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Control parameters for the quantum Deutsch algorithm

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Abstract: An example of two-qubit scenario for finding an optimal control parameters on a spin chain to implement the quantum Deutsch algorithm is provided. Two cases are studied in this paper: two-qubit and three-qubit systems. The latter case is used to study the impact of interaction with an environment on the outcome of the algorithm.

Keywords: quantum informatics, quantum information processing, quantum algorithms

1. Introduction

The goal of this work is to analyse influence of environment on implementation on a Heisenberg spin chain of the quantum Deutsch algorithm. A toy model of interaction is proposed. Its application shows the importance of considering external interactions when implementing quantum algorithms.

The paper is organised as follows. In section 2 some mathematical preliminaries needed to understand the work are presented. In section 3 the idea of the Deutsch algorithm is presented. Section 4 contains detailed description of the simulated system. Section 5 contains results from numerical simulations preformed on the proposed system. In section 6 the final conclusions are drawn.

2. Preliminaries

A qubit is an elementary unit of information in quantum computation. Its state is defined as a normalised vector in a two-dimensional Hilbert space

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{1}$$

where vectors $|0\rangle$ and $|1\rangle$ form the so called computational basis, $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha, \beta \in \mathbb{C}$. Their explicit form is

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2}$$

Numbers $|\alpha|^2$ and $|\beta|^2$ denote probabilities of finding the qubit in state $|0\rangle$ or $|1\rangle$ respectively when measured in computational basis.

If the system is composed of two or more qubits, its composite state is the tensor product of all qubits forming it. For a two qubit system where one is in state $|\psi\rangle$ and the other in state $|\phi\rangle$, the state of the entire system is

$$|\xi\rangle = |\psi\rangle \otimes |\phi\rangle = |\psi\phi\rangle, \tag{3}$$

the latter part of equation (3) is a shorter version of the notation.

Operations on qubits are realised through unitary matrices, called quantum gates. An example of such gate acting on a single qubit is the quantum not gate. One has

$$NOT|0\rangle = |1\rangle,\ NOT|1\rangle = |0\rangle. \tag{4}$$

Another example is the Hadamard gate H, which does not have a classical analogue

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},\ H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{5}$$

There also exists a number of gates acting on two qubits. One important example is the controlled not gate [1], usually denoted $CNOT$. It applies the NOT operation to the second qubit if and only if the first qubit is in $|1\rangle$ state. All possible results of applying this gate are summarised in equations (6)–(9)

$$CNOT|00\rangle = |00\rangle, \tag{6}$$
$$CNOT|01\rangle = |01\rangle, \tag{7}$$
$$CNOT|10\rangle = |11\rangle, \tag{8}$$
$$CNOT|11\rangle = |10\rangle. \tag{9}$$

Systems referred to as spin chains [2] and spin graphs [3] are currently deeply studied in terms of implementing quantum algorithms. Time evolution of such systems is described by Schrödinger's equation:

$$H|\psi\rangle = i\hbar\frac{\partial|\psi\rangle}{\partial t}, \tag{10}$$

where $\hbar$ is the Dirac constant. For simplicity a system of units is chosen in which $\hbar = 1$. H is a Hermitian matrix, the Hamiltonian of the system.

It can be shown from (10) that the time evolution is governed by

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle. \tag{11}$$

It is a well known fact that if H is hermitian, then e^{-iH} is unitary [4], so one can define a time evolution operator $U = e^{-iHt}$.

To implement a quantum algorithm, one can construct a Hamiltonian consisting of two terms

$$H = H_0 + H_c, \tag{12}$$

where H_0 is called the drift Hamiltonian and can be obtained using the Heisenberg model of interaction between qubits [5]. H_c is the control Hamiltonian, which describes interaction with external factors like laser or radio pulses. Depending on the type of interaction described by H_0 the model is called either a spin chain or a spin graph. If one includes only the nearest neighbour interactions the system is called a spin chain. If a given qubit interacts with, in general, all the other qubits, the system is called a spin graph.

The drift Hamiltonian of Heisenberg spin chain of length N is given by:

$$H_0 = J \sum_{i=1}^{N-1} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z, \tag{13}$$

where

$$S_i^k = \mathbb{1}^{\otimes i-1} \otimes \sigma^k \otimes \mathbb{1}^{\otimes N-i}, \tag{14}$$

where σ^k are the Pauli matrices given by $\sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma^y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \sigma^z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

3. Algorithm

The algorithm was first proposed in 1985 by David Deutsch [6]. The problem is as follows. Suppose one has a black box, usually called the oracle, that calculates a function $f : \{0, 1\} \to \{0, 1\}$. In order to check if the function is constant ($f(0) = f(1)$) or balanced ($f(0) = \overline{f(1)}$). In the classical case it is needed to ask the oracle twice to determine to which of the classes f belongs. In the quantum case it is sufficient to ask the oracle only once.

