

L.B. MAGALAS^{*,‡}, M. MAJEWSKI^{*}**HILBERT-TWIN – A NOVEL HILBERT TRANSFORM-BASED METHOD TO COMPUTE ENVELOPE OF FREE DECAYING OSCILLATIONS EMBEDDED IN NOISE, AND THE LOGARITHMIC DECREMENT IN HIGH-RESOLUTION MECHANICAL SPECTROSCOPY HRMS****HILBERT-TWIN – NOWA METODA OBLICZEŃ OBWIEDNI DRGAŃ SWOBODNIE TŁUMIONYCH ZAWIERAJĄCYCH SZUM I LOGARYTMICZNEGO DEKREMENTU TŁUMIENIA W WYSOKOROZDZIELCZEJ SPEKTROSKOPII MECHANICZNEJ HRMS**

In this work, we present a novel Hilbert-twin method to compute an envelope and the logarithmic decrement, δ , from exponentially damped time-invariant harmonic strain signals embedded in noise. The results obtained from five computing methods: (1) the parametric OMI (Optimization in Multiple Intervals) method, two interpolated discrete Fourier transform-based (IpDFT) methods: (2) the Yoshida-Magalas (YM) method and (3) the classic Yoshida (Y) method, (4) the novel Hilbert-twin (H-twin) method based on the Hilbert transform, and (5) the conventional Hilbert transform (HT) method are analyzed and compared. The fundamental feature of the Hilbert-twin method is the efficient elimination of intrinsic asymmetrical oscillations of the envelope, $a_{HT}(t)$, obtained from the discrete Hilbert transform of analyzed signals. Excellent performance in estimation of the logarithmic decrement from the Hilbert-twin method is comparable to that of the OMI and YM for the low- and high-damping levels. The Hilbert-twin method proved to be robust and effective in computing the logarithmic decrement and the resonant frequency of exponentially damped free decaying signals embedded in experimental noise. The Hilbert-twin method is also appropriate to detect nonlinearities in mechanical loss measurements of metals and alloys.

Keywords: Logarithmic decrement, mechanical spectroscopy, Hilbert transform, envelope, interpolated discrete Fourier transform

W pracy przedstawiono nową metodę Hilbert-twin, opartą na dyskretniej transformacji Hilberta, do obliczeń obwiedni wykładniczo tłumionych sygnałów odkształceń sprężystych zawierających w sobie szum oraz do estymacji logarytmicznego dekrementu tłumienia. Przeanalizowano i porównano wyniki obliczeń uzyskane z pięciu różnych metod: (1) metoda parametryczna OMI (Optimization in Multiple Intervals), dwie metody bazujące na interpolowanej dyskretniej transformacji Fouriera (IpDFT): (2) metoda Yoshida-Magalas (YM) i (3) klasyczna metoda Yoshidy (Y), (4) nowa metoda Hilbert-twin (H-twin), którą po raz pierwszy przedstawiono w niniejszej pracy oraz (5) klasyczna metoda obliczeń obwiedni z transformaty Hilberta (HT).

Zaletą i fundamentalną cechą charakterystyczną metody H-twin jest skuteczne usunięcie typowych dla dyskretniej transformaty Hilberta asymetrycznych oscylacji obwiedni. Z tego właśnie względu metoda H-twin zapewnia bardzo dobrą estymację logarytmicznego dekrementu tłumienia, która jest porównywalna z metodami OMI i YM zarówno dla niskich, jak i wysokich poziomów tłumienia. Metoda H-twin jest niewrażliwa na szum i jest wyjątkowo skuteczna w precyzyjnym wyznaczeniu logarytmicznego dekrementu tłumienia oraz częstotliwości rezonansowej wykładniczo tłumionych drgań swobodnie tłumionych zawierających szum eksperymentalny.

Metoda H-twin może również służyć do detekcji i analizy efektów nieliniowych występujących w trakcie pomiarów rozpraszania energii mechanicznej w metalach i stopach metali badanych metodą spektroskopii mechanicznej.

1. Introduction

In mechanical spectroscopy studies, it is often difficult to access experimental fine details of mechanical losses, such as the asymmetrical broadening of the Snoek-Köster peaks in deformed Fe-C alloys [1,2] and various steels containing martensite, hardly discernable overlapping relaxation peaks [3], phase transition peaks, other non-linear mechanical losses, transient internal friction peaks, etc. Accurate analysis of these mechanical losses is critical for the understanding of the physical

mechanisms that govern internal friction peaks in metals and alloys. For this reason, further progress in precise estimation of the logarithmic decrement is required. New methods and algorithms to compute the logarithmic decrement are obligated to pave the way for the high-resolution mechanical spectroscopy, HRMS.

