

## Heat transfer through the regular polyhedrons with asymmetric boundary conditions

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**Abstract** During heat transport through the walls of a hollow sphere, the heat stream can achieve extreme values. The same processes occur in regular polyhedrons. We can calculate the maximum heat transfer rate, the so-called critical heat transfer rate. We must assume here identical conditions of heat exchange on all internal and external walls of a regular polyhedron. The transfer rate of heat penetrating through the regular polyhedron with different heat transfer coefficients on the walls is called the heat transfer rate with asymmetric boundary conditions. We show that the heat transfer rate in this case will grow up if we replace those coefficients with their average values.

**Keywords:** Heat transfer rate; Regular polyhedrons; Asymmetric boundary condition

### Nomenclature

|           |   |                               |
|-----------|---|-------------------------------|
| $d$       | – | diameter, m                   |
| $n$       | – | number of sides of polyhedron |
| $P$       | – | area, m <sup>2</sup>          |
| $r$       | – | radius, m                     |
| $t$       | – | temperature of fluid, °C      |
| $\dot{Q}$ | – | heat transfer rate, W         |
| $x_i$     | – | real number                   |

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### Greek symbols

- $\alpha$  – heat transfer coefficient, W/m<sup>2</sup>K
- $\delta$  – thickness, m
- $\lambda$  – thermal conductivity, W/mK
- $v$  – temperature of surface, °C

### Subscripts and superscripts

- $as$  – asymmetric boundary condition
- $t$  – total
- $i, j$  –  $i$ -,  $j$ -function ( $i = 1, 2, \dots, n$ ), ( $j = 1, 2, \dots, n$ )
- $s$  – symmetric boundary condition
- 1, 2 – internal and external, respectively
- 0 – internal base

## 1 Introduction

Processes connected with heat transfer through spherical walls are described in [1,2]. In this context the problem of critical diameter for the hollow sphere can be noticed. It is analogical in the mathematical and physical meaning to the idea present for cylindrical and polygonal pipes [3,4].

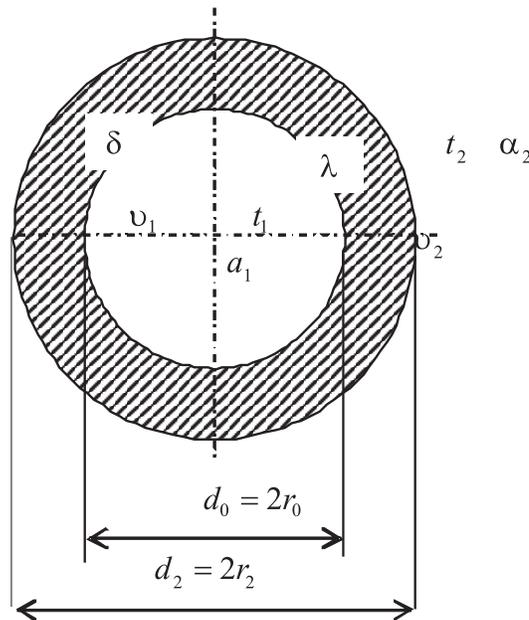


Figure 1. The cross-section of the hollow sphere with appropriate notation.

If we use Newton's and Fourier's equations for description of heat transfer through spherical walls (Fig. 1), the heat transfer rate through the hollow sphere [1] is expressed by the expression:

$$\dot{Q}_r = \frac{4\pi(t_1 - t_2)}{\frac{1}{\alpha_1 r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_2 r_2^2}}, \quad (1)$$

where  $\alpha_1$ ,  $\alpha_2$ , are the heat transfer coefficients,  $t_1$ ,  $t_2$  temperature of fluid on inner (1) and outer (2) surface, respectively, and  $\lambda$  is the thermal conductivity. For given data  $\alpha_1$ ,  $\alpha_2$ ,  $\lambda$ ,  $t_1$ ,  $t_2$  and specified radius of the interior area  $r_0$  we can select the radius  $r_2$  at which the heat transfer rate will achieve the maximum of its value (critical value), the so-called critical radius (diameter).

