 1$ denotes only the window's value forced to 0 at $t = \pm T/2$).

Analysing the 4th order polynomial three types of windows can be obtained marked as $w_{0,4}$, $w_{1,4}$ and $w_{2,4}$ with decay ratio 6, 12 and 18 dB/octave respectively with possible optimization of the HSLL only for the $w_{0,4}$ and $w_{1,4}$ windows, since the coefficients a_2 and a_4 for the $w_{2,4}$ window are determined from the system of two equations:

$$\begin{cases} 1 + 0.25 \cdot a_2 + 0.0625 \cdot a_4 = 0 \\ a_2 + a_4 \cdot 0.5 = 0, \end{cases} \tag{3}$$

so the following values are obtained: $a_2 = -8$ and $a_4 = 16$. Using only the first condition for $w_{1,4}$ window the optimization procedure leads to $a_2 = -8.2445$ and $a_4 = 18.984$ allowing to lower the HSLL from -27.7 dB to -39.5 dB as presented in the paper [4], where the details of the optimization procedure and comparison with classical windows are discussed.

3. Proposed family of windows and their design procedure

For the rational polynomial windows discussed in this paper the system of equations related to the asymptotic decay ratio is similar, but the coefficients of the denominator may also influence on the conditions necessary for the specified asymptotic decay ratio (according to the derivative calculation of the rational function). For example assuming the rational window (denoted as $rw_{k,p/q}$ where p and q stand for the polynomial order of the numerator and denominator respectively) containing two 2nd order polynomials with 18 dB/octave decay ratio ($rw_{2,2/2}$) the window's value and first derivative set to 0 at $|t| = T/2$ leads to:

$$\begin{cases} 1 + 0.25 \cdot a_2 = 0 \\ a_2 \cdot (1 + 0.25 \cdot b_2) = b_2 \cdot (1 + 0.25 \cdot a_2), \end{cases} \tag{4}$$

so we obtain the solution $b_2 = a_2 = -4$. Increasing the order of one of the polynomials the optimization of the HSLL is possible also for 18 dB/octave decay ratio. Nevertheless, the conditions for the 1st derivative obtained for the 2nd order of the denominator are independent on the coefficient b_2, since its value should not be equal to -4 because of the singularities at $t = \pm T/2$.

The complete system of equations for the $rw_{3,8/2}$ window with 24 dB/octave decay ratio (window's value and its two derivatives forced to zero at $t = \pm T/2$) can be expressed in the shortened cascading form:

$$
\begin{cases}
a_6 = -64 - \dfrac{3}{4} \cdot a_8 \\[2ex]
a_4 = 16 - \dfrac{3}{16} \cdot a_8 - \dfrac{1}{2} \cdot a_6 \\[2ex]
a_2 = -4 - \dfrac{1}{64} \cdot a_8 - \dfrac{1}{16} \cdot a_6 - \dfrac{1}{4} \cdot a_4 \,,
\end{cases}
\tag{5}
$$

or in the simplified way as:

$$
\begin{cases}
a_6 = -64 - \dfrac{3}{4} \cdot a_8 \\[2ex]
a_4 = 48 + \dfrac{3}{16} \cdot a_8 \\[2ex]
a_2 = -12 - \dfrac{1}{64} \cdot a_8 \,.
\end{cases}
\tag{6}
$$

The system of equations (6) illustrates the independency of the decay ratio on the coefficient b_2 and the necessity of using at least 8-th order polynomial in the numerator for its optimization regardless of the possible changes of the denominator.

4. Discussion of results

Another relevant aspect is the window length, since parameters of the continuous windows may differ from their discrete implementations. Nevertheless, the window length equal to 1024 points leads to a reasonably accurate approximation of the continuous windows, especially for lower level polynomials. The differences of the sidelobe attenuation of the standard polynomial windows are presented in Table 1 for the verification – for most windows they are very similar so all further results presented in the paper have been obtained for 1024-point windows.

Table 1. Comparison of the sidelobe attenuation for continuous polynomial windows and their discrete implementations using 1024 samples

Window type	HSLL [dB]		Window type	HSLL [dB]	
	continuous window [4]	discrete window (1024 samples)		continuous window [4]	discrete window (1024 samples)
$w_{0,2}$	−21.29	−21.30	$w_{1,8}$	−55.81	−55.82
$w_{0,4}$	−39.48	−39.41	$w_{1,10}$	−70.52	−70.52
$w_{0,6}$	−48.58	−48.58	$w_{2,6}$	−33.35	−33.35
$w_{0,8}$	−62.79	−62.39	$w_{2,8}$	−55.81	−55.81
$w_{0,10}$	−71.85	−70.52	$w_{2,10}$	−62.52	−62.48
$w_{1,4}$	−27.72	−27.72	$w_{3,8}$	−38.48	−38.48
$w_{1,6}$	−48.58	−48.54	$w_{3,10}$	−62.52	−62.28

