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On In-Network and Other Types of Amplifier Descriptions for Nonlinear Distortion Analysis

Andrzej Borys

***Abstract*—Basics of modelling analog weakly nonlinear amplifiers at higher frequencies for the purpose of nonlinear distortion analysis are addressed in this paper. First, the constitutive relation for this class of amplifiers, with the use of a Volterra series, is formulated. It is the basis for formulation and derivation of the so-called in-network and input-output type descriptions of an amplifier in the time domain, which are then transferred into the multi-frequency domains. Usefulness of the general models achieved, which were not published up to now in the literature, lies in the fact that they can be used for any topology in which the amplifier is incorporated and for any nonlinear distortion measure assumed. Some examples of calculations are given at the end of the paper for cascade and feedback topologies, and for harmonic distortion measure.**

***Keywords*—Weakly nonlinear amplifiers, nonlinear distortion analysis, harmonic distortion, constitutive relations, Volterra series.**

I. INTRODUCTION

IN THE LITERATURE on nonlinear circuits and systems, the Volterra series is viewed as a typical input-output representation, so also applied in the analysis in such a form. However, as we show in this paper, it can play other roles, too. One needs it, for example, in a correct formulation of the so-called constitutive relation [6] of a weakly nonlinear amplifier. In this case, the Volterra series is its part as we show here.

In this paper, to formulate the constitutive relation of a weakly nonlinear amplifier, we view it as a two-port circuit element. And we formulate for it two equations relating to each other the amplifier's port variables. These equations constitute the amplifier's constitutive relation.

The amplifier's constitutive relation is the basis for formulation of its two models called in-network and input-output type descriptions [3]. They are simply the element's constitutive relation in which a context is taken into account, with that context meaning here a port to which the input signal is applied. More precisely, when the input signal is applied directly to one of the amplifier's ports, we speak about the input-output type description. But, when the point of application of the input signal is "moved" to another port of a network in which a given amplifier is incorporated, we speak about the in-network type description.

The amplifier's in-network and input-output type descriptions in the time domain are formulated and discussed in detail in section III. In the next section, these descriptions are transformed into the multi-frequency domains.

Usefulness of the amplifier's in-network type model in consideration of different network topologies surrounding the amplifier is illustrated by two examples.

Also, we show here how this model can be applied in calculations of different nonlinear distortion measures. For illustration, its application in evaluation of harmonic distortion in a simple feedback structure of Fig. 3 is discussed in more detail in section VI.

The purpose of this paper is to develop fundamentals of modeling weakly nonlinear amplifiers for the needs of nonlinear distortion calculations. Also, the task is to put some already published results into a more general framework.

The main results achieved in this paper are the new general models of weakly nonlinear amplifiers developed for the needs of nonlinear distortion evaluation. Up to now, they were not published in the literature.

II. CONSTITUTIVE RELATION OF A WEAKLY NONLINEAR AMPLIFIER

In his paper [6] on fundamentals of electronic device modeling, Chua introduced the notion of a constitutive relation. He showed how it can be formulated for basic two-terminal circuit elements (resistor, inductor, capacitor, and memristor) as well as for internally more complex ones (possessing however also two terminals). In the latter case, the constitutive relations, apart from the terminal variables, involve also their derivatives and integrals, and sometimes internal variables (including their derivatives and integrals), too. Furthermore, their form is then also not of a simple algebraic relation, but of a system of algebraic, differential, and integral equations [6].

Here, we use the following terminological convention: in case of the constitutive relations having the form of algebraic relationships, we name them alternatively constitutive equations.

The constitutive relations were also defined in [6] for circuit elements possessing more terminals than two, or more or equal to two ports (that is for multi-terminal or multi-port elements). Hence, we can expect that formulation of the constitutive relation for such an element as (linear or nonlinear) amplifier is possible. Simply because it is a two-port circuit element (block, module) - of which consist larger objects (circuits), as for example multi-stage amplifiers.

Having formulated the constitutive relation of an amplifier as a circuit element, we show afterwards consistency of this notion.

Now, consider a nonlinear time-invariant amplifier with memory denoted by a capital letter H in Fig. 1.

The author is with the Institute of Telecommunications, Faculty of Telecommunications and Electrical Engineering, University of Technology and Life Sciences (UTP), S. Kaliskiego 7, 85-789 Bydgoszcz, Poland (e-mail: andrzej.borys@utp.edu.pl).

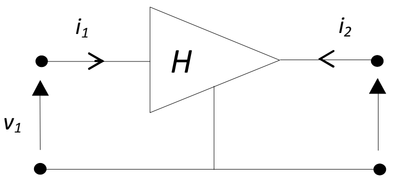

Fig. 1. Amplifier as a two-port circuit element.

It can be viewed, using the terminology formulated in [6], as a dynamic two-port described by two sets of variables: of port voltages $\{v_1, v_2\}$ and currents $\{i_1, i_2\}$, and by a set of algebraic and non-algebraic relations between their elements. So denoting the latter set by a bold-faced $\boldsymbol{f}_H$, meaning a vector, we can write

$$\boldsymbol{f}_H\left(v_1(t), v_2(t), i_1(t), i_2(t)\right) = \mathbf{0} \tag{1a}$$

or shortly, without showing a continuous time variable t in the expression, as

$$\boldsymbol{f}_H\left(v_1, v_2, i_1, i_2\right) = \mathbf{0} \tag{1b}$$

where $\mathbf{0}$ means a zero vector.

Equation (1b) constitutes an implicit form of the constitutive relation of an amplifier H in Fig. 1. In what follows, we assume that its explicit form can be derived uniquely from (1b).