The Hilbert transform (HT) is inherent in the theoretical concept of mechanical spectroscopy and in the physical picture of mechanical relaxation phenomena, *viz.*, the real and the imaginary part of the complex compliance are re-

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lated via the Hilbert transform [4,5]. This paper describes an endeavor to use the Hilbert transform [6,7] to compute the logarithmic decrement [2,10-13], δ , from exponentially damped time-invariant harmonic oscillations embedded in the experimental noise, $\varepsilon_w(t)$, [10-14] and recorded in a low-frequency mechanical spectrometer (inverted torsion pendulum [1,2,4,5,8-13]). In this paper, damped harmonic oscillations containing noise are referred to as the ‘free-elastic decay’. It is demonstrated that the Hilbert transform of free-elastic decay yields deteriorated envelope containing detrimental ripples (oscillations) [15-18]. This then give rise to substantial scatter of the δ values obtained from the envelope estimated according to the Hilbert transform. To circumvent this problem, a novel Hilbert-twin method (herein referred to as the H-twin) is developed in the present work. The H-twin method provides the ‘true envelope’, and thus shows excellent performance in the computation of the logarithmic decrement, δ . In the present study the δ values are computed according to the following methods:

(1) the OMI (Optimization in Multiple Intervals) [8,9], (2) the YM (Yoshida-Magalas) [12-14] and (3) the Y (Yoshida) based on interpolated discrete Fourier transform, IpDFT [12-14,19], (4) the H-twin (Hilbert-twin), and (5) the HT (the Hilbert transform) [6,7]. The estimated values of δ are compared. To ensure meaningful comparison of the results, all computations are performed with (1) the same experimental conditions, (2) the same parameters used in signal processing, such as the length of the analyzed signal, the number of oscillations N_{osc} (the number of periods), the sampling frequency f_s , (3) the signal-to-noise ratio, S/N, (4) the maximal strain amplitude, A_0 , and the same signal processing technique [8-14].

2. Experimental

Hitherto computations of the mechanical loss in terms of the dimensionless logarithmic decrement, δ , and the elastic modulus, M , in terms of the resonant frequency, f_0 , ($M \sim f_0^2$) discard the presence of noise in free decaying oscillations recorded in resonant mechanical spectrometers. The logarithmic decrement, δ , can be determined from Eq. (1) using the digitized data $A_i(t)$ and t_i from free-elastic decay:

$$A(t) = A_0 e^{-\delta f_0 t} \cos(2\pi f_0 t + \phi) + \varepsilon_w(t) + dc. \quad (1)$$

Here, A_0 stands for the maximal strain amplitude, t states for a continuous time in seconds, $-\pi < \varphi \leq \pi$ is the phase in radians, and dc denotes an offset or the slow-varying trend. The zero-point drift (non-harmonic distortion), is neglected here [10]. The additive white noise, $\varepsilon_w(t)$, is described by the signal-to-noise ratio S/N [8-14].

The free-elastic decaying signal $A(t)$ is only one of possible projections, that is, a real part of some analytical signal $z(t)$ [7]. The quadrature projection of this signal (the imaginary part $\hat{A}(t)$) is conjugated according to the Hilbert transform [6,7]. Instantaneous attributes of the free-elastic decaying signal $A(t)$, that is, its envelope and instantaneous phase are discussed in this work to estimate the logarithmic decrement. The complex analytic signal:

