ARCHIVES OF ELECTRICAL ENGINEERING VOL. 64(3), pp. 371-377 (2015)

DOI 10.2478/aee-2015-0029

Time-frequency responses of generalized first order parametric sections*

ANNA PIWOWAR

Silesian University of Technology, Faculty of Electrical Engineering
44-100 Gliwice, ul. Akademicka 10, Poland
e-mail: Anna.Piwowar@polsl.pl

(Received: 31.09.2014, revised: 12.12.2014)

Abstract: This paper presents the method of analysis of parametric systems in frequency domain. These systems are also referred to as linear time varying systems (LTV). The article includes a description of an analytical method for determining the frequency response of the first order parametric circuit with non-periodically variable parameters. The results have been illustrated by an example.

Key words: LTV system, linear time varying, parametric system, time-frequency response

1. Introduction

The parametric sections, also referred to as the linear time varying (LTV) sections, are systems with time-variable parameters. The parametric systems can be treated as the first linear approximation of nonlinear systems [3]. Any non-stationary LTV system is described by the ordinary differential equations with variable coefficients. The coefficients are called parametric functions and can be treated as the variable parameters of the system.

The basic dynamic characteristics of linear stationary systems, such as a step response, impulse response and frequency response can be generalized to LTV systems. The knowledge of methods applied to determinate the frequency responses is useful in the analogue [1] and digital [5] LTV filter synthesis.

This work is the continuation of the article series devoted to the analysis of parametric systems with non-periodical parameters. The aim of this paper is to present the time-frequency characteristics of a LTV system and an exemplar method determining the time-frequency response. The obtained results have been illustrated by an example of plotting the amplitude and phase characteristics of the generalized first order LTV section. Their analysis in the time domain is included in the work [7]. The generalization, in comparison to the earlier works, consists in the significant extension of the parametric function class.

---

* This is extended version of a paper which was presented at the 23th Symposium on Electromagnetic Phenomena in Nonlinear Circuits, Pilsen, Czech Republic 02-04.07, 2014.

2. Description of the generalized LTV section in the time domain

In the time domain, the parametric systems are described in two different ways [3, 8]. The former of them consists in using ordinary linear parametric differential Equations [7]. The processes in the parametric systems are described by differential equations where coefficients depend on time:

$$y'(t) + \omega(t)y(t) = kx(t), \tag{1}$$

where: $x(t)$ – input signal of LTV section, $y(t)$ – output signal of LTV section, $\omega(t)$ – parametric function interpreted as time-varying angular cut off frequency (Fig. 1); k – time-invariant gain coefficient for constant component. To simplify the calculations it is assumed that $k = 1$ or $k = \omega_g$, depending on calculation needs.

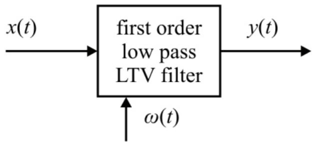


Fig. 1. Example model of the first order LTV system

Any limited energy of the non-periodic function describing the time-variable parameter of the system from Fig. 1 can be approximated [2] by the series:

$$\omega(t) = \omega_g + \sum_{\substack{k=-N \\ k\neq 0}}^{k=N} C_k \mathrm{e}^{-p_k t}, \quad \omega_g > 0, C_k, p_k \in \mathrm{C}, \tag{2}$$

thus:

$$C_k = |C_k| e^{j\psi_k} = A_k + \mathrm{j}B_k, \qquad C_k = C^*_{-k}, \tag{3}$$

$$p_k = \gamma_k + \mathrm{j}\Omega_k, \qquad p_k = p^*_{-k}. \tag{4}$$

The Equation (2) describing the variable parameter $\omega(t)$ can be transformed to the following form:

$$\omega(t) = \omega_g + \sum_{k=1}^{k=N} \mathrm{e}^{-\gamma_k t}\left(A_k \cos(\Omega_k t) + B_k \sin(\Omega_k t)\right), \tag{5}$$

where:

$$A_k = 2|C_k|\cos\psi_k, \tag{6}$$

$$B_k = -2|C_k|\sin\psi_k. \tag{7}$$

For assumptions (2)-(4), the parametric functions fulfills the conditions:

$$\lim_{t\to\infty} \omega(t) = \omega_g > 0. \tag{8}$$

The variability of the parametric function described by equation (5) has been considered. Coefficients A_k and B_k are a response for the functions of values for t = 0, coefficients Ω_k describe the oscillations of the parametric function whereas coefficients γ_k describe the time in which the stationary value ω_g is reached. Examples of the functions $\omega(t)$ obtained for a few sets of coefficients (see Tab. 1) have been shown in Figure 2.