An example of the explicit form of the constitutive relation of an element closely related to that in Fig. 1 was given in [6] (see page 1026 therein). That is for an algebraic n-port called in [6] an $\boldsymbol{x}$-controlled voltage source, where $\boldsymbol{x}$ stands for a vector of controlling variables. Specialized for n=2 (two ports) with a controlling variable being the voltage at the port 1, it reads as

$$\begin{cases} i_1 = 0 \\ v_2 = f\left(v_1\right) \end{cases} \tag{2}$$

where f denotes an ordinary function, not necessarily a linear one.

Note now that if the amplifier of Fig. 1 were a memoryless element its description by (2) would constitute an adequate model for it. Also, observe that, to allow in such a model for H of Fig. 1 being a dynamic element (with memory), we need only to modify slightly (2). Simply, we then rewrite the second equation in (2) in a more general form as shown below

$$\begin{cases} i_1 = 0 \\ v_2 = H\left(v_1\right) \end{cases} \tag{3}$$

with H meaning now a generally non-algebraic operator (describing the dynamic phenomena).

Remarks:

1. The explicit forms of the constitutive relations of amplifiers: without and with memory (a dynamic one) considered in this paper are given, respectively, by (2) and (3).
2. Note that knowing (3) we can detail (1b). Then, we get

$$\boldsymbol{f}_H\left(v_1, v_2, i_1, i_2\right) = \begin{bmatrix} i_1 \\ v_2 - H\left(v_1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

3. Note that all the forms of the amplifier constitutive relations presented are "truly internal" relations, relating to each other exclusively amplifier port variables. For these relations, a place of application of an input signal to a network, in which a given amplifier is embedded, is of absolutely no importance.
4. Note that similar means of amplifier modeling as given by (2) or (3) were used without referring to notion of the constitutive relation, in many papers, for example see the most recent works [5], [10], [11]. Here, we provide however a general framework for all these particular results published in the literature up to now.

In what follows, we specialize our general model of an amplifier given by (3) to the case of a weakly nonlinear amplifier. That is we specialize it to such an amplifier for which the operator $H\left(v_1\right)$ in the second equation in (3) can be expanded in a Volterra [4] series giving

$$v_2 = H\left(v_1\right) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_H^{(n)}(\tau_1, \ldots, \tau_n) \prod_{k=1}^{n} v_1(t - \tau_k) d\tau_k \,. \tag{5}$$

In (5), $h_H^{(n)}(\tau_1, \ldots, \tau_n)$ means the amplifier nonlinear impulse response of the n-th order.

Furthermore, the term "weakly nonlinear" means that the series (5) can be well approximated by its first three components for the values of amplitudes and frequencies of the voltage signals $v_1(t)$ that are used in practice at the amplifier port 1. In other words, we assume that we can successfully use in the analysis (not exceeding a certain amount of error) the following truncated expansion:

$$v_2 = H\left(v_1\right) = v_{2H}^{(1)} + v_{2H}^{(2)} + v_{2H}^{(3)} + \ldots \cong v_{2H}^{(1)} + v_{2H}^{(2)} + v_{2H}^{(3)} \tag{6a}$$

with

$$v_{2H}^{(n)}\left(t\right) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_H^{(n)}(\tau_1, \ldots, \tau_n) \prod_{k=1}^{n} v_1(t - \tau_k) d\tau_k \tag{6b}$$

for $n = 1, 2, 3, \ldots$. For simplicity of notation, the time variable t is dropped in (6a). Moreover, it is clear from (6a) and (6b) that $v_{2H}^{(n)}(v_1) = v_{2H}^{(n)}(t)$, n=1,2,3,…, are the partial responses in the voltage $v_2(t)$ that are associated with the corresponding orders n of amplifier nonlinearities. Further, the second subscript "H" in these partial responses indicates that they are calculated according to the formula of an amplifier constitutive relation (they regard the amplifier constitutive relation).

Remarks:

1. Note that the Volterra series given by (6a) and (6b) does not play in (3) its usual role of an input-output

representation for a nonlinear circuit or system. In (3), it is a part of this constitutive relation of a circuit element called a weakly nonlinear amplifier. And it is important to be aware of this fact because a constitutive relation, from its nature, is an "internal" description.
2. The implicit constitutive relation of a weakly nonlinear amplifier given by (3), (6a), and (6b) is that relation one needs to derive its in-network and input-output type descriptions (defined in [3]). Derivation of these descriptions is the objective of the next section.

III. In-Network and Input-Output Type Descriptions of a Weakly Nonlinear Amplifier

In [3], in-network and input-output type descriptions of nonlinear circuit elements were defined and illustrated for some basic two-terminal elements by presenting their detailed derivations. More complicated elements, as for example multi-port ones, were not dealt with. Here, we consider the latter on an example of a weakly nonlinear amplifier (being a two-port element).

We can observe that the fundamental difference between amplifier's in-network and input-output type descriptions is the following:
1. In an in-network type description, none of the port variables is its input signal. That is in this case the input signal is applied to a place different from any of the element's ports.
2. In an input-output type description, one port variable is its input signal, and the second one is its output signal.

(This follows immediately from the application of definitions given in [3] to a two-port circuit element.)