$$z(t) = A(t) + i \hat{A}(t), \quad (2)$$

can be represented in its exponential form,

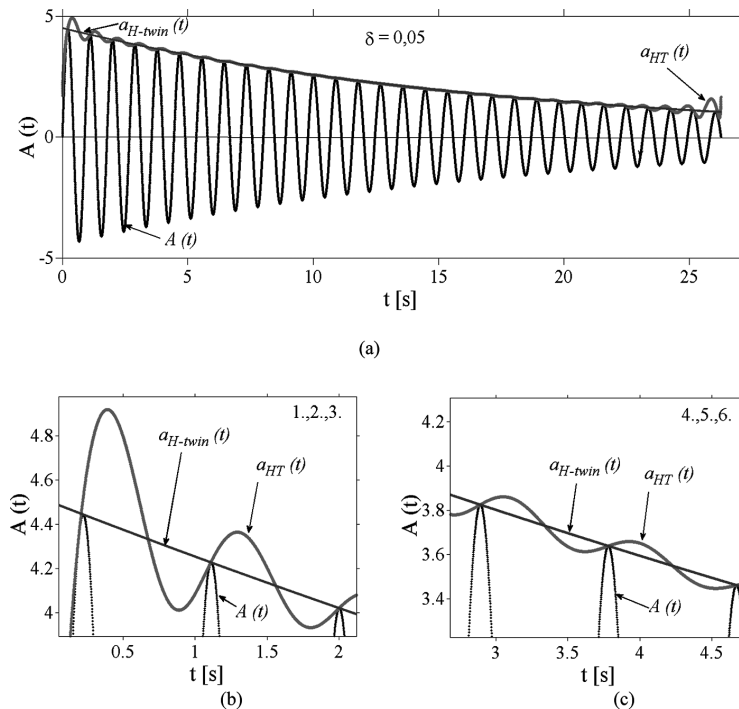


Fig. 1. Illustration of the theoretical exponentially damped harmonic signal $A(t)$ (black) ($\delta = 0.05$, the resonant frequency $f_0 = 1.12345$ Hz, the sampling frequency $f_s = 1$ kHz) and two envelope-lines obtained from the Hilbert transform, $a_{HT}(t)$ (red), and from the Hilbert-twin method, $a_{H-twin}(t)$ (blue). Persistent ripples on the $a_{HT}(t)$ envelope (red) obtained from the discrete Hilbert transform are aptly demonstrated for the first six oscillations (Figs. 1b and 1c). The flattened true envelope, $a_{H-twin}(t)$ (blue), is estimated from the H-twin method. (For interpretation of the references to color in this figure legend, readers are referred to the web version of this article.)

$$z(t) = a_{HT}(t) e^{i\varphi_{inst}(t)}, \quad (3)$$

where $a_{HT}(t)$ is the envelope of the free-elastic decay $A(t)$ (Fig. 1) and $\varphi_{inst}(t)$ denotes the instantaneous phase of the Hilbert transform. The instantaneous phase $\varphi_{inst}(t)$ [6,7] is given by

$$\varphi_{inst}(t) = \arctg\left(\frac{\hat{A}(t)}{A(t)}\right). \quad (4)$$

The envelope contains information about the energy of the free-elastic decay $A(t)$. The logarithmic decrement, δ , can be obtained from the envelope $a_{HT}(t)$ of the free-elastic decay:

$$a_{HT}(t) = \sqrt{A(t)^2 + \hat{A}(t)^2} = A_0 e^{-\delta \cdot f_0 \cdot t}. \quad (5)$$

The plot of $\log[a_{HT}(t)]$ versus t is a straight line. Thus, the least-squares curve fitting yields the logarithmic decrement and the maximum strain amplitude A_0 . The imaginary part of the free-elastic decay, $\hat{A}(t)$, is obtained numerically from the Fourier transform [7]. It is tacitly assumed that the resonant frequency, f_0 , is estimated from the YM or Hilbert-twin method.

Two envelopes obtained from the Hilbert transform, $a_{HT}(t)$, and from the Hilbert-twin method, $a_{H-twin}(t)$, are shown in Figs. 1 and 2. Selected parts of the envelopes are enlarged to demonstrate oscillatory behavior of the envelope computed from the Hilbert transform, $a_{HT}(t)$, and the true envelope, $a_{H-twin}(t)$. It is anticipated that two envelopes yield different values of the logarithmic decrement. The comparison of the performance of five different computing methods of the logarithmic decrement is carried out for the same signal, $A(t)$, characterized by the logarithmic decrement $\delta = 0.05$, and the resonant frequency $f_0 = 1.12345$ Hz. The signal, $A(t)$, is digitally sampled with the sampling frequency $f_s = 1$ kHz. The length of the digitized signal $A(t)$ is finite, determined by an experimentalist.

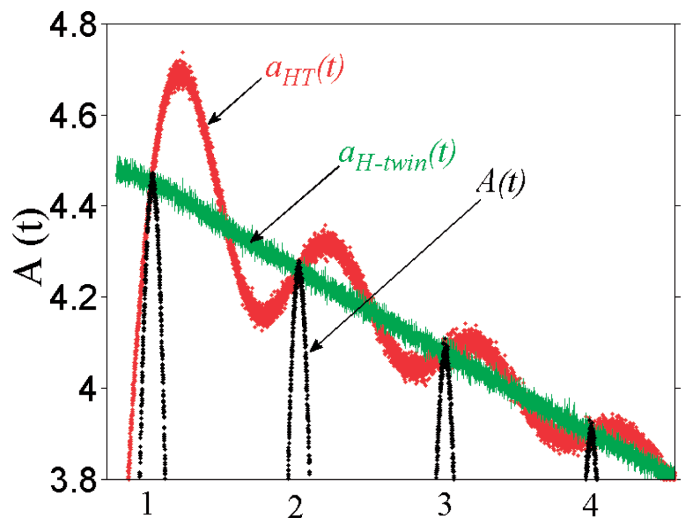
In the following section it is demonstrated that the logarithm of the true envelope, $a_{H-twin}(t)$, yields excellent estimation of the logarithmic decrement for the exponentially damped harmonic strain signal $A(t)$, embedded in noise, as described by Eq. (1). The results are demonstrated for a theoretical signal (Fig. 1) and the signal with white noise, S/N = 32 dB (Fig. 2).