Regular polyhedrons are composed of identical, truncated pyramids, which adhere by lateral surfaces. Bases of the pyramids are regular polygons, homothetic with respect to the centre of similarity in point  $O$  (Fig. 2). Lateral surfaces are insulated (heat exchange does not occur between adjoining pyramids). In polyhedron we can inscribe the hollow sphere. It is tangential to the respective bases of pyramids, which form the internal and external surface of the total solid. Material of the polyhedron is anisotropic with respect to thermal conductivity, which value is finite in perpendicular planes, and is infinite in planes parallel to the bases of pyramids.

If we use (for the above mentioned regular polyhedrons) heat transmission laws, geometrical relationships between specific areas and radii of a hollow sphere which is inscribed in a polyhedron,  $\frac{P_2}{P_0} = \frac{r_2^2}{r_0^2}$ , and boundary conditions (Fig. 2), after necessary transformations we obtain a heat transfer rate through a single truncated pyramid which is a repetitive element of the structure of the polyhedron, therefore, the total heat transfer rate is:

$$\dot{Q}_t = \frac{r_0^2}{n P_0} \frac{t_1 - t_2}{\left[ \frac{1}{\alpha_1 r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_2 r_2^2} \right]}, \quad (2)$$

where  $n$  is the number of polyhedron sides.

Now we analyze the influence of the heat transfer coefficient changes of the regular polyhedron circumference on the value of the heat transfer rate. The aim is to compare the values of the heat transfer rate through the regular polyhedron in two cases. In the first one the heat transfer coefficients are different on each side, while in the second one the heat transfer coefficients are identical and equal to their average value.

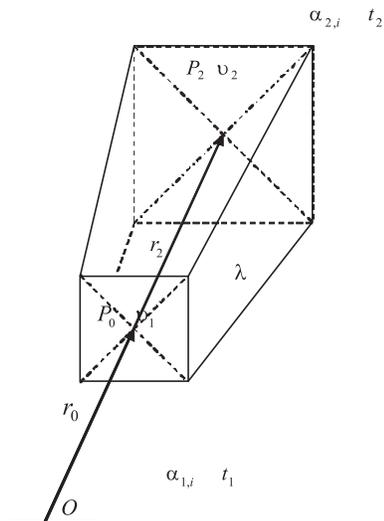


Figure 2. Truncated pyramids with square bases as an example of repeatable element for the regular hexahedron.

## 2 Calculations

For any real number  $x_i$  and for any concave function,  $f$ , the Jensen inequality is true [5]:

$$\sum_{i=1}^n w_i f(x_i) \leq f\left(\sum_{i=1}^n w_i x_i\right), \quad (3)$$

where the so-called Jensen's weights  $w_i$  are non-negative. In our case Jensen's weights are identical on each wall and equal  $w_i=1/n$ . This is due to the geometrical properties of regular polyhedrons:

$$\begin{cases} P_0 = const, \\ \sum_{i=1}^n P_0 = nP_0 = P, \\ w_i = \frac{P_0}{P} = \frac{1}{n}, \\ \sum_{i=1}^n w_i = \sum_{i=1}^n \frac{1}{n} = 1. \end{cases} \quad (4)$$

We assume in addition that the heat-transfer coefficients  $\alpha_{1,i}$ ,  $\alpha_{2,i}$  for individual walls have different values. Taking this into account, using the inequality (3) and formula (2) we get for the case with asymmetric bound-

ary conditions the relation

$$\frac{w_1 \frac{r_0^2}{P_0} \left[ \frac{1}{\alpha_{1,1} r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_{2,1} r_2^2} \right] + \dots + w_n \frac{r_0^2}{P_0} \left[ \frac{1}{\alpha_{1,n} r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_{2,n} r_2^2} \right]}{w_1 + \dots + w_n} \leq$$