We present derivation of the in-network type description of a weakly nonlinear amplifier first. So for this, we assume now that this amplifier (denoted as well by H as in Fig. 1) is embedded in a network to which an input voltage signal $v_i(t)$ is applied at a port $i \neq 1,2$ (for convenience, we assume here that ports of H embedded in a network are numbered in the same way as shown in Fig. 1). Furthermore, we assume that the port voltages $v_1(t)$ and $v_2(t)$ in this network can be expressed by a Volterra series related to the input voltage signal $v_i(t)$. That is we can express $v_1(t)$ and $v_2(t)$ as

$$v_1 = H_1(v_i) = v_{1i}^{(1)} + v_{1i}^{(2)} + v_{1i}^{(3)} + \ldots \tag{7a}$$

with

$$v_{1i}^{(n)}(t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_1^{(n)}(\tau_1,\ldots,\tau_n) \prod_{k=1}^{n} v_i(t-\tau_k) d\tau_k \tag{7b}$$

and

$$v_2 = H_2(v_i) = v_{2i}^{(1)} + v_{2i}^{(2)} + v_{2i}^{(3)} + \ldots \tag{8a}$$

with

$$v_{2i}^{(n)}(t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_2^{(n)}(\tau_1,\ldots,\tau_n) \prod_{k=1}^{n} v_i(t-\tau_k) d\tau_k \tag{8b}$$

respectively, for $n = 1,2,3,\ldots$. In (7a), $H_1(v_i)$ denotes an operator describing the relation existing in a network considered between the port voltage $v_1(t)$ (can be viewed as an output signal in this relation) and the port voltage $v_i(t)$ (which is the network input signal, so also the input signal in this relation). At this point, we point also out a fundamental difference existing between H and H_1: the first operator depends exclusively upon the amplifier's parameters, contrary to the second one, which depends, generally, upon all the network's parameters (that is of the embedded amplifier and of the network part connected to it). Further, $H_2(v_i)$ from (8a) is defined similarly, and differs also similarly from H, as $H_1(v_i)$. Moreover, $v_{1i}^{(n)} = v_{1i}^{(n)}(t)$ and $v_{2i}^{(n)} = v_{2i}^{(n)}(t)$, n=1,2,3,…, denote the partial responses in the voltages $v_1(t)$ and $v_2(t)$, respectively, that are associated with the corresponding orders of the network nonlinearities (as the whole) and are related to the network input signal placed at port i. We say also that they are of degree n with respect to the variable v_i. The time variable t is dropped in (7a) and (8a) for simplicity of notation. Furthermore, $h_1^{(n)}(\tau_1,\ldots,\tau_n)$ and $h_2^{(n)}(\tau_1,\ldots,\tau_n)$ mean the network nonlinear impulse responses of the n-th order for the relations between ports 1 and i, and 2 and i, respectively.

Before going further, let's observe that the second equation of the amplifier implicit constitutive relation (3) is expressed now by (6a) and (6b). Furthermore, note that according to a rule given in [3], to get the element's in-network type description, we have to introduce its terminal or port variables related with the network input signal into its constitutive equation. So substituting the voltages $v_1(t)$ and $v_2(t)$ given by the expressions on the right-hand sides of (7a) and (8a), we get

$$\begin{aligned} &v_2(t) = v_{2i}^{(1)}(t) + v_{2i}^{(2)}(t) + v_{2i}^{(3)}(t) + \ldots = \\ &= \int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)\Big[v_{1i}^{(1)}(t-\tau_1) + v_{1i}^{(2)}(t-\tau_1) + \\ &+ v_{1i}^{(3)}(t-\tau_1) + ..\Big] d\tau_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(2)}(\tau_1,\tau_2)\Big[v_{1i}^{(1)}(t-\tau_1) + \\ &+ v_{1i}^{(2)}(t-\tau_1) + ..\Big]\Big[v_{1i}^{(1)}(t-\tau_2) + v_{1i}^{(2)}(t-\tau_2) + ..\Big] d\tau_1 d\tau_2 + \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(3)}(\tau_1,\tau_2,\tau_3)\Big[v_{1i}^{(1)}(t-\tau_1) + \\ &+ v_{1i}^{(2)}(t-\tau_1) + ..\Big]\Big[v_{1i}^{(1)}(t-\tau_2) + v_{1i}^{(2)}(t-\tau_2) + ..\Big] \cdot \\ &\cdot \Big[v_{1i}^{(1)}(t-\tau_3) + v_{1i}^{(2)}(t-\tau_3) + ..\Big] d\tau_1 d\tau_2 d\tau_3 + \ldots \end{aligned} \tag{9}$$

In the next step, comparison of the components of the same order (degree) with respect to the input signal v_i on both sides of (9) gives

$$v_{2i}^{(1)}(t)=\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)v_{1i}^{(1)}(t-\tau_1)d\tau_1 \tag{10a}$$

$$v_{2i}^{(2)}(t)=\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)v_{1i}^{(2)}(t-\tau_1)d\tau_1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(2)}(\tau_1,\tau_2)\cdot \cdot v_{1i}^{(1)}(t-\tau_1)v_{1i}^{(1)}(t-\tau_2)d\tau_1 d\tau_2 \tag{10b}$$

$$v_{2i}^{(3)}(t)=\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)v_{1i}^{(3)}(t-\tau_1)d\tau_1+ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(2)}(\tau_1,\tau_2)\Big[v_{1i}^{(1)}(t-\tau_1)v_{1i}^{(2)}(t-\tau_2)+ +v_{1i}^{(2)}(t-\tau_1)v_{1i}^{(1)}(t-\tau_2)\Big]d\tau_1 d\tau_2+ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(3)}(\tau_1,\tau_2,\tau_3)v_{1i}^{(1)}(t-\tau_1)v_{1i}^{(1)}(t-\tau_2)\cdot \cdot v_{1i}^{(1)}(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \tag{10c}$$

and so on.

Consider now the first equation in (3). Obviously, its both sides can be also viewed as the Volterra series, as follows

$$i_1=i_1^{(1)}+i_1^{(2)}+i_1^{(3)}+\ldots=0+0+0+\ldots \tag{11}$$

where the time variable t is dropped for simplicity of notation and $i_1^{(n)}$, n=1,2,3,…, are the partial responses (of the corresponding orders n) in the current i_1. Furthermore, note that i_1 is independent of any other port variable of the amplifier. Also, observe that the current i_1 is independent of all the network port variables when the amplifier is embedded in it. In other words, equation (11) is exactly the same: as a part of the amplifier constitutive relation and as an equation relating the current $i_1(t)$ with the network input signal $v_i(t)$.