Table 2. Comparison of the sidelobe attenuation for standard polynomial windows and rational polynomial windows with 12 and 18 dB/octave decay for 1024 samples

Window type	Polynomial coefficients						HSLL [dB]	WML [1/T]
	b_2	a_2	a_4	a_6	a_8	a_{10}		
$w_{1,4}$	–	−8	16	–	–	–	−27.72	1.835
$rw_{1,2/2}$	6.5761	−4	–	–	–	–	**−34.78**	1.938
$w_{1,6}$	–	−11.691	50.513	−79.003	–	–	−48.54	2.357
$rw_{1,4/2}$	10.912	−8.004	16.016	–	–	–	−58.95	4.250
$w_{1,8}$	–	−15.010	92.183	−273.039	321.879	–	−55.82	2.748
$rw_{1,6/2}$	4.630	−11.652	47.5616	−67.8176	–	–	**−66.07**	2.969
$w_{1,10}$	–	−16.084	113.960	−457.860	1049.839	−1073.507	−70.52	2.985
$rw_{1,8/2}$	4.259	−15.206	90.484	−249.98	269.37	–	**−74.86**	3.375
$w_{2,6}$	–	−12	48	−64	–	–	−33.35	2.219
$rw_{2,4/2}$	5.452	−8	16	–	–	–	**−43.55**	2.344
$w_{2,8}$	–	−14.990	91.937	−271.983	320.293	–	−55.81	2.750
$rw_{2,6/2}$	9.323	−11.942	47.539	−63.078	–	–	−69.11	4.688
$w_{2,10}$	–	−18.267	143.302	−605.988	1365.73	−1286.10	−62.48	3.125
$rw_{2,8/2}$	4.259	−15.206	90.484	−249.98	269.37	–	**−74.86**	3.375

The comparison of the rational polynomial windows should be made assuming similar computational complexity of standard polynomial windows. Since the computation of the polynomial's value requires 2·N operations, according to Horner's scheme, the rational polynomial window with *N*-th and *M*-th order polynomials in the numerator and denominator respectively should be compared with $(N + M)^{th}$ order standard polynomial windows (considering also the additional floating-point division).
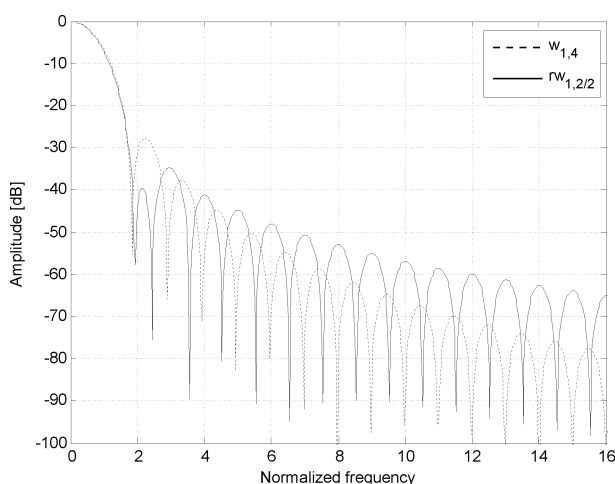

Fig. 1. Comparison of the frequency characteristics of 4th order polynomial and optimized rational polynomial window based on two 2nd order polynomials for decay ratio 12 dB/octave

The comparison of the parameters of the optimized rational windows is presented in Table 2. The frequency characteristics of the best of obtained windows (marked in Table 2) in comparison to the most similar standard polynomial are illustrated in Figures 1-5. The order of the denominator is limited to 2, since higher order polynomials cause the undesired increase of the WML during the optimization procedure.
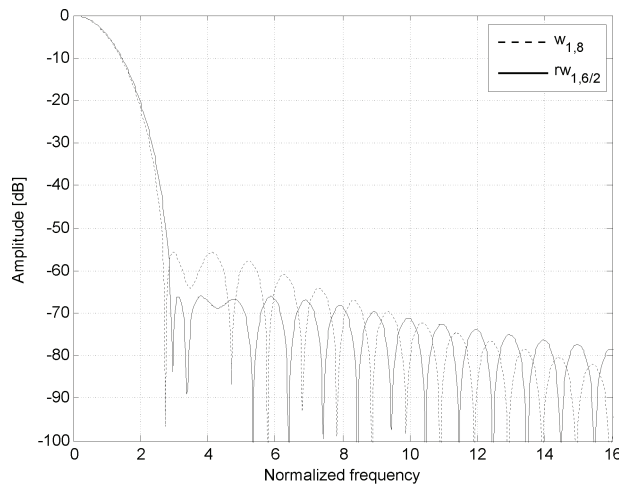