3. The true envelope of exponentially damped time-invariant harmonic oscillations embedded in noise

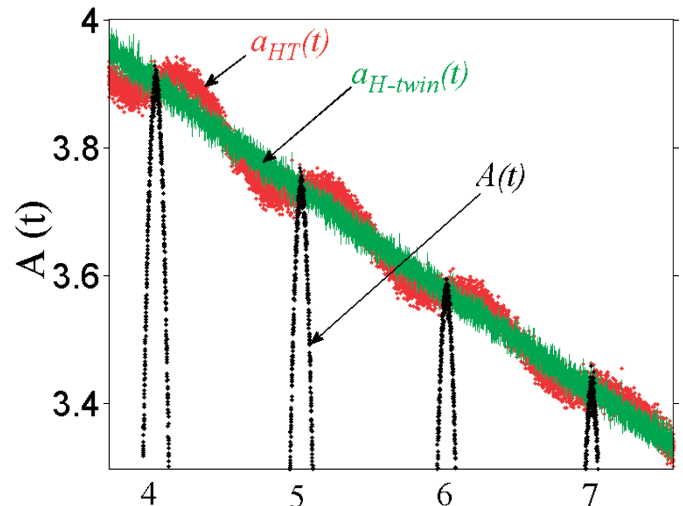
Computation of the true envelope, which must be a straight line in a semi-logarithmic plot of envelope versus time is a mathematically challenging task. This problem is unequivocally solved, in this work, by using the ‘twinning procedure’ (double twinning of the digitized real-time free-elastic decaying signal $A(t)$ in the time-domain), and further, by computing the imaginary part, $\hat{A}(t)$, for the new input data. Thus, the Hilbert twin method doubles the length of the analyzed signal and computes the discrete Hilbert transform for the entire signal. We impose a restriction, that is, the number of samples in the $A(t)$ signal is equal to the number of samples in the twinned signal. It is therefore essential to use appropriate

sampling frequency, f_s . These have proven to be sufficient to obtain the logarithmic decrement with the requested negligible estimation error as required in high-resolution mechanical spectroscopy, HRMS. The envelope $a_{HT}(t)$ has a maximum magnitude of oscillatory behavior at the beginning and the end of the signal $A(t)$, with decreasing oscillatory behavior in the vicinity of the middle part of the signal. Figures 1b, 2a, and 2b indicate that the H-twin method eliminates effectively oscillatory behavior of envelope $a_{HT}(t)$ at both ends. For high- and very high-damping levels, a slight deviation of the true envelope, $a_{H-twin}(t)$, is visible at the very beginning of the $A(t)$ signal (Fig. 2a; green envelope, $a_{H-twin}(t)$, corresponding to the beginning of the first period).

Figure 2 demonstrates the so-called true envelope, $a_{H-twin}(t)$, obtained from the H-twin method. The fundamental feature of the Hilbert-twin method is the efficient elimination of ‘ripples’, that is, intrinsic asymmetrical oscillations of the envelope, $a_{HT}(t)$, obtained from discrete Hilbert transform (HT) of the signal $A(t)$, illustrated by red curves in Figs. 1 and 2. Thereby the logarithm of the true envelope yields excellent estimation of the logarithmic decrement for low- and high-damping levels alike. In addition, the least-square linear



(a)



(b)