It follows from (11) that

$$i_1^{(1)}=0,\ \ i_1^{(2)}=0,\ \ i_1^{(3)}=0,\ \ldots. \tag{12}$$

Having (10a-c) and (12), we can now define the in-network type description of a nonlinear amplifier in the time domain as a series of pairs

$$\begin{cases} i_{1i}^{(n)}(t) \\ v_{2i}^{(n)}(t) \end{cases},\ \ n=1,2,3,\ldots \tag{13}$$

where $i_{1i}^{(n)}(t)$ are given by (12) (the second subscript "i" is added here to for underlying the consideration of the case of an in-network type model) and $v_{2i}^{(n)}(t)$ by (10a-c), for $n=1,2,3,\ldots$. And, as mentioned above, when we can restrict ourselves to the first three pairs in (13), we have an in-network type model for a weakly nonlinear amplifier.

Remarks:
1. Evidently, (13) represents an iterative process. That is we calculate first $v_{2i}^{(1)}$, then $v_{2i}^{(2)}$, and next $v_{2i}^{(3)}$, and so on. This process constitutes an iterative model of a nonlinear amplifier.
2. It follows from (10a-c) that the order of calculations of the aforementioned terms is fixed in the amplifier in-network type model.

Note that equations (10a-c) can be also written in another form when we express in them the partial responses $v_{1i}^{(n)}(t)$, n=1,2,3,…, according to (7b). Then, we get

$$v_{2i}^{(1)}(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)h_1^{(1)}(\tau_a)v_i(t-\tau_1-\tau_a)d\tau_a d\tau_1 \tag{14a}$$

$$v_{2i}^{(2)}(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)h_1^{(2)}(\tau_a,\tau_b)v_i(t-\tau_1-\tau_a)\cdot \cdot v_i(t-\tau_2-\tau_b)d\tau_a d\tau_b d\tau_1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(2)}(\tau_1,\tau_2)h_1^{(1)}(\tau_a)\cdot \cdot h_1^{(1)}(\tau_b)v_i(t-\tau_1-\tau_a)v_i(t-\tau_2-\tau_b)d\tau_a d\tau_b d\tau_1 d\tau_2 \tag{14b}$$

$$v_{2i}^{(3)}(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)h_1^{(3)}(\tau_a,\tau_b,\tau_c)v_i(t-\tau_1-\tau_a)\cdot \cdot v_i(t-\tau_1-\tau_b)v_i(t-\tau_1-\tau_c)d\tau_a d\tau_b d\tau_c d\tau_1+ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(2)}(\tau_1,\tau_2)\Big[h_1^{(1)}(\tau_a)h_1^{(2)}(\tau_b,\tau_c)v_i(t-\tau_1-\tau_a)\cdot \cdot v_i(t-\tau_2-\tau_b)v_i(t-\tau_2-\tau_c)+h_1^{(2)}(\tau_a,\tau_b)h_1^{(1)}(\tau_c)\cdot \cdot v_i(t-\tau_1-\tau_a)v_i(t-\tau_1-\tau_b)v_i(t-\tau_2-\tau_c)\Big]\cdot \cdot d\tau_a d\tau_b d\tau_c d\tau_1 d\tau_2++\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_H^{(3)}(\tau_1,\tau_2,\tau_3)\cdot \cdot h_1^{(1)}(\tau_a)h_1^{(1)}(\tau_b)h_1^{(1)}(\tau_c)v_i(t-\tau_1-\tau_a)v_i(t-\tau_2-\tau_b)\cdot \cdot v_i(t-\tau_3-\tau_c)d\tau_a d\tau_b d\tau_c d\tau_1 d\tau_2 d\tau_3 \tag{14c}$$

and so on.

Derivation of the input-output type description [3] for the nonlinear amplifier is much simpler. In this case, the input signal $v_i(t)$ is applied directly at the amplifier port 1 (see Fig. 1). So the counterparts of (14a-c) are achieved directly from (6a) and (b) by substitution of $v_1(t)=v_i(t)$ and dropping the second index "H" by the symbols of partial responses (indicating thereby that they regard the amplifier input-output type model). For completeness, we provide also the corresponding expressions

$$v_2^{(1)}(t)=\int_{-\infty}^{\infty} h_H^{(1)}(\tau_1)v_i(t-\tau_1)d\tau_1 \tag{15a}$$

$$v_2^{(2)}(t)=\int_{-\infty}^{\infty} h_H^{(2)}(\tau_1,\tau_2)v_i(t-\tau_1)v_i(t-\tau_2)d\tau_1 d\tau_2 \quad (15b)$$

$$v_2^{(3)}(t)=\int_{-\infty}^{\infty} h_H^{(3)}(\tau_1,\tau_2,\tau_3)v_i(t-\tau_1)v_i(t-\tau_2)\cdot \\ \cdot v_i(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \quad (15c)$$

and so on. And finally, the whole input-output type model of the amplifier is given by

$$\begin{cases} i_1^{(n)}(t) \\ v_2^{(n)}(t) \end{cases},\ n=1,2,3,\ldots \quad (16)$$

where now $i_1^{(n)}(t)$ are given by (12), but $v_2^{(n)}(t)$ by (15a-c), for $n=1,2,3,\ldots$.

Remarks:

1. Relation (16), similarly as (13), represents an iterative process. So this process constitutes an iterative model of a nonlinear amplifier.
2. It follows from (15a-c) that the order of calculations of the terms $v_2^{(n)}(t)$, $n=1,2,3,\ldots$, is not fixed in the amplifier input-output type model.