$$\frac{r_0^2}{P_0} \left[ \frac{1}{\frac{r_0^2 (w_1 \alpha_{1,1} + \dots + w_n \alpha_{1,n})}{w_1 + \dots + w_n}} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\frac{r_2^2 (w_1 \alpha_{2,1} + \dots + w_n \alpha_{2,n})}{w_1 + \dots + w_n}} \right], \quad (5)$$

which after rearrangements can be transformed to:

$$\frac{1}{n} \sum_{i=1}^n \frac{r_0^2}{P_0} \left[ \frac{1}{\alpha_{1,i} r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_{2,i} r_2^2} \right] \leq$$

$$\frac{r_0^2}{P_0} \left[ \frac{1}{r_0^2 \sum_{i=1}^n \frac{1}{n} \alpha_{1,i}} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{r_2^2 \sum_{i=1}^n \frac{1}{n} \alpha_{2,i}} \right]. \quad (6)$$

Finally, we get

$$\sum_{i=1}^n \frac{r_0^2}{P_0} \left[ \frac{1}{\alpha_{1,i} r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_{2,i} r_2^2} \right] \leq n \frac{r_0^2}{P_0} \left[ \frac{1}{\alpha_1 r_0^2} + \frac{1}{\lambda} \left( \frac{1}{r_0} - \frac{1}{r_2} \right) + \frac{1}{\alpha_2 r_2^2} \right], \quad (7)$$

where

$$\begin{cases} \alpha_1 = \frac{\sum_{j=1}^n \alpha_{1,j}}{n}, \\ \alpha_2 = \frac{\sum_{j=1}^n \alpha_{2,j}}{n}. \end{cases} \quad (8)$$

The expressions (8) represent the average values of heat transfer coefficients on the inner and outer surfaces of the solid, respectively.

The expression on the left hand side of (7) is the formula for the heat transfer through the regular polyhedron, taking into consideration the asymmetry of boundary conditions on the inner and outer surface,  $\dot{Q}_t^{as}$ , (so called heat transfer rate with asymmetric boundary conditions), while the expression on the right,  $\dot{Q}_t^s$ , shows the formula for the rate of heat with average heat transfer coefficients (8), which are constant (heat transfer rate with symmetric boundary conditions), therefore

$$\dot{Q}_t^{as} \leq \dot{Q}_t^s, \quad (9)$$

and the structure of  $\dot{Q}_t^s$  is identical with Eq. (2).

We can see that Jensen's inequality goes into the equation only if the heat transfer coefficients are equal on the inside and outside of the body. Such conditions generally does not occur in nature. However we can obtain them in laboratory conditions.

## 2.1 Simulation results

In the example below we analyzed the heat transfer for the tetrahedron. The simulation assumes the following data:  $t_1 - t_2 = 100$  K,  $r_0 = 1$  m,  $\lambda = 100$  W/mK,  $\alpha_{1,1} = 10$  W/m<sup>2</sup>K,  $\alpha_{1,2} = \alpha_{1,3} = \alpha_{1,4} = 30$  W/m<sup>2</sup>K,  $\alpha_{2,1} = 10$  W/m<sup>2</sup>K,  $\alpha_{2,2} = \alpha_{2,3} = \alpha_{2,4} = 30$  W/m<sup>2</sup>K,  $\bar{\alpha}_1 = 25$  W/m<sup>2</sup>K,  $\bar{\alpha}_2 = 25$  W/m<sup>2</sup>K.