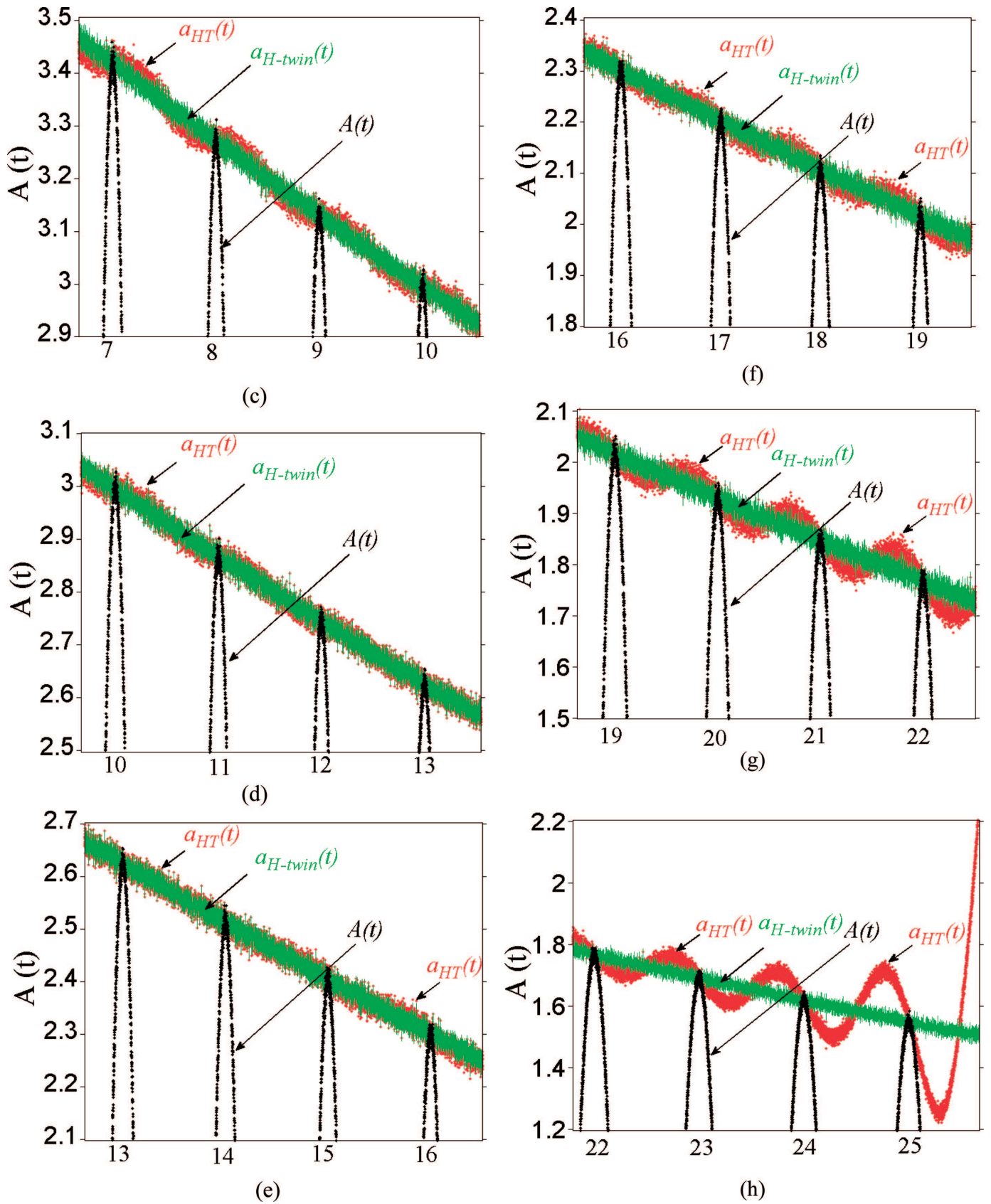


Fig. 2. Two envelope-lines calculated according to the Hilbert transform, $a_{HT}(t)$ (red), and the Hilbert-twin method, $a_{H-twin}(t)$ (green), for free-elastic decaying signal $A(t)$ (black); $\delta = 0.05$, $S/N = 32$ dB, $f_0 = 1.12345$ Hz, $f_s = 1$ kHz. Persistent ripples (oscillations) on the envelope, $a_{HT}(t)$, are clearly shown for a sequence of periods (Figs. 2 a-h). Detrimental effect of ripples is negligible in the middle part of the signal $A(t)$ (Fig. 2e). The H-twin method yields the ‘true envelope’ $a_{H-twin}(t)$ (green) for all damping levels. (For interpretation of the references to color in this figure legend, readers are referred to the web version of this article.)

regression used to compute the logarithmic decrement from Eq. (5) averages out zero-mean noise.