It is interesting to note that the input signal $v_i(t)$ can be also expressed as the following Volterra series

$$v_i(t)=v_i^{(1)}(t)+v_i^{(2)}(t)+v_i^{(3)}(t)+\ldots=v_i(t)+0+0+\ldots \quad (17)$$

where $v_i^{(n)}$, n=1,2,3,…, mean the partial responses (of the corresponding orders n) in the voltage v_i. And (17) can be used to derive the amplifier input-output type description from its in-network type one. To this end, we assume that the input signal v_i is applied directly at port 1 (see Fig. 1) what together with (17) allows us to write

$$v_{1i}^{(1)}(t)=v_i^{(1)}(t)=v_i(t) \quad (18a)$$

$$v_{1i}^{(2)}(t)=v_i^{(2)}(t)=0 \quad (18b)$$

$$v_{1i}^{(3)}(t)=v_i^{(3)}(t)=0 \quad (18c)$$

and so on. Applying then (18a-c) in (10a-c), we get really the expressions occurring on the right hand sides of (15a-c).

Finally in this section, note also that in the case of linear circuits there is no need for introducing their in-network type descriptions. This is evident, for example, from consideration of a linear amplifier that is described by (10a) or (15a) and of which all the partial responses $v_{2i}^{(n)}(t)$ or $v_2^{(n)}(t)$ for $n\geq 2$, and $i_{1i}^{(n)}(t)$ or $i_1^{(n)}(t)$ for $n\geq 1$ are identically equal to zero. The form of the right hand sides of (10a) and (15a) is the same.

IV. In-Network and Input-Output Type Descriptions in Multi-Frequency Domains

To transform the expressions given by (10a-c), (13), (14a-c), (15a-c), and (16) into the multi-frequency domains, it is necessary to introduce in them artificial auxiliary time variables. For this purpose, we apply a standard procedure used in this area, as described, for instance, in [4], [9]. That is, for example, we apply

$$v_2^{(n)}=v_2^{(n)}(t)\rightarrow v_2^{(n)}(t_1,..,t_n) \quad (19a)$$

n=1, 2, 3,…, for voltages on the left hand sides of (15a-c) and

$$v_i^{n}=v_i^{n}(t)\rightarrow v_i^{(n)}(t_1,..,t_n) \quad (19b)$$

with

$$v_i^{(n)}(t_1,..,t_n)=v_i(t_1)\cdots v_i(t_n) \quad (19c)$$

for powers of v_i which occur on the right-hand sides of (15a-c). In (19a-c), $t_1,..,t_n$ mean the artificial auxiliary time variables, and $n=1,2,3,\ldots$.

Having the artificial auxiliary time variables introduced (where needed) in all the aforementioned expressions, we apply in the next step the so-called multidimensional Fourier transforms [1], [4], [7] to them. These Fourier transforms, for the successive indices $n=1,2,3,\ldots$, are defined by

$$G^{(n)}(f_1,\ldots,f_n)=\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} g^{(n)}(t_1,\ldots,t_n)\cdot \\ \cdot\exp(-j2\pi(f_1t_1+\ldots+f_nt_n))dt_1\cdots dt_n \quad (20)$$

where $G^{(n)}(f_1,\ldots,f_n)$ means the n-dimensional Fourier transform of a function $g^{(n)}(t_1,\ldots,t_n)$ having n arguments being artificial auxiliary time variables. Moreover, $f_1,\ldots,f_n$ in (20) are the frequencies from the n-dimensional frequency space.

Because lack of space, we omit here the details of derivations. We present only the results, which are as follows.

So from (10a-e), we get

$$V_{2i}^{(1)}(f_1)=H_H^{(1)}(f_1)V_{1i}^{(1)}(f_1) \quad (21a)$$

$$V_{2i}^{(2)}(f_1,f_2)=H_H^{(1)}(f_1+f_2)V_{1i}^{(2)}(f_1,f_2)+ \\ +H_H^{(2)}(f_1,f_2)V_{1i}^{(1)}(f_1)V_{1i}^{(1)}(f_2) \quad (21b)$$

$$V_{2i}^{(3)}(f_1,f_2,f_3)=H_H^{(1)}(f_1+f_2+f_3)V_{1i}^{(3)}(f_1,f_2,f_3)+ \\ +H_H^{(2)}(f_1,f_2+f_3)V_{1i}^{(1)}(f_1)V_{1i}^{(2)}(f_2,f_3)+ \\ +H_H^{(2)}(f_1+f_2,f_3)V_{1i}^{(2)}(f_1,f_2)V_{1i}^{(1)}(f_3)+ \\ +H_H^{(3)}(f_1,f_2,f_3)V_{1i}^{(1)}(f_1)V_{1i}^{(1)}(f_2)V_{1i}^{(1)}(f_3) \quad (21c)$$

and so on. In (21a-c), f_1, f_2, and f_3 represent the frequency variables in the corresponding multidimensional frequency spaces. That is f_1 belongs to the one-dimensional frequency space, f_1 and f_2 belong to the two-dimensional frequency space, f_1, f_2, and f_3 belong to the three-dimensional frequency space, and so on. Moreover, $H_H^{(1)}$ and $V_{1i}^{(1)}$, $H_H^{(2)}$ and $V_{1i}^{(2)}$, $H_H^{(3)}$ and $V_{1i}^{(3)}$, and so on denote the one-, two-, and three-dimensional, and so on, respectively, Fourier transforms of the corresponding impulse responses and partial voltage responses of the successive orders, occurring in (21a-c). They depend upon the frequencies f_1, f_2, and f_3 as indicated. $H_H^{(1)}$, $H_H^{(2)}$, $H_H^{(3)}$, and so on, are also called the nonlinear transfer functions of the corresponding orders $n=1,2,3,\ldots$ (for n=1 meaning a standard linear transfer function).