Table 1. Coefficients of parametric functions

	ω_g	A_1	A_2	A_3	B_1	B_2	B_3	Ω_1	Ω_2	Ω_3	γ_1	γ_2	γ_3
$\omega_1(t)$	10	–10	10	20	10	10	20	1	2	3	1	2	3
$\omega_2(t)$	10	–10	10	20	0	0	0	10	2	3	1	2	3
$\omega_3(t)$	10	10	5	10	10	20	2	5	5	5	10	1	1

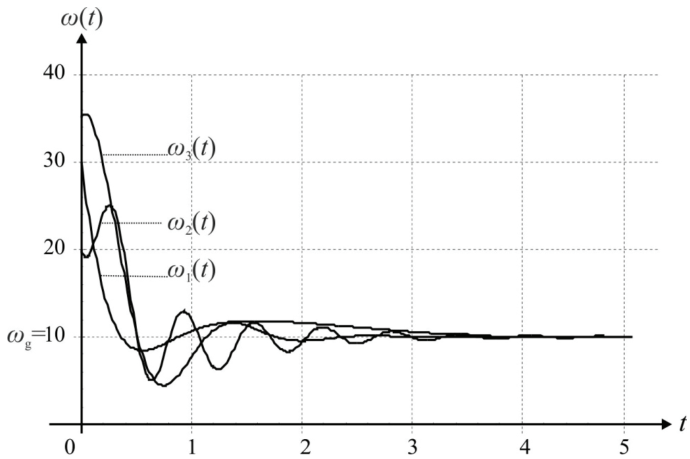


Fig. 2. Examples of parametric function variability

If the above requirement is met and the considered parametric sections are stable, after sufficiently long time the LTV sections become equivalent to the stationary low-pass sections with a cut-off angular frequency ω_g. In such cases, these sections become classic low-pass LTI (linear time invariant) filters.

The latter of the description methods in the time domain consists in using the parametric convolution with the kernel defined by the impulse response $h(t)$ expressed by [8]:

$$y(t) = \int_0^t h(t, t-\zeta)x(\zeta)d\zeta, \tag{9}$$

where: $h(t, \tilde{t}\ \zeta) = h(t, \zeta)$ – impulse response of the system, ζ – moment of application of the excitation to the input of the system.

The impulse responses of a LTV system can be obtained by solving Equation (1), for the exaction in the Diracc impulse: $x(t) = \delta(t - \zeta)$, where $t = \zeta$ is the moment of time when excitation $\delta(t)$ is applied.

A detailed analysis of time domain has been carried out in works [7]. The solution to a differential equation with zero initial conditions results in an impulse response $h(t, \zeta)$ of the parametric system and is expressed as:

$$h(t,\zeta) = \exp(-\omega_g(t-\zeta)),$$
$$\exp\left(\sum_{k=1}^{N} D_{1k}\left(\cos(\Omega_k\zeta) - \cos(\Omega_k t)\right)\exp(\gamma_k(t-\zeta))\right), \tag{10}$$
$$\exp\left(\sum_{k=1}^{N} D_{2k}\left(\sin(\Omega_k\zeta) - \sin(\Omega_k t)\right)\exp(\gamma_k(t-\zeta))\right),$$

where:

$$D_{1k} = \frac{-A_k\gamma_k - B_k\omega_k}{\gamma_k^2 + \omega_k^2}, \tag{11}$$

$$D_{2k} = \frac{A_k\omega_k - B_k\gamma_k}{\gamma_k^2 + \omega_k^2}. \tag{12}$$

The above formula has been carried out with the assumption that $\omega(t)$ is non-periodically varying according to relation (5). If the coefficients of the equation are varying, the responses will be different for different moments of time $t = \zeta$ in which exaction $\delta(t - \zeta)$ is applied to the input of the system. The mentioned effect is typical for LTV systems and does not exist in classic LTI (linear time invariant) systems.