The algorithm is as follows:

1. Prepare a state: $|\Psi\rangle = |0\rangle \otimes |1\rangle$,

2. Apply the Hadamard gate $H^{\otimes 2}$ to the state $|\Psi\rangle$ what yields

$$|\Psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{15}$$

3. Apply a unitary gate $U_f : |x\rangle \otimes |y\rangle \rightarrow |x\rangle \otimes |f(x) \oplus y\rangle$ to the state $|\Psi_1\rangle$ what gives

$$|\Psi_2\rangle = \begin{cases} \pm\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f \text{ is constant,} \\ \pm\frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f \text{ is balanced.} \end{cases} \tag{16}$$

4. Apply $H \otimes \mathbb{1}$ to the state $|\Psi_2\rangle$ what gives:

$$|\Psi_3\rangle = \begin{cases} \pm|0\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f \text{ is constant,} \\ \pm|1\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f \text{ is balanced.} \end{cases} \tag{17}$$

5. Measure the value of the first qubit. If the outcome is $|0\rangle$ then the function is constant, if the outcome is $|1\rangle$ the function is balanced.

The quantum circuit for the algorithm is shown in Figure 1. Gate U_f implementing

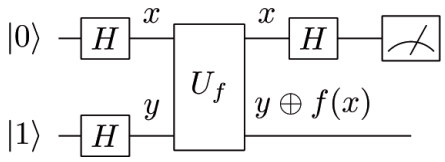

Fig. 1. Quantum circuit for the Deutsch algorithm.

function f is one of the following:

$$U_f = CNOT, \tag{18}$$
$$U_f = \mathbb{1} \otimes \mathbb{1}, \tag{19}$$
$$U_f = \mathbb{1} \otimes NOT, \tag{20}$$
$$U_f = (\mathbb{1} \otimes NOT)CNOT. \tag{21}$$

So the total unitary evolution of the system is described by one of the four gates:

$$U_T = (H \otimes \mathbb{1})CNOT(H \otimes H), \tag{22}$$
$$U_T = (H \otimes \mathbb{1})(\mathbb{1} \otimes \mathbb{1})(H \otimes H), \tag{23}$$
$$U_T = (H \otimes \mathbb{1})(\mathbb{1} \otimes NOT)(H \otimes H), \tag{24}$$
$$U_T = (H \otimes \mathbb{1})(\mathbb{1} \otimes NOT)CNOT(H \otimes H). \tag{25}$$

4. System and simulation

The control part of the Hamiltonian given by (12) is chosen to be

$$H_c(t) = h_x(t)S_1^x + h_y(t)S_1^y. \tag{26}$$

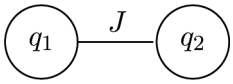

Fig. 2. Spin chain for the quantum Deutsch algorithm.

Hereafter parameters $h_{x,y}$, are called controls and are chosen to be piecewise constant. Additionally, they are applied in sequence, so for $t = 0$ we have $H_c(0) = h_x(0)S_1^x$ which is kept constant until $t = \Delta t$. Next, for $t = \Delta t$ we have $H_c(\Delta t) = h_y(\Delta t)S_1^y$ and so on. Full evolution of the system, schematically depicted in Figure 2 is described by a unitary matrix U, of the form:

$$\begin{aligned} U &= \prod_{n=0}^{L-1} e^{-i\Delta t(H_0+H_c(n\Delta t))} && (27)\\ &= e^{-i\Delta t(H_0+H_c((L-1)\Delta t))} \dots e^{-i\Delta t(H_0+H_c(0))} && (28)\\ &= U((L-1)\Delta t) \dots U(\Delta t)U(0), && (29) \end{aligned}$$

where L if the total number of controls.

The goal is to find such sequence of controls that maximizes gate fidelity

$$F = \frac{1}{2^N}|U_T^\dagger U|, \tag{30}$$

where U_T is the target gate, given by one of the equations (22), (23), (24) or (25) and † denotes the hermitian conjugate of matrix U.

After the best sequence of control parameters for given L was found, the same parameters were applied to a system with an additional qubit. A schematic representation of this system is shown in Figure 3. In this case, the third qubit was modeling the interaction of the two-qubit system with an environment.

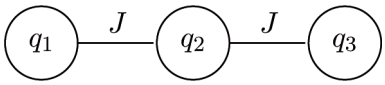

Fig. 3. Spin chain for the quantum Deutsch algorithm and simulated environment.

5. Results

The following parameters were used in the numerical simulation: $J = 1$ and $L\Delta t = 3$. In the case of three-qubit simulation, the additional qubit was coupled to the second qubit with a coupling constant $J = 1$.