Furthermore from (13), we have

$$\begin{cases} I_{1i}^{(n)}\left(f_1,\ldots,f_n\right) \\ V_{2i}^{(n)}\left(f_1,\ldots,f_n\right) \end{cases}, \quad n=1,2,3,\ldots \tag{22}$$

where $I_{1i}^{(n)}\left(f_1,\ldots,f_n\right)$ and $V_{1i}^{(n)}\left(f_1,\ldots,f_n\right)$, $n=1,2,3,\ldots$, denote the successive n-dimensional Fourier transforms of the partial responses in the corresponding port current and voltage regarding the amplifier in-network type model.

And from (14a-c), we obtain

$$V_{2i}^{(1)}\left(f_1\right)=H_H^{(1)}\left(f_1\right)H_1^{(1)}\left(f_1\right)V_i\left(f_1\right) \tag{23a}$$

$$\begin{aligned} V_{2i}^{(2)}\left(f_1,f_2\right)=\Big[&H_H^{(1)}\left(f_1+f_2\right)H_1^{(2)}\left(f_1,f_2\right)+\\ &+H_H^{(2)}\left(f_1,f_2\right)H_1^{(1)}\left(f_1\right)H_1^{(1)}\left(f_2\right)\Big]V_i\left(f_1\right)V_i\left(f_2\right) \end{aligned} \tag{23b}$$

$$\begin{aligned} V_{2i}^{(3)}\left(f_1,f_2,f_3\right)=\Big[&H_H^{(1)}\left(f_1+f_2+f_3\right)H_1^{(3)}\left(f_1,f_2,f_3\right)+\\ &+H_H^{(2)}\left(f_1,f_2+f_3\right)H_1^{(1)}\left(f_1\right)H_1^{(2)}\left(f_2,f_3\right)+\\ &H_H^{(2)}\left(f_1+f_2,f_3\right)H_1^{(2)}\left(f_1,f_2\right)H_1^{(1)}\left(f_3\right)+H_H^{(3)}\left(f_1,f_2,f_3\right)\cdot\\ &\cdot H_1^{(1)}\left(f_1\right)H_1^{(1)}\left(f_2\right)H_1^{(1)}\left(f_3\right)\Big]V_i\left(f_1\right)V_i\left(f_2\right)V_i\left(f_3\right) \end{aligned} \tag{23c}$$

where $H_1^{(n)}$ stands for the n-dimensional Fourier transform of the network nonlinear impulse response $h_1^{(n)}$, $n=1,2,3,\ldots$. Moreover, V_i stands for the one-dimensional Fourier transform of the input signal v_i.

From (15a-c), we get

$$V_2^{(1)}\left(f_1\right)=H_H^{(1)}\left(f_1\right)V_i\left(f_1\right) \tag{24a}$$

$$V_2^{(2)}\left(f_1,f_2\right)=H_H^{(2)}\left(f_1,f_2\right)V_i\left(f_1\right)V_i\left(f_2\right) \tag{24b}$$

$$V_2^{(2)}\left(f_1,f_2,f_3\right)=H_H^{(2)}\left(f_1,f_2,f_3\right)V_i\left(f_1\right)V_i\left(f_2\right)V_i\left(f_3\right)\,. \tag{24c}$$

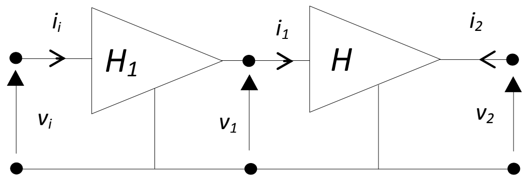

Fig. 2. Cascade connection of two circuit blocks.

And finally, we have from (16)

$$\begin{cases} I_1^{(n)}\left(f_1,\ldots,f_n\right) \\ V_2^{(n)}\left(f_1,\ldots,f_n\right) \end{cases}, \quad n=1,2,3,\ldots \tag{25}$$

where $I_1^{(n)}\left(f_1,\ldots,f_n\right)$ and $V_1^{(n)}\left(f_1,\ldots,f_n\right)$, $n=1,2,3,\ldots$, denote the successive n-dimensional Fourier transforms of the partial responses in the corresponding port current and voltage regarding the amplifier input-output type model.

V. Applications of In-Network Type Model of Nonlinear Amplifier

In-network type model of a nonlinear amplifier is destined for calculations of nonlinear transfer functions of circuits (networks) containing such an element. In the previous section, we developed two versions of this model. The first one, given by (21a-c), is suited to computer-aided calculations, for example, as an element stamp to be built in a computer program of nonlinear analysis [2], [12]. On the other hand, the second version, given by (23a-c), is suited to “hand and pencil” calculations (such as, for example, those presented in [5], [10], [11]). Here, we will present some remarks only on the latter.

First, observe that (23a-c) show transparently the dependence of the in-network type model of a nonlinear amplifier upon its context. This dependence is through the nonlinear transfer functions $H_1^{(1)}\left(f_1\right)$, $H_1^{(2)}\left(f_1,f_2\right)$, $H_1^{(3)}\left(f_1,f_2,f_3\right)$. We show now, on two illustrative examples, how the aforementioned model changes with changes in network topology.

In first example, we consider the topology of a cascade connection of two circuit blocks, as shown in Fig. 2, in two versions.