3. Generalized LTV section in the frequency domain

Parametric system frequency responses are defined in two ways [4, 8]. The first of them is identical to the one concerning stationary systems and consists in defining the frequency characteristics as the ratio of the complex output signal to the complex monoharmonic input signal. The second way of the frequency characteristic description requires knowledge of the impulse response and it defines time frequency response as:

$$H(\mathrm{j}\omega,t) = \mathcal{F}_\tau\{h(t,\tau)\}, \tag{13}$$

where: $\mathcal{F}_\tau$ – Fourier transform operator.

A method of determining of the time-frequency responses of the LTV sections is described below. Assuming, that the impulse response of the LTV system is given and further denoted as $h(t - \zeta, \zeta)$ (10) [7], the response for any exaction is expressed by (9). For a complex monoharmonic signal $X(t) = e^{j\omega t}$ formula (9) can be presented as:

$$Y(j\omega,t) = e^{j\omega t}\int_{-\infty}^{t} h(t-\zeta,\zeta)e^{-j\omega(t-\zeta)}d\zeta. \tag{14}$$

As in the case of the stationary systems, magnitude and phase characteristics are defined as a ratio of the output signal $Y(\mathrm{j}\omega, t)$ and the input signal $e^{j\omega t}$.

$$H(j\omega,t)=\frac{Y(j\omega,t)}{e^{\mathrm{j}\omega t}}=\int_0^\infty h(t-\zeta,\zeta)e^{-j\omega(t-\zeta)}d\zeta\,. \tag{15}$$

Taking into account that $t-\zeta=\tau$ and $h(t,\ \tau)=h(\tau,\ t-\tau)$ one can obtain a transfer function in normal form (compare with eq. (13))

$$H(j\omega,t)=\int_0^\infty h(t,\tau)e^{-j\omega\tau}\mathrm{d}\tau=\mathcal{F}_\tau\{h(t,\tau)\}. \tag{16}$$

Expanding Equation (10) by its functional Taylor series and applying a generalized version of Newton's binomial formula, one can get the relation describing the time-frequency responses (16) in closed form. From a mathematical point of view, the Equation (16) describes the classic Fourier transform of the impulse response with τ as a variable. Contrary to the stationary systems, the parametric system frequency characteristics are the complex functions of time and frequency. Moreover,

$$H(\mathrm{j}\omega,t)=\left|H(\mathrm{j}\omega,t)\right|e^{\mathrm{j}\varphi(\omega,t)}\,. \tag{17}$$

Functions $|H(\mathrm{j}\omega,\ t)|$ and $\varphi(\omega,\ t)$ are called time-frequency magnitude characteristics in literature [8] while time – frequency phase characteristics. Based on formula (16), in this paper, time-frequency characteristics of the first order of the parametric sections (Eq. 1) with a parameter variable following formula (5) were determined.

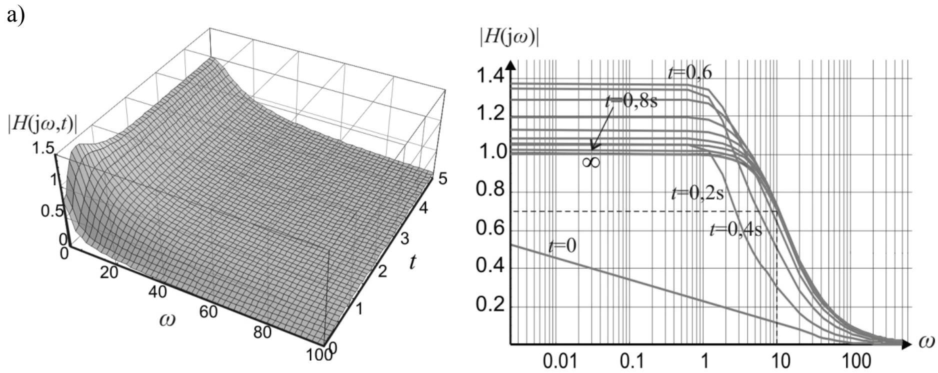
It should be noted that in the frequency domain, another description of the parametric system is possible. It consist in the application of the double Fourier transform with respect to the impulse function $h(t,\ \tau)$. It is called a bispectral frequency response [4]. The mentioned method of describing LTV system frequency properties is not used in this paper.