Results obtained from numerical experiments are shown in Figures 4, 5. The procedure for obtaining data points was as follows. For each L there were made 100 optimizations using `scipy.optimize` [7]. Python module. After this steps, the result with the highest fidelity was chosen as the data point.

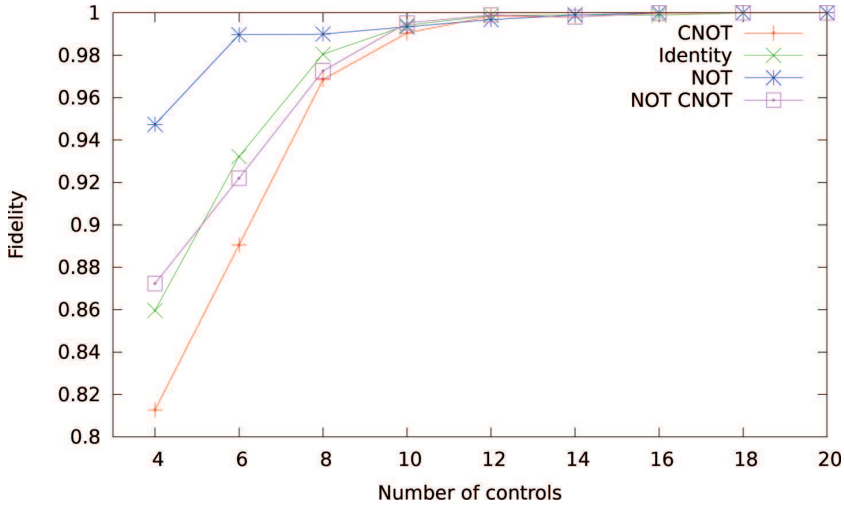

Fig. 4. Dependence of fidelity on number of control steps.

6. Conclusions and future work

Figures 4 and 5 show the results of numerical simulations. They show results only for even number of controls. They show that for $L = 12$ the Deutsch algorithm is implemented with very high fidelity. Adding additional control steps does not seem to have a significant impact on the quality of the algorithm. Also the tables show that it is almost certain that the algorithm will yield the correct result.

Adding a third qubit, of which the implementation of the algorithm is unaware, has drastic impact on the results. Probabilities of obtaining correct results from the algorithm we between 0.25 and 0.89 with a mean value around 0.5. Adding more control steps did not improve the situation. So, adding an "environment" renders the algorithm useless.

Further work may include changing the coupling constant between the second and third qubit, or maybe even changing the entire system into a spin graph, where both

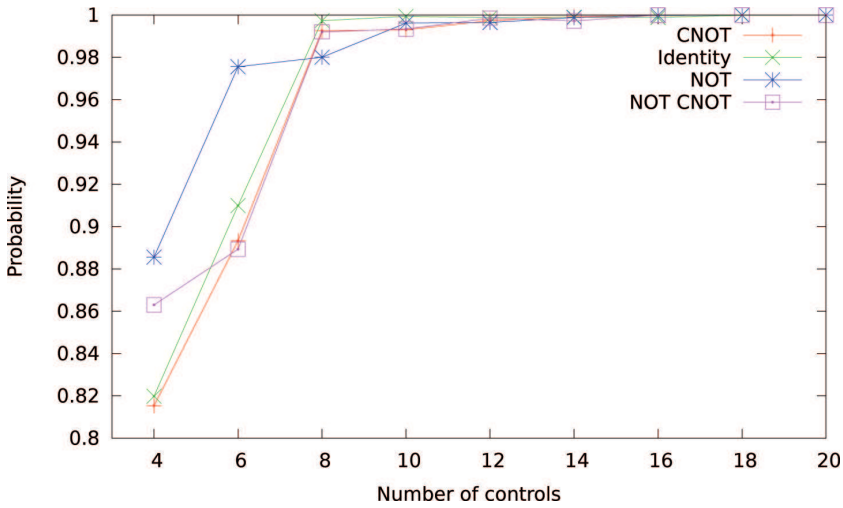

Fig. 5. Dependence of probability of the correct outcome on number of control steps (two qubits).

qubits interact with the "environment". It may be possible to find such values of J_{13} and J_{23} that the algorithm will be stable, what is suggested by [8].

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Parametry kontrolne dla kwantowego algorytmu Deutscha

Streszczenie

Przedstawiono przykład dwu-kubitowego znajdowania optymalnych wartości kontrolnych łańcucha spinowego implementującego kwantowy algorytm Deutscha. W tym artykule rozważane są dwa przypadki: systemy dwu-kubitowe i trój-kubitowe. Ostatni przypadek jest użyty do zbadania wpływu oddziaływania ze środowiskiem na wynik algorytmu.