In first version, let H_1 in Fig. 2 denote a purely linear circuit (for example, a strictly linear amplifier preceding a weakly nonlinear one). The nonlinear transfer functions of higher orders ($n\geq 2$) of such a circuit are identically equal to zero, what simplifies (23b) and (23c) to

$$V_{2i}^{(2)}\left(f_1,f_2\right)=H_H^{(2)}\left(f_1,f_2\right)H_1^{(1)}\left(f_1\right)H_1^{(1)}\left(f_2\right)V_i\left(f_1\right)V_i\left(f_2\right) \tag{26}$$

and

$$\begin{aligned} V_{2i}^{(3)}\left(f_1,f_2,f_3\right)=&H_H^{(3)}\left(f_1,f_2,f_3\right)H_1^{(1)}\left(f_1\right)H_1^{(1)}\left(f_2\right)\cdot\\ &\cdot H_1^{(1)}\left(f_3\right)V_i\left(f_1\right)V_i\left(f_2\right)V_i\left(f_3\right) \end{aligned} \tag{27}$$

respectively. Obviously, the form of (23a) does not change in this case.

In second version, let H_1 in Fig. 2 denote a nonlinear circuit of which nonlinear transfer functions of higher orders ($n \geq 2$) are not identically equal to zero (at least those of the orders n=2 and n=3). Then, the form of expressions one uses is exactly that given by (23a-c).

In second example, we consider another fundamental topology, a feedback structure, as shown in Fig. 3.

In Fig. 3, the voltages v_i, v_1, and v_2 denote the port voltages, at the corresponding ports: *i*, 1, and 2. The feedback block K stands for a linear dynamic circuit (a linear circuit with memory) so it is fully characterized by its (linear) transfer function $K(f)$, with *f* meaning a frequency variable.

The nonlinear transfer functions $H_1^{(1)}$, $H_1^{(2)}$, and $H_1^{(3)}$ can be calculated for the feedback structure of Fig. 3 using one of the well-known methods published in the literature [4], [8]. The details of these derivations are omitted here. We present only the final results:

$$H_1^{(1)}(f_1) = \frac{1}{1 + H_H^{(1)}(f_1)K(f_1)} \tag{28a}$$

$$H_1^{(2)}(f_1,f_2) = \frac{-H_H^{(2)}(f_1,f_2)K(f_1+f_2)}{\left[1+H_H^{(1)}(f_1+f_2)K(f_1+f_2)\right]} \cdot \frac{1}{\left[1+H_H^{(1)}(f_1)K(f_1)\right]\left[1+H_H^{(1)}(f_2)K(f_2)\right]} \tag{28b}$$

$$\begin{aligned} H_1^{(3)}(f_1,f_2,f_3) = \\ = \frac{K(f_1+f_2+f_3)}{\left[1+H_H^{(1)}(f_1+f_2+f_3)K(f_1+f_2+f_3)\right]} \cdot \\ \cdot \frac{1}{\left[1+H_H^{(1)}(f_1)K(f_1)\right]\left[1+H_H^{(1)}(f_2)K(f_2)\right]} \cdot \\ \cdot \frac{1}{\left[1+H_H^{(1)}(f_3)K(f_3)\right]} \Big\{ -H_H^{(3)}(f_1,f_2,f_3) + \\ + \frac{H_H^{(2)}(f_1+f_2,f_3)H_H^{(2)}(f_1,f_2)K(f_1+f_2)}{\left[1+H_H^{(1)}(f_1+f_2)K(f_1+f_2)\right]} + \\ + \frac{H_H^{(2)}(f_1,f_2+f_3)H_H^{(2)}(f_2,f_3)K(f_2+f_3)}{\left[1+H_H^{(1)}(f_2+f_3)K(f_2+f_3)\right]} \Big\} \end{aligned} \tag{28c}$$

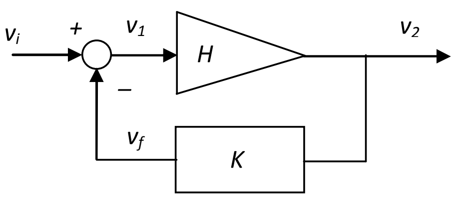

Fig. 3. Feedback topology consisting of a nonlinear amplifier H and a linear block K.

Applying (28a-c) in (23a-c) gives

$$H_2^{(1)}(f_1) = \frac{V_{2i}^{(1)}(f_1)}{V_i(f_1)} = \frac{H_H^{(1)}(f_1)}{1+H_H^{(1)}(f_1)K(f_1)} \tag{29a}$$

$$\begin{aligned} H_2^{(2)}(f_1,f_2) = \frac{V_{2i}^{(2)}(f_1,f_2)}{V_i(f_1)V_i(f_2)} = \\ = \frac{H_H^{(2)}(f_1,f_2)}{\left[1+H_H^{(1)}(f_1+f_2)K(f_1+f_2)\right]} \cdot \\ \cdot \frac{1}{\left[1+H_H^{(1)}(f_1)K(f_1)\right]\left[1+H_H^{(1)}(f_2)K(f_2)\right]} \end{aligned} \tag{29b}$$

$$\begin{aligned} H_2^{(3)}(f_1,f_2,f_3) = \frac{V_{2i}^{(3)}(f_1,f_2,f_3)}{V_i(f_1)V_i(f_2)V_i(f_3)} = \\ = \frac{1}{\left[1+H_H^{(1)}(f_1+f_2+f_3)K(f_1+f_2+f_3)\right]} \cdot \\ \cdot \frac{1}{\left[1+H_H^{(1)}(f_1)K(f_1)\right]\left[1+H_H^{(1)}(f_2)K(f_2)\right]} \cdot \\ \cdot \frac{1}{\left[1+H_H^{(1)}(f_3)K(f_3)\right]} \Big\{ H_H^{(3)}(f_1,f_2,f_3) - \\ - \frac{H_H^{(2)}(f_1+f_2,f_3)H_H^{(2)}(f_1,f_2)K(f_1+f_2)}{\left[1+H_H^{(1)}(f_1+f_2)K(f_1+f_2)\right]} - \\ - \frac{H_H^{(2)}(f_1,f_2+f_3)H_H^{(2)}(f_2,f_3)K(f_2+f_3)}{\left[1+H_H^{(1)}(f_2+f_3)K(f_2+f_3)\right]} \Big\} \end{aligned} \tag{29c}$$

Finally, note that the right hand sides of (29a-c) are the expressions determining the first three nonlinear transfer functions $H_2^{(1)}$, $H_2^{(2)}$, and $H_2^{(3)}$ of the whole feedback structure of Fig. 3, from port *i* to port 2.