3. Example

The low pass LTV first order filter presented in Figure 1 has been analyzed for a parameter varying as the function $\omega_1(t)$ variability expressed by Equation 5. The waveform of the time varying cut-off angular frequency $\omega_1(t)$ of the system has been shown in Figure 2. The magnitude and phase responses of the analyzed systems determined on the base of the formula (16) have been presented in figures 3 and 4.

Any change in behaviour of the parameter $\omega(t)$ results in the formation of time-frequency responses of the system. According to the variability of the parametric function, the cut-off frequency and attenuation ratio are varied. When the parametric function achieves its steady value ω_g, the LTV system becomes equivalent to a classic, first order low pass filter with constant a value of the cut-off angular frequency ω_g.

a)



b)

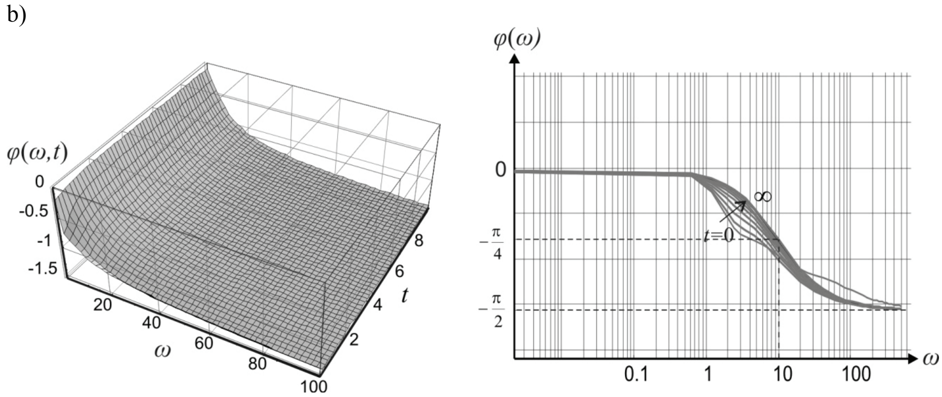


Fig. 3. Time-frequency responses of first order LTV (a) magnitude response and its shaping process in time (b) phase response and its shaping process in time

4. Conclusions

The method of frequency response determination presented in this paper requires a closed form of the section impulse response. The method can be used both for frequency characteristic calculations of the first order low-pass LTV sections and for higher order sections with an arbitrary variability of a parameter.

The time-frequency response of a non-stationary system is the Fourier transform of an impulse response. Moreover, it is a function not only of frequency (as in the stationary systems case), but also of time. The variability of the system parameters allows to freely shape the magnitude and phase characteristics of parametric systems.

References

[1] Belmont M.R., Matthews J.J., *Generalized frequency response as applied to circuits with time varying elements.* IEE Proc. Circuits Devices and Systems 142: 217-222 (1995).

[2] Cllement P.R., *On completeness of basis function used for signal analysis*. SIAM Reviev 5(2): 167-172 (1963).

[3] D'Angelo H., *Linear Time-Varying Systems. Analysis and Synthesis*. Allyn and Bacon, Inc. Boston (1970).

[4] Erfani S., Bayan N., *On linear time-varying system characterizations.* IEEE Int. Conf. on Electro/Information Technology, pp. 207-210 (2009).

[5] Gersho A., *Characterization of time-varying linear systems*. Proc. of the IEEE 51(1): 238 (1963).

[6] Peters S.D., Fahrny M.M., *A design technique for recursive linear time-varying digital filters.* IEEE Int. Symp. Circuits and Systems 1, Ontario, pp. 779-782 (1998).

[7] Piwowar A., Walczak J., *Impulse responses of generalized first order LTV sections, Analysis and Symulation of Electrical and Computer Systems.* Lect. Notes Electrical Egn. 324, Chapter 6, Springer (2013).

[8] Zadeh L.A., *Frequency analysis of variable networks*. Proc. IRE 32: 291-299 (1950).