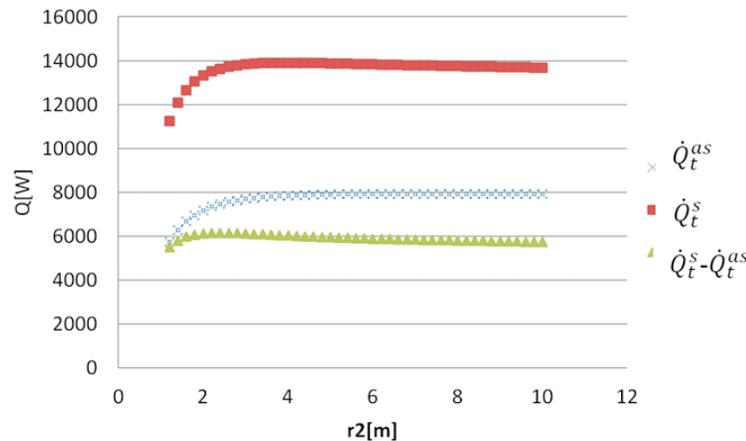


Figure 3. Dependency of the heat transfer rate on the inner area radius:  $\dot{Q}_t^s = f(r_2)$ ,  $\dot{Q}_t^{as} = f(r_2)$ ,  $\dot{Q}_t^s - \dot{Q}_t^{as} = f(r_2)$ .

The heat transfer rate in the asymmetric case is smaller than in the case with the average heat transfer coefficients (Fig. 3).

We can show the changes of the heat transfer rate together with the changes of the heat transfer coefficients for the regular polyhedron with  $n = 20$  sides. The simulation assumes the following data:  $t_1 - t_2 = 100$  K,  $r_0 = 1$  m,  $P_0 = 1$  m<sup>2</sup>,  $\lambda = 100$  W/mK,  $r_2 = 4$  m, and the heat transfer coefficients changes for each internal surface in the following way:  $i = 1$  means  $\alpha_{1,1} = 10$  W/m<sup>2</sup>K for 1 wall and  $\alpha_{1,i} = 20$  W/m<sup>2</sup>K for 9 walls,  $j = 2$  means  $\alpha_{1,1} = \alpha_{1,2} = 10$  W/m<sup>2</sup>K for 2 walls and  $\alpha_{1,i} = 20$  W/m<sup>2</sup>K for 8 walls and so on. Heat transfer coefficients change only on the inner surface

of the polyhedron, and have a value  $10 \text{ W/m}^2\text{K}$  or  $20 \text{ W/m}^2\text{K}$  (step change), and  $j$  is the number of walls on the inner surface with the lower coefficient of heat transfer. In this case relation (9) is true (Fig. 4).

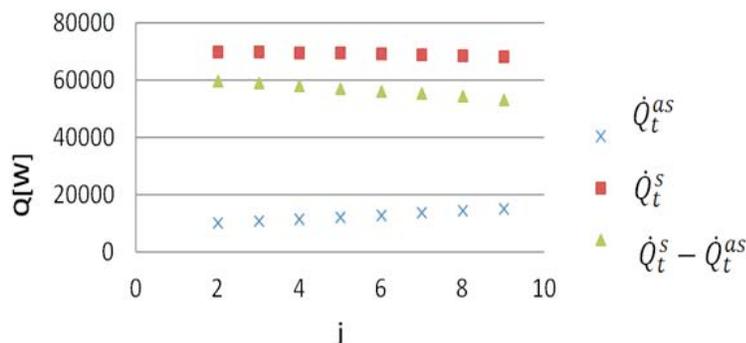


Figure 4. Influence of heat transfer coefficients on the inner surface of the regular polyhedron with  $n = 10$  sides on the values of the heat transfer rate. The dependencies from the top respectively:  $\dot{Q}_t^s = f(j)$ ,  $\dot{Q}_t^s - \dot{Q}_t^{as} = f(j)$ ,  $\dot{Q}_t^{as} = f(j)$ , where  $j$  – the number of pairs of walls with the lower coefficient of heat transfer.

The same effect as in (9) occurs when the heat transfer coefficients change on the external surface of the solid polyhedron only.