Remarks:

1. The nonlinear transfer functions (29a-c) represent the input-output type description (in the multi-frequency domains) of the whole feedback configuration of Fig. 3.
2. Similarly, the nonlinear transfer functions $H_1^{(1)}$, $H_1^{(2)}$, and $H_1^{(3)}$ describing the "context" network of the amplifier H in Fig. 3, and given by (28a-c), represent the input-output type model, too. Because they regard the relation existing between the (output) port voltage v_1 and the (input) voltage v_i.
3. The examples presented above show how the in-network and input-output types of descriptions are involved with each other.

Characteristic for the descriptions (models) developed here for a (weakly) nonlinear amplifier is that they are general, opposite to those presented in [5], [10], [11]. So they can be applied not only to calculations of harmonic distortion, but also of any other nonlinear distortion measure, as for example intermodulation distortion, signal compression, cross modulation etc. Each of these measures has own peculiarities, which however can be exploited in a way for making their calculations more effective. We illustrate this point in the next section sowing how to simplify the harmonic distortion calculations in the feedback structure of Fig. 3.

VI. Specialization of General Relations for Harmonic Distortion and Feedback Structure of Fig. 3

As known [4], the Volterra series can be expressed in a more convenient form for circuit analysis by using in it nonlinear transfer functions instead of nonlinear impulse responses. So applying this approach to (8a) and (8b), we get

$$\begin{aligned}v_2(t) &= \int_{-\infty}^{\infty} H_2^{(1)}(f_1)V_i(f_1)\exp(j2\pi f_1 t)df_1 + \\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_2^{(2)}(f_1,f_2)V_i(f_1)V_i(f_2)\exp(j2\pi(f_1t+f_2t))df_1df_2 + \\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_2^{(2)}(f_1,f_2,f_3)V_i(f_1)V_i(f_2)\cdot \\ &\cdot V_i(f_3)\exp(j2\pi(f_1t+f_2t+f_3t))df_1df_2df_3 + \dots \quad .\end{aligned} \tag{30}$$

Then, we assume that the input signal $v_i(t)$ in the feedback structure of Fig. 3 has the following form

$$v_i(t) = A_i\exp(j2\pi f_o t) \Leftrightarrow V_i(f) = A_i\delta(f-f_o) \tag{31}$$

where a relation with its ordinary (one-dimensional) Fourier transform $V_i(f)$ is shown, too. A_i in (31) is a real number and means the amplitude of this harmonic signal, but $f_o = \omega_o/(2\pi)$ means its frequency (and ω_o its angular frequency). Furthermore, δ means the Dirac impulse and *f* is the current frequency in its Fourier transform.

Applying (31) in (30) with $H_2^{(1)}$, $H_2^{(2)}$, and $H_2^{(3)}$ given by (29a-c), we obtain

$$\begin{aligned}v_2(t) &= a_{1f}(j\omega_o)v_i(t) + a_{2f}(j\omega_o)(v_i(t))^2 + \\ &+ a_{3f}(j\omega_o)(v_i(t))^3 + \dots \quad .\end{aligned} \tag{32}$$

where the coefficients $a_{1f}(j\omega_o)$, $a_{2f}(j\omega_o)$, and $a_{3f}(j\omega_o)$ are given by

$$a_{1f}(j\omega_o) = H_2^{(1)}(f_o) = \frac{H_H^{(1)}(f_o)}{1+H_H^{(1)}(f_o)K(f_o)} \tag{33a}$$

$$\begin{aligned}a_{2f}(j\omega_o) &= H_2^{(2)}(f_o,f_o) = \\ &= \frac{H_H^{(2)}(f_o,f_o)}{\left[1+H_H^{(1)}(f_o)K(f_o)\right]^2\left[1+H_H^{(1)}(2f_o)K(2f_o)\right]}\end{aligned} \tag{33b}$$

$$\begin{aligned}a_{3f}(j\omega_o) &= H_2^{(3)}(f_o,f_o,f_o) = \\ &= \frac{1}{\left[1+H_H^{(1)}(f_o)K(f_o)\right]^3\left[1+H_H^{(1)}(3f_o)K(3f_o)\right]}\cdot \\ &\cdot\left\{H_H^{(3)}(f_o,f_o,f_o) - \frac{H_H^{(2)}(2f_o,f_o)H_H^{(2)}(f_o,f_o)K(2f_o)}{\left[1+H_H^{(1)}(2f_o)K(2f_o)\right]} - \right. \\ &\left. - \frac{H_H^{(2)}(f_o,2f_o)H_H^{(2)}(f_o,f_o)K(2f_o)}{\left[1+H_H^{(1)}(2f_o)K(2f_o)\right]}\right\} \quad .\end{aligned} \tag{33c}$$

Note that (32) is a power series like representation found by Palumbo and Pennisi [10] to be a very useful tool in harmonic distortion analysis of analog dynamic weakly nonlinear circuits. Moreover, (33a-c) are the expressions for the *closed-loop nonlinear coefficients* (so called in [10]) of a feedback structure in Fig. 3.

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