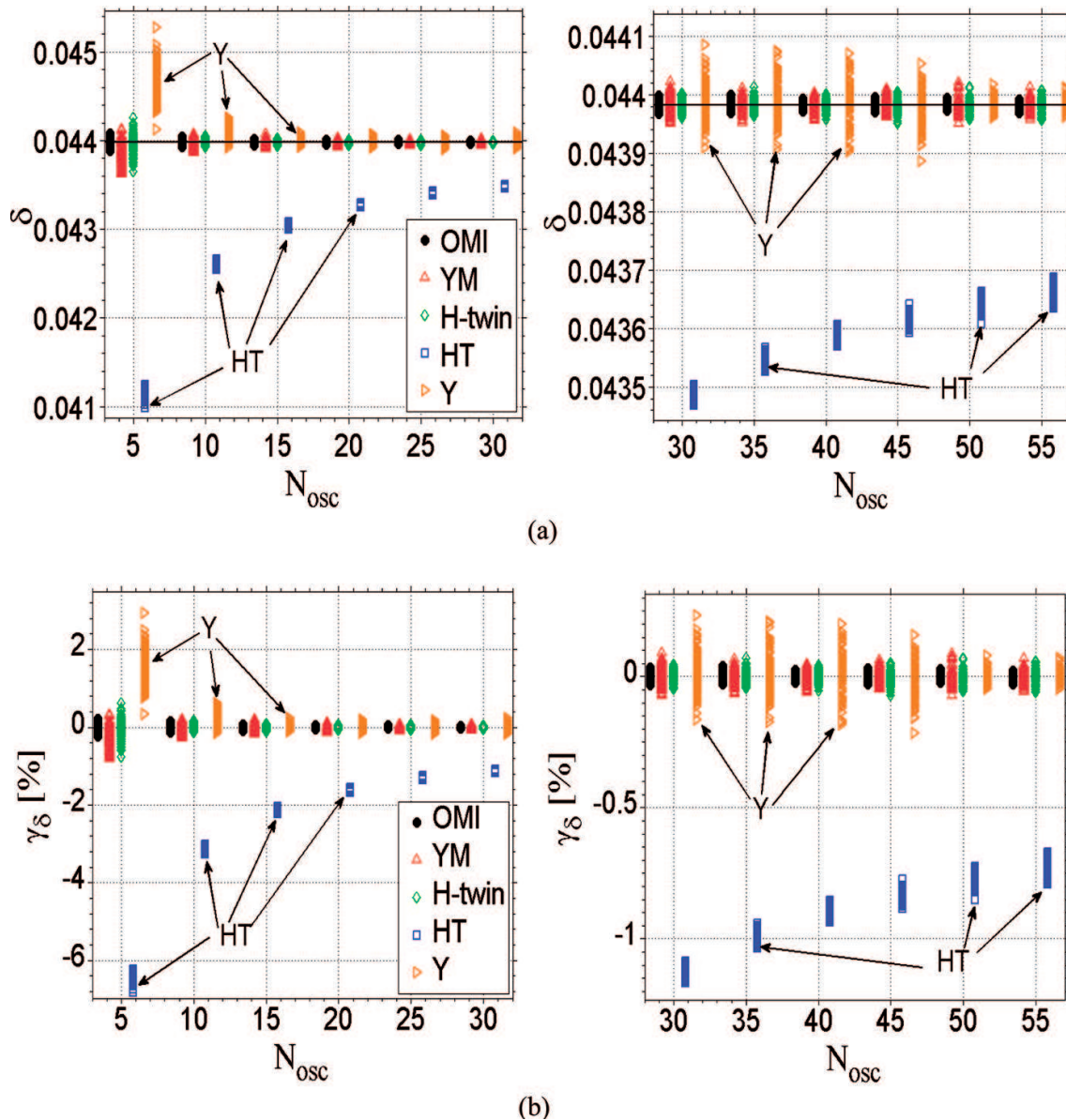
We note that oscillatory behaviour of the envelope, $a_{HT}(t)$, for low values of the logarithmic decrement is fairly easy to be eliminated by the H-twin method.

We should emphasize that the Hilbert-twin method does not use any low-pass filtering or averaging [20-24]. But nevertheless, the true envelope (straight-line envelope) estimated from the Hilbert-twin method, depicted in Figs. 1 and 2, is free of ripples and distortions.

The Hilbert-twin method can also be routinely used to check whether the envelope analyzed for digitized data obtained in a resonant mechanical spectrometer is accurately exponential to distinguish nonlinearities either in the mechanical loss (non-linear internal friction phenomena) or in the measurement system.