Now the heat transfer coefficients change on the internal and external surface in the following way:  $j = 1$  means  $\alpha_{1,1} = 10 \text{ W/m}^2\text{K}$  and  $\alpha_{2,1} = 10 \text{ W/m}^2\text{K}$  for 1 pair of inner and outer walls and  $\alpha_{1,i} = 20 \text{ W/m}^2\text{K}$ ,  $\alpha_{2,i} = 20 \text{ W/m}^2\text{K}$  for 9 pairs of walls,  $j = 2$  means  $\alpha_{1,1} = \alpha_{1,2} = \alpha_{2,1} = \alpha_{2,2} = 10 \text{ W/m}^2\text{K}$  for 2 pairs of walls and  $\alpha_{1,i} = \alpha_{2,i} = 20 \text{ W/m}^2\text{K}$  for 8 pairs of walls and so on. The heat transfer coefficients have the value  $10 \text{ W/m}^2\text{K}$  or  $20 \text{ W/m}^2\text{K}$  (step change) and  $j$  is the number of pairs of parallel to each other walls with the lower coefficient of heat transfer. In this case relation (9) is true also (Fig. 5).

### 3 Conclusions

1. Physical phenomena of heat transfer through regular polyhedrons have (excluding laboratory conditions) an asymmetric nature only. The value of heat transfer rate is obtained by Eq. (2) with heat transfer coefficient as in (8). Mathematically it is easy to calculate. In the practice of engineering we need accurate measurements of the heat transfer coefficients on each wall of the body and it is difficult,

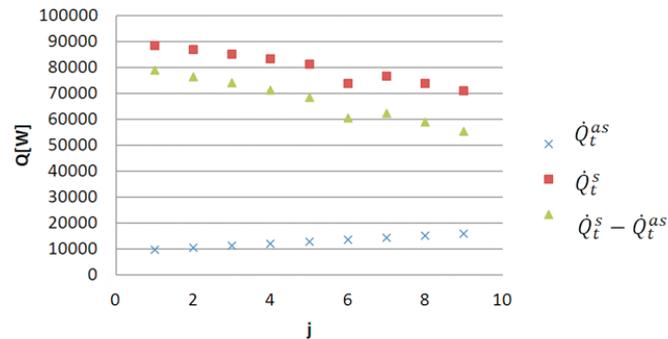


Figure 5. Influence of heat transfer coefficients on the inner and outer surface of the regular polyhedron with  $n = 10$  sides on the values of the heat transfer rate. The dependencies from the top respectively:  $\dot{Q}_t^s = f(j)$ ,  $\dot{Q}_t^s - \dot{Q}_t^{as} = f(j)$ ,  $\dot{Q}_t^{as} = f(j)$ , where  $j$  – the number of pairs of walls with the lower coefficient of heat transfer.

because there is no equipment for measuring the local heat transfer coefficient (we are able to measure this factor only in certain cases).

2. In theory the right and the left hand sides of relation (9) are equal only if the heat transfer coefficients on all the walls of polyhedrons are equal. Such conditions generally does not occur in nature, although we can obtain them in laboratory tests. Generally, as a rule, the left-hand side of (9) is less than the right one:  $\dot{Q}_t^{as} < \dot{Q}_t^s$ .
3. The difference  $\dot{Q}_t^s - \dot{Q}_t^{as} = f(r_2)$  decreases to asymptotic limit together with the growth of the external polyhedrons sizes.
4. The concept of the critical diameter described in the literature [1] in the case of the asymmetric boundary conditions makes sense for the average value of the heat transfer coefficient on the internal and external areas of the walls only.
5. The results obtained can be generalized to all solids, which are described on the sphere.
6. The issues considered in the paper may have some practical applications for example in measuring techniques. The apparatus are often protected against excessive temperature and heat stream. The easily calculated  $\dot{Q}_t^s$  with the correctly obtained average heat transfer coefficient ensures improvement of the conditions of service for the often

expensive electronic equipment and its safety. The second area of application may be estimating the maximum heat losses in buildings or in other structures with the shape of a polyhedron. The work can also be useful for the construction of the apparatus for measuring heat transfer coefficients too, so called 'alpha-measurement'.

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