Fig. 2. Comparison of the frequency characteristics of 8th order polynomial and optimized rational polynomial window based on 6th and 2nd order polynomials for decay ratio 12 dB/octave
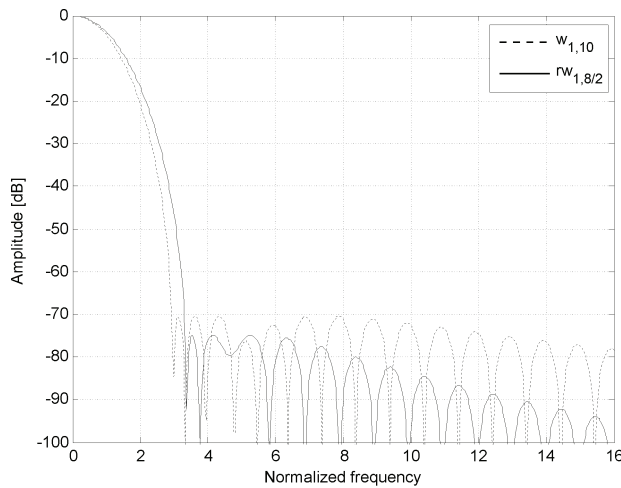

Fig. 3. Comparison of the frequency characteristics of 10th order polynomial and optimized rational polynomial window based on 8th and 2nd order polynomials for decay ratio 12 dB/octave

5. Conclusions

The rational polynomial windows family presented in the paper can be an interesting alternative for standard polynomial windows, since using only the additional division slightly increasing the computational complexity significantly better sidelobes attenuation can be achieved. For some combinations of the polynomial orders in the numerator and the desired

decay ratio the obtained results are unsatisfactory due to the substantial increase of the width of the main lobe e.g. for $rw_{1,4/2}$ and $rw_{2,6/2}$ windows.
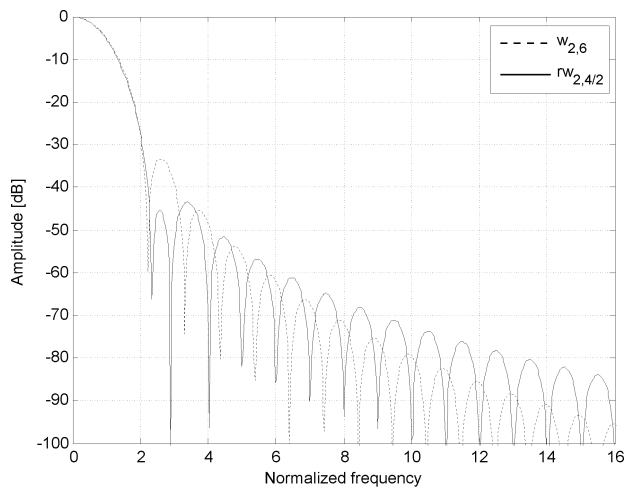

Fig. 4. Comparison of the frequency characteristics of 6th order polynomial and optimized rational polynomial window based on 4th and 2nd order polynomials for decay ratio 18 dB/octave
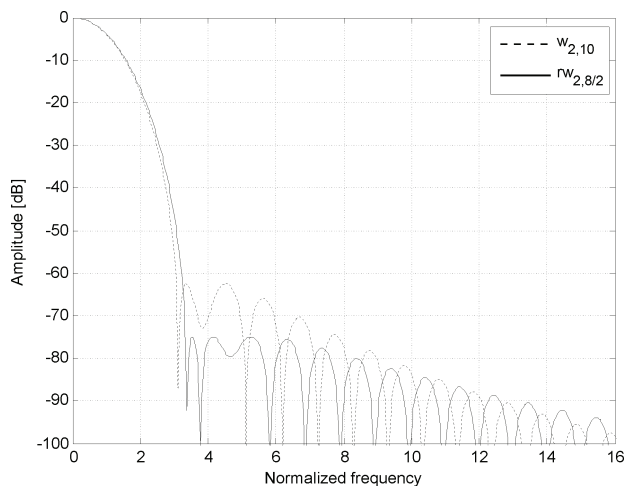

Fig. 5. Comparison of the frequency characteristics of 10th order polynomial and optimized rational polynomial window based on 8th and 2nd order polynomials for decay ratio 18 dB/octave

Nevertheless the usage of some other combinations leads to much lower level of the highest sidelobe preserving the desired decay ratio and only slightly wider main lobe. For example for $rw_{2,4/2}$ window over 10dB better HSLL can be obtained with only about 5% wider main lobe in comparison to standard 6th order polynomial window with the same decay speed.

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