4. Comparison of methods

Figure 3 illustrates the performance of the OMI, YM, H-twin, HT, and Y methods used to estimate the internal friction, $Q^{-1} = 0.014$. Dispersion in internal friction values around $Q^{-1} = 0.014$ is reported in Refs. [25,26] and assessed in [14]. The relationship between the internal friction, Q^{-1} , and the logarithmic decrement, δ , is well known [4,10,27]. The conventional Hilbert transform method is available in many commercial software packages. The HT method, however, has major drawbacks caused by the oscillatory behavior of the envelope (i.e., presence of undesirable ripples on the envelope), which strongly affect calculation of the logarithmic decrement. It is important to emphasize that the discrete Hilbert transform method does not yield correct values of δ for both short and long free decaying signals in the time scale, as clearly shown in Fig. 3. Note that the HT method underestimates the logarithmic decrement (Fig. 3a). The results shown in Fig. 3 unequivocally demonstrate that an increase



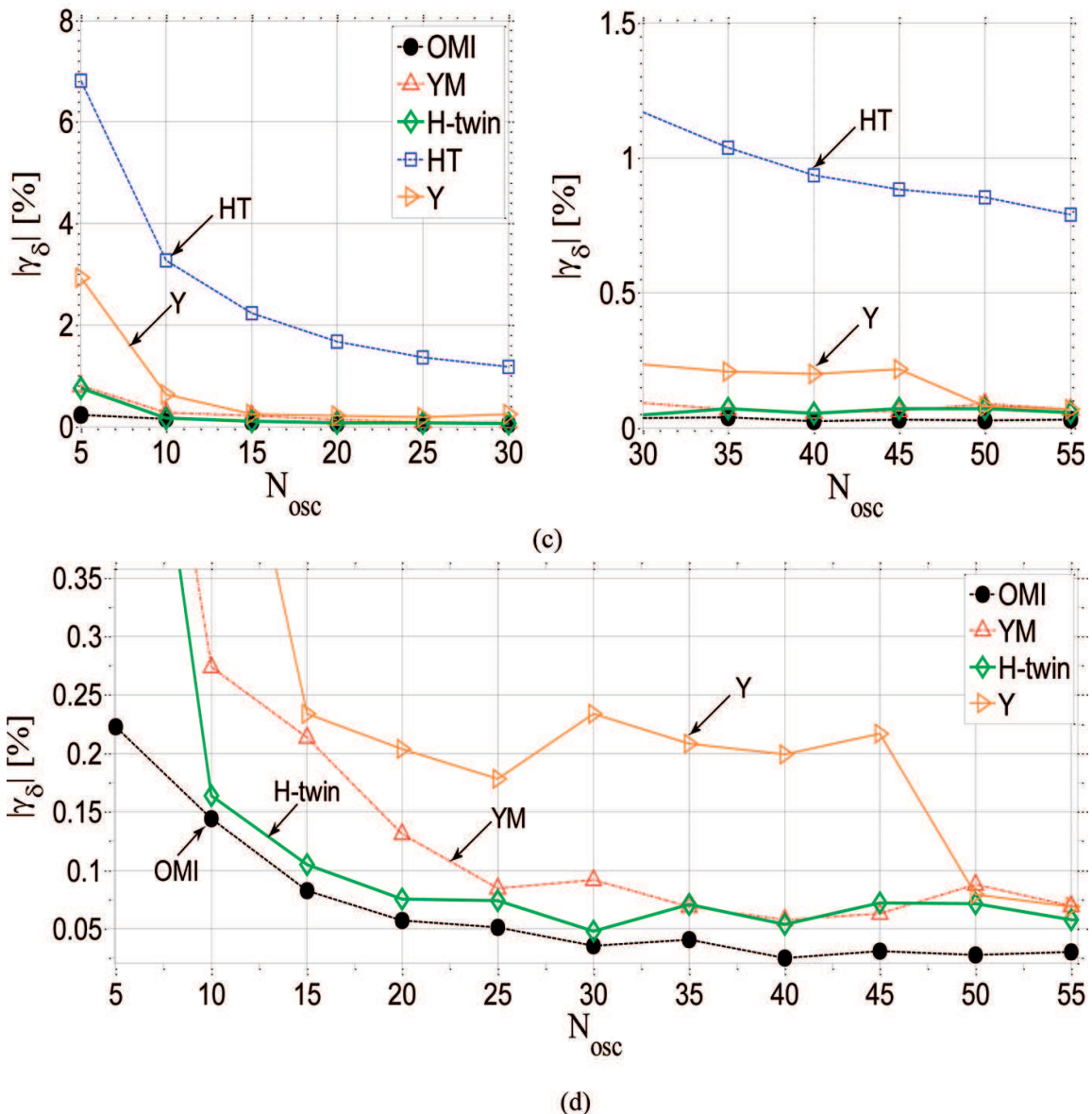


Fig. 3. (a) Dispersion of the logarithmic decrement, δ , estimated for 100 free-elastic decays for $\delta = 0.04398229$ (i.e., $Q^{-1} = 0.014$ [28,29]), $f_0 = 1.12345$ Hz, $f_s = 3$ kHz, $S/N = 32$ dB according to the OMI (\bullet , black), YM (Δ , red), H-twin (\diamond , green), Yoshida (\triangleright , brown), and HT (\square , blue) methods as a function of the number of free decaying oscillations N_{osc} (i.e., the number of periods). Each vertical row comprises 100 points of the estimated logarithmic decrement for each method, from left to right. (b) The relative error in estimation of the logarithmic decrement. 100 results are illustrated in vertical rows, from left to right. (c) The maximum relative error, $|\gamma_\delta|$, of the logarithmic decrement, δ . (d) The maximum relative error, $|\gamma_\delta|$, below the level of 1%, is illustrated with zoom (the largest relative error values obtained for the Hilbert transform method are outside the figure.)

in the number of oscillations N_{osc} from 5 to 15 drastically reduces the relative error in the estimation of the logarithmic decrement for Fourier transform-based methods. This effect is strong for the HT and Y methods (Fig. 3c) and less pronounced for the YM and H-twin methods (Fig. 3d). These features may be explained by the intrinsic property of the discrete Fourier transform employed in these methods, that is, the leakage and distortion effects, which are typical for free decaying signals that have short time span. The H-twin method is robust against this phenomenon (Fig. 3d). Further increase in the number of oscillations from 15 to 35 reduces the relative error, whereas further increase from 35 to 60 improves precision only slightly (Fig. 3d). To conclude, the computational precision of different methods depends on the length of exponentially damped free decaying oscillations or the number of periods.

Figure 3d indicates that the H-twin method is slightly superior to the YM method. Note that the relative error is less for the proposed H-twin method, than the interpolated discrete Fourier transform-based YM method for time span lower than 35 periods. Increasing the number of periods over 30 results in only modest improvement in the accuracy obtained.

It is important to mention that the parametric OMI method and the YM method are considered as gold standards in mechanical spectroscopy. This work demonstrates that, in certain situations, the Hilbert-twin method shows similar, excellent performance (e.g., for the length of free-elastic decaying signal higher than 10 periods). It should be mentioned, however, that the H-twin algorithm is computationally intensive compared to the Hilbert transform and the interpolated discrete Fourier transform-based methods (YM and Y).

The OMI and YM methods have been successfully tested under various experimental conditions [8-14,28]. The H-twin method has been tested in numerous mechanical loss measurements and investigation of Snoek dislocation-enhanced Snoek peak and Snoek-Köster peak, since 1996 [2]. Further work is needed, however, to validate the H-twin method for other extreme experimental conditions, that is, for the low- (Q^{-1} from 10^{-4} to 10^{-5}), and extreme low- (Q^{-1} from 10^{-8} to 10^{-9}) damping levels.

In conclusion, it is worthwhile to reiterate the fact that the OMI, H-twin and YM methods are substantially more reliable and precise than the classical methods used to compute the logarithmic decrement [8-11] and the conventional Hilbert transform method [6,7].

5. Conclusions

This paper describes practical advantages of the novel Hilbert-twin method to compute the logarithmic decrement from the true envelope, $a_{H-twin}(t)$, of exponentially damped time-invariant free decaying oscillations embedded in experimental noise. A fairly robust numerical procedure is developed that is based on the use of the twinning procedure in the time-domain of free-decaying signal, followed by computation of the Hilbert transform of the original free decaying signal extended by its twinned signal. The H-twin method eliminates oscillatory behavior of the envelope, $a_{HT}(t)$, computed from the conventional discrete Hilbert transform. To put it in a different way, the H-twin method yields the **true envelope**, which is perfectly linear on a semi-logarithmic plot of the envelope versus time. Hence, not surprisingly, the performance of the Hilbert-twin method is comparable to the parametric OMI method and the interpolated discrete Fourier transform YM method. It is unequivocally demonstrated that the Hilbert-twin method outperforms two non-parametric methods such as the conventional discrete Hilbert transform (HT) method and the Yoshida (Y) method. The OMI, YM, H-twin, HT, and Y methods are collected in [28].

It is advocated that the Hilbert-twin method can also be used to detect nonlinearities in resonant and subresonant mechanical loss measurements of a sample (non-linear internal friction phenomena in solids) or in the measurement system of a mechanical spectrometer. We have not pursued this line of investigation in the current work, since non-linear mechanical spectroscopy has not yet established solid theoretical and experimental backgrounds.

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